Vlasov Theory of Equilibria

→ $\boxed{\text{Vlasov - Poisson} + \begin{array}{c}\text{Weak/strong} \\ \text{Jeans Thm}\end{array}}$

$f = f(J_1, J_2, J_3)$ is soln. of V.E

3 "isolating integrals", at most

so $\nabla^2\phi = 4\pi G \int d^3v \, f(J_1, J_2, J_3)$ ⇒ eqbrm.

n.b. maximum entropy ?
minimum energy (w/ wo constraints) ?

BGK soln vs. equilibrium ⇒ stability ? (i.e. Jeans)

→ Examples

① Spherical (~~basic models~~)
- ~~simplest~~ galaxy = disk + core
- globular clusters ...

$f = f(\mathcal{E}, r, |L|)$
$= f(\mathcal{E}, r)$
$= n(r) \, f(\mathcal{E})$ separable

- Polytropes / Plummer Model
- Lane - Emden
- isothermal sphere
- King model

model $f(\mathcal{E})$, obtain $n(r)$ ⇒ luminosity, observations

② Disk - Mestel

③ instead $f(\mathcal{E}) \Rightarrow n(r)$
then $n(r) \Rightarrow f(\mathcal{E})$

Eddington's Eqn.

then

→ which equilibrium ⇒ Violent Relaxation, etc.
(Lynden-Bell)

⇓

true $\begin{cases} \text{max/min} \\ \text{most probable} \end{cases}$

→ Preliminary – Solutions

→ Jeans $f = f(\text{IoM})$ solves Vlasov Eqn.

i.e. $\frac{df}{dt} = \frac{\partial f}{\partial I} \frac{dI}{dt} = 0$

→ strong Jeans

If almost all orbits $\begin{cases} \text{regular} \\ \text{non-resonant frequencies} \\ (\underline{m} \cdot \underline{\omega} \neq 0) \end{cases}$ $\Rightarrow f = f(J_1, J_2, J_3)$

3 actions

Proof Lemma: (Ergodic Thm)

⇒ When regular orbit is non-resonant, average time T that orbit spends in any region D of its torus is $\sim \underset{\text{volume}}{V(D)} \sim \int_D d^3\Theta$

(time avg ~ phase space avg.)

N.B. $\rightarrow$ orbits : $(\underline{J}, \underline{\theta})$

$$
\begin{aligned}
&H = H(\underline{J}, \underline{\theta}) \\
&\begin{cases}
\dfrac{d\underline{J}}{dt} = -\dfrac{\partial H}{\partial \underline{\theta}} \\[2ex]
\dfrac{d\underline{\theta}}{dt} = \underline{\omega}(\underline{J}) = \dfrac{\partial H}{\partial \underline{J}}
\end{cases}
\end{aligned}
$$

$\rightarrow$ isolating integral

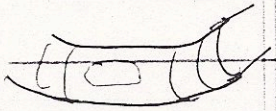
(~ independent separation constant)

integrability $\Rightarrow$ separability of H-J equation

$\Rightarrow$ separation constant $\Leftrightarrow$ IOM, [6D $\Rightarrow$ 3 constants]

$\rightarrow$ Pf, cont'd. $\quad f_D$ s/t $\begin{cases} f_D(\underline{\theta}) = 1 & \underline{\theta} \in D \\ f_D = 0, & \text{otherwise} \end{cases}$

δ distribution



$$f_D(\underline{\theta}) = \sum_{\underline{n}=-\infty}^{+\infty} F_{\underline{n}} e^{i\underline{n}\cdot\underline{\theta}}$$

ampl, $F_n = F_n(J)$, only

$$\int_{\text{torus}} d^3\theta \, f_D(\underline{\theta}) = \int_D d^3\theta \, \underset{1}{\cancel{f}} = V(D) \rightarrow \boxed{\text{Volume}}$$

now, also, $$V(D) = \int_{\text{torus}} d^3\theta \, f_D(\underline{\theta}) = \sum_{\underline{n}=-\infty}^{+\infty} F_{\underline{n}} \prod_{k=1}^{3} \int_0^{2\pi} d\psi \, \underset{(\text{phase})}{e^{in_k\psi}}$$

$$= (2\pi)^3 F_{\underline{0}} \qquad (\text{norm}).$$

now time of dwell

$$T_T(D) = \frac{1}{T}\int_0^T dt\, f_D(\overset{\text{position}}{\underline{\theta}(t)})$$

$$= \frac{1}{T}\int_0^T dt \sum_{\underline{n}} F_n e^{i(\underline{n}\cdot\underline{\theta})}$$

$$= \frac{1}{T}\int_0^T dt \sum_{\underline{n}} F_n e^{i\underline{n}\cdot\underline{\omega}t} e^{i\underline{n}\cdot\theta(0)}$$

$$= F_0 + \frac{1}{T}\sum_{n\neq 0} e^{i(\underline{n}\cdot\theta(0))} F_n \frac{e^{i(\underline{n}\cdot\underline{\Omega})T} - 1}{i(\underline{n}\cdot\underline{\Omega})}$$

if $\underline{n}\cdot\underline{\Omega} \neq 0$ (no resonances) $\rightarrow$ 3 integrals isolating

$$\boxed{\begin{aligned}\lim_{T\to\infty} T_T(D) &= F_0 = \left[(2\pi)^3\right]^{-1} V(D) \\ &= V(D)/(2\pi)^3\end{aligned}}$$

Dwell time

$\therefore$ $\underline{n}\cdot\underline{\omega} \neq 0$ $\Rightarrow$ $\boxed{\text{Volume }(D) \text{ (phase space)} \sim \text{Dwell Time }(D)}$

Ergodic Thm.

N.B. If resonance

N.B. if resonance $\underline{n} \cdot \underline{\Omega} = 0$

$\Rightarrow n_1 \theta_1 + n_2 \theta_2 + n_3 \theta_3 = \text{const}$ for some n_1, n_2, n_3

$\Rightarrow$ star confined to spiral $\leftrightarrow$ additional isolating integral

$\leftrightarrow \mu = 0$ set.

c) strong Jeans: for any observable Q

$$\langle Q \rangle = \int d^3x \int d^3v \, Q f \qquad f(\underline{x}, \underline{v}, t)$$

$Q(\underline{x}, \underline{v})$ smooth

$\Rightarrow$

$$\langle Q \rangle = \int d^3\theta \int d^3J \, Q(\underline{x}, \underline{v}) \, f(\underline{x}, \underline{v}, t)$$

$\Rightarrow$ in steady state $f(\underline{x}, \underline{v}, t) \to \bar{f}(\underline{x}, \underline{v}) \to \bar{f}(\underline{\theta}, \underline{J})$ (@ time order)

$$\langle Q \rangle = \bar{Q} = \int d^3\theta \int d^3J \, Q(\underline{\theta}, \underline{J}) \, \bar{f}(\underline{\theta}, \underline{J})$$

$$\bar{f} = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \, f(\underline{x}, \underline{v}, t)$$

but $\bar{f} \, d^3\underline{J} \, d^3\theta \to$ probability in $d^3J d^3\theta$ volume

by ergodic $\sim \frac{d^3\theta}{(2\pi)^3}$ prob. actions in d^3J

eff action dist'
i.e. $\langle F_0(\underline{J})\rangle$

$$\Rightarrow \quad \int \frac{d^3\theta}{(2\pi)^3} \left[\int d^3\underline{J} \; f_J(\underline{J}) \right]$$

$F_0 \rightarrow J$ only

$$\Rightarrow \quad \langle Q\rangle \sim \int_V d^3\theta \int d^3J \; Q(\theta, J) \; f_J(J)$$

$$\text{i.e.} \quad \bar{F}(\theta, J) = \frac{1}{T}\int^T dt\, f(\underline{x}, \underline{v}, t)$$

$$= f_J(\underline{J})$$

F_0, $\langle f(J)\rangle$ $\Rightarrow$ [Fixation with J^{th} integral]

Spherical Systems

$$f = f\left(\frac{mv^2}{2} + \Phi, |L|\right)$$

so

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 4\pi G \int d^3v\, f\left(\frac{v^2}{2} + \Phi, |\underline{r}\times\underline{v}|\right)}$$

$\Rightarrow$ class of solutions ?

$$\Psi = -\Phi + \Phi_0 \quad , \quad \mathcal{E} = -E + \Phi_0$$

relative potential $\qquad$ energy $= \Psi - v^2/2$

B.C.: $\Psi \rightarrow \Phi_0$ at ∞.

$$\boxed{\nabla^2 \Psi = -4\pi G \rho}$$

Now, if $f = f(\mathcal{E})$

$= f(\Psi - v^2/2)$

isotropic dispersion (cf Jeans Eqns)

$$\Rightarrow \boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) = -16\pi^2 G \int_0^{\sqrt{2\Psi}} f(\Psi - v^2/2)\, v^2 dv = -16\pi^2 G \int_0^{\Psi} f(\mathcal{E}) \sqrt{2(\Psi - v^2/2)}\, d\mathcal{E}}$$

Φ_0 s/t $f(\mathcal{E})$ for $\mathcal{E} < 0$ (un(trapped / bound) zone)

$\rightarrow$ NL eqn. for Ψ (given $f(\mathcal{E})$) $\rightarrow$ Plummer, King, iso sphere

(vel $\rightarrow$ position)

$\rightarrow$ linear eqn. for $f(\mathcal{E})$ given Ψ (Eddington)

position $\rightarrow$ velocity

→ Polytropes / Plummer

$$f(\mathcal{E}) = F\,\mathcal{E}^{n-3/2} \qquad \mathcal{E}>0$$
$$= 0 \qquad \mathcal{E}<0$$

so

$$\rho = 4\pi \int_0^\infty f\left(\Psi - \frac{v^2}{2}\right) v^2\,dv$$

$$= 4\pi F \int_0^{\sqrt{2\Psi}} \left(\Psi - v^2/2\right)^{n-3/2} v^2\,dv$$

now, if $\quad v^2 = 2\Psi \cos^2\theta$

$$\Rightarrow \rho = C_n \Psi^n$$

$$\left\{ \begin{aligned} C_n &= 2^{7/2}\, F\pi \left[\int_0^{\pi/2} \sin^{2n-2}\theta\, d\theta - \int_0^{\pi/2} \sin^{2n}\theta\, d\theta \right] \\ &= \frac{(2\pi)^{3/2}\,(n-3/2)!}{n!}\, F \end{aligned} \right.$$

→ NL Poisson Eqn.

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) = -4\pi G\, C_n \Psi^n}$$

<u>Now</u>, $s \equiv r/b$

$\psi = \Psi / \Psi_0$

$\Psi_0 = \Psi(0)$

$b \equiv (4\pi G \Psi_0^{n-1} C_n)^{-1/2}$

$$\Rightarrow \boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi}{ds}\right) = \begin{cases} -\psi^n & \psi > 0 \\ 0 & \psi \leq 0 \end{cases}}$$

Lane-Emden Eqn.

→ b.c. $\left.\frac{d\psi}{ds}\right|_0 = 0$

(no grav. force at center)

$\psi(0) = 1$, defn.

- Lane-Emden familiar from stellar structure

$$\frac{dP}{dr} = -\rho \frac{d\Phi}{dr} \qquad \text{i.e.} \quad \frac{dP}{dr} = \rho g(r)$$

(hydro eqbrm)

$P = K \rho^\gamma$

polytropic gas

i.e.

$$K\gamma\rho^{\gamma-2}\frac{d\rho}{dr} = \frac{d\Psi}{dr}$$

$\Psi = 0$ on edge

$$\rho^{\gamma-1} = \frac{\gamma-1}{K\gamma}\Psi$$

$$\rightarrow \rho = \left(\frac{\gamma-1}{K\gamma}\right)^{1/\gamma-1}\Psi^{1/\gamma-1}$$

$$\rightarrow \rho = C_n \Psi^n$$

$n \Leftrightarrow 1/\gamma-1$

obvious correspondence

$\rightarrow \gamma_{eff} = 1 + 1/n$

i.e. density of stellar dynamics / Vlasov polytrope sphere same as gas sphere with $\gamma_{eff} = 1 + 1/n$.

Solutions:

→ general unknown

→ $n = 1$ → Helmholtz → Hw.

→ $n = 5$ → Schuster

$$\Psi = \frac{1}{\left(1 + \frac{s^2}{3}\right)^{1/2}}$$ → solves L-E with $n=5$

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$$\boxed{\begin{aligned}\rho &= c_5 \Psi^5 \\ &= c_5 \Psi_0^5 \Big/ \left(1+ r^2/b^2\right)^{5/2}\end{aligned}}$$

→ Plummer model

density finite everywhere

but $\underset{\substack{\downarrow \\ \text{total}}}{\underline{M}}$ finite

$$M = \sqrt{3}\, \Psi_0 b / G$$

→ Plummer model fits globular clusters moderately well

→ generally falls off too fast $\rho \sim r^{-5}$ for many

→ M finite, $\rho \neq 0$ everywhere

Now, $n < 5$ ⟨ $\rho \to 0$, finite radius

$n = 5$ ⟨ ρ finite; M finite

$n > 5$ ⟨ ρ decay so slowly ↓ M infinite

→ Isothermal Sphere

$n \to \infty \Leftrightarrow \gamma \to 1 \Rightarrow$ "isothermal" system

i.e. $f(\mathcal{E}) = \dfrac{\overset{\text{const.}}{\rho_1}\, e^{\mathcal{E}/\sigma^2}}{(2\pi\sigma^2)^{3/2}}$ ↳ const dispersion

$= \dfrac{\rho_1\, e^{(\Psi - v^2/2)/\sigma^2}}{(2\pi\sigma^2)^{3/2}}$

$\Rightarrow \quad \rho_* = \rho_1 e^{\Psi/\sigma^2}$

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial \Psi}{\partial r} = -4\pi G \rho_1 e^{\Psi/\sigma^2}$$

$$\Rightarrow \quad \boxed{\frac{d}{dr}\left(r^2 \frac{d\ln\rho}{dr}\right) = \frac{-4\pi G}{\sigma^2} r^2 \rho} \quad *$$

but for $\gamma = 1$ polytope,

$$p = \frac{k_B T}{m}\rho$$

$$\frac{dP}{dr} = \frac{k_B T}{m}\frac{d\rho}{dr} = -\rho\frac{GM(r)}{r^2}$$

$$\Rightarrow \boxed{\frac{d}{dr}\left(r^2\frac{d\ln\rho}{dr}\right) = -\frac{Gm}{k_B T}4\pi r^2\rho} \quad \cancel{**}$$

* same as ** ! $\quad \sigma^2 = \frac{k_B T}{m}$ (no surprise)

$\Rightarrow n\to\infty$, $f(\varepsilon) = e^{\varepsilon/\sigma^2}$ is collisionless system with same structure as collisional one

$\Rightarrow$ Gaussian, with const dispersion $\Rightarrow$ isothermal sphere structure

Solution:

$$\rho \sim r^\alpha$$

$$r^2 r^\alpha (G) \sim \frac{d}{dr}\left(r^2 r^{-\alpha} r^{\alpha-1}\right)(\alpha-1)$$

$$\sim (1)\ (\alpha-1)$$

$\Rightarrow \alpha = -2$ is solution

$\Rightarrow \boxed{\rho(r) = \sigma^2 / 2\pi G r^2}$ singular isothermal sphere density

$\rightarrow \rho \rightarrow \infty$ as $r \rightarrow 0$ $\Rightarrow$ needs cut-off

$\rightarrow$ King Model / King radius

II.) General Theory of Equilibria and Stability for Self-Gravitating Systems.

A.) Equilibria

i.) Vlasov / Jeans Equation

→ seek phase space evolution equation

→ gravity N body ⇒ Hamiltonian

so N-body Hamiltonian system ⇒ $\underline{\nabla} \cdot \underline{V}_{\substack{phase\\space}} = 0$
(Liouville's Thm.)

so if define:

$$N' = \sum_{i=1}^{N} \delta(\underline{x} - \underline{x}_i(t))\, \delta(\underline{v} - \underline{v}_i(t))$$

- formally exact dist. for N point system
- Liouville distrib. function

$$\Rightarrow \quad \frac{\partial N'}{\partial t} + \underline{\nabla}_\Gamma \cdot (\underline{V}_\Gamma N') = 0$$

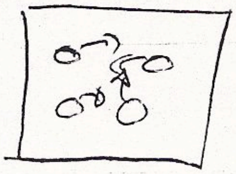
so

$$\frac{\partial N'}{\partial t} + \underline{V}_\Gamma \cdot \underline{\nabla}_\Gamma N' = 0$$

i.e. Liouvillian conserved along particle trajectories

- but $\acute{N}$ not useful → N points in 6N-dim space, but ultimately seek <u>collective</u> description:

<u>i.e.</u>



→ <u>i.e.</u> 'follows individual peas'

better: 'density of pea soup, from "crushed peas"'

<u>i.e.</u>,

$$\acute{N} = \sum_{i=1}^{N} \delta(\underline{x}-\underline{x}_i(t))\,\delta(\underline{v}-\underline{v}_i(t))$$

↓

N-point distribution function replaced by:

$f(\underline{x},\underline{v},t) \equiv$ 1 point distribution (phase space density)

As $f = \text{const.}$ along phase space trajectories:

$$\frac{d\underline{x}}{dt} = \underline{v}, \qquad \frac{d\underline{v}}{dt} = -\underline{\nabla}\Phi \qquad (m=1)$$

$$\boxed{\nabla^2\phi = 4\pi G\rho = 4\pi G\int d^3v\, f(\underline{x},\underline{v},t)}$$

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f - \underline{\nabla}\phi\cdot\frac{\partial f}{\partial \underline{v}} = 0}$$

Vlasov/Jeans - Poisson system

→ Key questions: $\begin{cases}\text{Gas} \\ \text{Gravity}\end{cases}$

1) how get from Liouville N to f ?

2) relation to Boltzmann equation ?
(i.e. collisions ?!).

1) Deriving the Vlasov Equation — $\begin{cases}\text{Gas} \\ \text{Gravity}\end{cases}$ (Hard sphere potential)

– Dilemma: when does $\frac{dN}{dt} = 0 \Rightarrow \frac{df}{dt} = 0$

Consider N particle system $\begin{cases}\text{Hamiltonian} \\ V = V(1,2)\end{cases}$

for N particle distribution f^N

$$\frac{\partial f^N}{\partial t} + \sum_{i=1}^{N}\left(\dot{x}_i \cdot \frac{\partial f^N}{\partial \underline{x}_i} + \dot{p}_i \cdot \frac{\partial f^N}{\partial \underline{p}_i}\right) = 0$$

$$\Rightarrow \frac{\partial f^N}{\partial t} + \sum_{i=1}^{N} \underline{v}_i \cdot \frac{\partial f^N}{\partial \underline{x}_i} - \frac{\partial f^N}{\partial \underline{p}_i} \cdot \sum_{j<i} \frac{\partial V_{ij}}{\partial \underline{x}_i} = 0$$

as $\dot{\underline{x}}_i = \underline{v}_i$

$$\dot{\underline{p}}_i = -\partial \sum_{j<i} V_{ij} / \partial \underline{x}_i$$

Now construct (BBGKY) hierarchy by integrating out points, i.e.

$$f(t, \underline{x}_1, \underline{p}_1) = \int d\Gamma_2\, d\Gamma_3 \ldots d\Gamma_N\, f^N \quad \to 1\text{ pt.}$$

$$f(t, \underline{x}_1, \underline{p}_1; \underline{x}_2, \underline{p}_2) = \int d\Gamma_3 \ldots d\Gamma_N\, f^N \quad \to 2\text{pt.}$$

$\Rightarrow$

$$\frac{\partial f^{(1)}}{\partial t} + \underline{v}_1 \cdot \frac{\partial f^{(1)}}{\partial \underline{x}_1} = \underbrace{(N-1)}_{\substack{N-1 \\ \text{binary} \\ \text{pairs}}} \int \left\{ \underbrace{\frac{\partial V_{1,2}}{\partial \underline{x}_1}}_{\text{interaction}} \cdot \frac{\partial f^{(2)}}{\partial \underline{p}_1}\, d\Gamma_2 \right\}$$

$\hookrightarrow$ 2-particle dist ($3 \to N$ intn out).

Similarly;

$$\frac{\partial}{\partial t} f^{(2)} + \underline{v}_1 \cdot \frac{\partial}{\partial \underline{x}_1} f^{(2)} + \underline{v}_2 \cdot \frac{\partial}{\partial \underline{x}_2} f^{(2)} - \frac{\partial V_{1,2}}{\partial \underline{x}_1} \cdot \frac{\partial f^{(2)}}{\partial \underline{p}_1} - \frac{\partial V_{1,2}}{\partial \underline{x}_2} \cdot \frac{\partial f^{(2)}}{\partial \underline{p}_2}$$

$$= (N-2) \int d\Gamma_3 \left[\frac{\partial f^{(3)}}{\partial \underline{p}_1} \cdot \frac{\partial V_{1,3}}{\partial \underline{r}_1} + \frac{\partial f^{(3)}}{\partial \underline{p}_2} \cdot \frac{\partial V_{2,3}}{\partial \underline{r}_2} \right]$$

etc.

$\Rightarrow$ hierarchy (coupled) $\leftrightarrow$ Liouville Eqn. non-linear

$\Rightarrow$ to truncate ?! $\to$ examine RHS of $f^{(2)}$.

Now, have for dilute gas:

$d < \bar{r} < l_{mfp} < l_{sys}$

absent in gravity!

d → range of force

$\bar{r}$ → mean interparticle spacing

l_{mfp} → mean free path

l_{sys} → system size

⟺ if: $\bar{V}$ thermal speed (i.e. avg. speed) ⟹

$$\frac{\bar{V}}{d} > \frac{\bar{V}}{\bar{r}} > \frac{\bar{V}}{l_{mfp}} > \frac{\bar{V}}{l_{sys}}$$

$\frac{\bar{V}}{d}$ → $1/\tau_{coll.}$ → collision time

$\frac{\bar{V}}{l_{mfp}}$ → ν (colln freq.)

$\frac{\bar{V}}{l_{sys}}$ → ν_{relax} (system relaxation)

Lemma

$$\text{Now, } \int d\Gamma_{3V} \frac{\partial f^{(3)}}{\partial p} \sim \frac{1}{V} \frac{\partial f^{(2)}}{\partial p} A(3)$$

$A(3)$ → dim-less fctn. containing spatial dependence.

$d\Gamma_{3V}$ → velocity dependence i.e. $d\Gamma_3 = d\Gamma_{3x}\, d\Gamma_{3V}$

$\frac{1}{V}$ → spatial normalization factor

⟹ assert

$$RHS \sim \frac{\partial V}{\partial r} \frac{\partial f^{(2)}}{\partial p} \underbrace{\frac{d^3}{\bar{r}^3}}$$

→ interaction volume filling factor

i.e. $\frac{\text{Volume } V \text{ significant}}{\text{Total volume}}$

i.e. as:

$$N\int d\Gamma_3 \frac{\partial f^{(3)}}{\partial p}\frac{\partial V}{\partial r} \sim N \frac{\partial V}{\partial r}\frac{d^3}{V}\frac{\partial f^{(2)}}{\partial p}$$

$$\sim \frac{V}{\bar{r}^3}\frac{\partial V}{\partial r}\frac{d^3}{V}\frac{\partial f^{(2)}}{\partial p}$$

$$= \frac{d^3}{\bar{r}^3}\frac{\partial V}{\partial r}\frac{\partial f^{(2)}}{\partial p}$$

$\therefore$ $RHS/LHS \sim \underbrace{d^3/\bar{r}^3}_{\to \text{ leverage is diluteness parameter}} \ll 1$

$\Rightarrow$

$$\frac{df^{(2)}}{dt}(t, \Gamma_1, \Gamma_2) = 0 \qquad \text{truncation!}$$

$\to$ consistent with notion of freely moving particles, interacting only within $d < \bar{r}$

$\to$ {molecular chaos assumption (Boltzmann)} $\to$ i.e. for $f^{(2)}$ eqn.

$$f^{(2)}(1,2,t)\Big|_{t=t_0} = f^{(1)}(1,t)\Big|_{t=t_0} f^{(1)}(2,t)\Big|_{t=t_0} \to \text{initially uncorrelated}$$

$\Rightarrow$ so:

$f^{(2)}$ remains uncorrelated, if start so!

(no growth $f^{(2)}$ - no intrinsic correlation develops)

{ ⊛ collisions / collisional dissipation →
inability to retain info. in 1 pt. function
thru strong 2-body event

2) Boltzmann Eqn. ⇒

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} = N \int d\Gamma_2 \, \frac{\partial V_{1,2}}{\partial \underline{r}_1} \cdot \frac{\partial f^{(2)}}{\partial \underline{p}_1} \qquad \text{(1 pt. eqn.)}$$

$$= N \int d\Gamma_2 \, \frac{\partial V_{1,2}}{\partial \underline{r}_1} \cdot \frac{\partial}{\partial \underline{p}_1} \left[f^{(1)}(1,t) \, f^{(1)}(2,t) \right]$$

→ Boltzmann collision operator

$$\equiv C(f)$$

Note:

- not tied to assumption of "near equilibrium"!
- collisional dissipation of Boltzmann equation arises from purely conservative 2-body interactions! ⊛ → consequence of discreteness (i.e. N-pt. distribution) ⇔ jibes with use detailed balance in H-thm.
- time scale formulation (Bogoliubov) equivalent to spatial scale ordering

i.e. Key: $d \ll \bar{r} \ll l_{mfp} \ll l_{sys}$

⇒ But, for gravity, have no small interaction range d!!?

: { Basic Time-Scales for Gravitational Interaction
{ BBGKY Validity

For general N-body system, can re-express hierarchy ordering as:

$\mathcal{T}_* \sim r_0/\bar{v}$ → time of collisional interaction (r_0 → typical range of "strong collision" → { entropy created / info. lost)

$\mathcal{T}_0 \sim l_{mfp}/\bar{v}$ → time between collisions

$\Theta_0 \sim L/\bar{v}$ → macro-relaxation time

i.e. group evolution by stages, a/a' multiple time scale expansion

$t \lesssim \mathcal{T}_*$ → rapid, chaotic evolution → tracks initial information

$\mathcal{T}_* < t <$ few $\mathcal{T}_0$ → smoothing, kinetic evotion → collisions eliminate i.c. info

few $\mathcal{T}_0 < t < \Theta$ → hydrodynamic evolution → all phase space info. lost

Estimating the scales:

1) $r_0 \sim b$ s/t $v^2 \sim \dfrac{Gm}{b}$ ⇒ $r_0 \sim \dfrac{Gm}{v^2}$

$\boxed{\mathcal{T}_* \sim Gm/v^3}$ → time of collision

→ time between collisions

$$\tau_0 = \frac{l_{mfp}}{\bar{v}} \quad ; \quad l_{mfp} = \frac{1}{n\sigma}$$

for star cluster: (blob) $\quad n = \frac{N}{R^3}$

$R \to$ cluster radius

$N \to$ # stars

$$\sigma = \pi b^2$$

$$l_{mfp} = \frac{R^3}{\pi N b^2} = \frac{v^4}{\pi (Gm)^2 N}$$

so

$$\boxed{\tau_0 = v^3 R^3 / G^2 m^2 N}$$

so $\quad \tau_* \ll \tau_0 \Longleftrightarrow \frac{Gm}{v^3} \ll \frac{v^3 R^3}{G^2 m^2 N}$

$$\frac{G^3 m^3}{R^3} \ll \frac{v^6}{N} \Rightarrow \begin{cases} \frac{Gm}{R} < \frac{v^2}{N^{1/3}} \\ \frac{GmN}{R v^2} < N^{2/3} \end{cases}$$

relevant, frequently encountered quasi-stationary limit

note: virialization ⟷ {approx. bulk energy balance}

if $\quad \frac{GmN}{R} \sim \overline{v^2}$

→ i.e. {virialization not required for hierarchy}

$$\Rightarrow \begin{rcases} \tau_* \cong \tau_G N^{-3/2} \\ \tau_0 \cong \tau_G N^{1/2} \end{rcases} \quad \tau_G \equiv \left(R^3 / Gm\right)^{1/2}$$

↳ so virialized state

rd:

③ $\Theta_0 = R/\bar{V}$

So For Ⓝ virialized system:

$$\tilde{T}_* / \tilde{T}_0 \sim 1/N^2$$ → infrequent collisions

but: → truncation of BBGKY hierarchy OK → i.e. Vlasov eqn. valid.

$$\tilde{T}_0 / \Theta_0 \sim N$$

↳ macroscopic relaxation governed by collisionless kinetic theory

i.e. in practice, no hydrodynamic regime even accessed!

Bottom line: Vlasov eqn. valid, but macro-dynamics kinetic.

Cautionary Tale:

→ initial velocity distribution a la' "Hubble Law"

$v \sim rH$ with:

↳ relative velocity of 2 galaxies

$$H \sim \left(Gm N / R^3\right)^{1/2} \sim \tilde{T}_G^{-1} N^{1/2}$$

i.e. $V \sim \frac{r}{R} \left(\frac{GmN}{R}\right)^{1/2}$

$\sim \frac{r}{R} V_v$

⇒

$$\left.\begin{aligned} T_* &= T_G N^{-1/2} \\ \tilde{T}_0 &= T_G N^{-1/2} \\ \Theta_0 &= T_G N^{-1/2} \end{aligned}\right\}$$

No Separation!!

→ no truncation of BBGKY hierarchy

e bottom line:

- need satisfy $\tau_* < T_0 < \Theta_0$ for BBGKY hierarchy game.

- for quasi-virialized clusters, have

$$\tau_* \ll T_0 \quad \text{but} \quad \Theta_0 < T_0 \Rightarrow \left\{ \begin{matrix} \text{collisionless} \\ \text{dynamics} \end{matrix} \right. !$$

well-described by Vlasov equation, if initial correlations weak

Vlasov description not always valid.

ii.) Using the Vlasov Equation

Vlasov Equation:

② equilibrium theory
$\nabla^2 \Phi = 4\pi G \int d^3v \, f(\mathrm{IOM}[\Phi])$
$\Rightarrow f, \Phi$!

① moment equations → macro-description (local)
Virial equation → gross description (global)

Now,
$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial x} - \frac{\partial \Phi}{\partial \underline{x}} \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

$$L = \frac{1}{2} \left(\dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2 \right) - \Phi(r, z, \theta)$$

general

$$\boxed{\text{Jeans' Thm: Solution of Vlasov equation is function of Integrals of Motion (IOMs)}}$$

$I = I(x(t), v(t)) \equiv$ IOM for (all) trajectories $x(t), v(t)$. IOM is function of trajectories

then, definition of IOM $\Rightarrow$

$$\left.\frac{dI}{dt}\right)_{\text{trajectory}} = 0 \quad \leftrightarrow \quad \frac{\partial I}{\partial t} + \frac{d\underline{x}}{dt} \cdot \frac{\partial I}{\partial \underline{x}} + \frac{d\underline{v}}{dt} \cdot \frac{\partial I}{\partial \underline{v}} = 0$$

+, Vlasov equation $\Rightarrow$ $\dfrac{df}{dt} = 0$

so clearly $f = f(I)$ solves Vlasov equation ✓.

Key Point:

$\Rightarrow$ for equilibrium, need solve

$$\nabla^2 \Phi = 4\pi G \int d^3v\, f.$$

As $f = f(\text{IOM}) = f(\text{IOM}(\Phi))$, understanding IOM constrains parametrization of f and nature of equilibrium.

$\dot{v}_r \quad \frac{d\dot{r}}{dt} = \ddot{r} = \frac{-\partial\Phi}{\partial r} + r\dot{\phi}^2 = -\frac{\partial\Phi}{\partial r} + \frac{v_\phi^2}{r}$

$$\frac{d\dot{\phi}}{dt} = \frac{d}{dt}\left(r^2\dot{\phi}\right) = \frac{-\partial\Phi}{\partial\phi}$$

$$= \ddot{\phi} + 2\dot{r}\dot{\phi} = \frac{-\partial\Phi}{\partial\phi}$$

$$\dot{v}_\phi = -\frac{v_r v_\phi}{r} - \frac{1}{r}\frac{\partial\Phi}{\partial\phi}$$

$$\dot{v}_z = -\frac{\partial\Phi}{\partial z}$$

so

$$\boxed{\frac{\partial f}{\partial t} + v_r\frac{\partial f}{\partial r} + \frac{v_\phi}{r}\frac{\partial f}{\partial\phi} + v_z\frac{\partial f}{\partial z} - \left(+\frac{\partial\Phi}{\partial r} - \frac{v_\phi^2}{r}\right)\frac{\partial f}{\partial v_r} - \left(\frac{v_r v_\phi}{r} + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\right)\frac{\partial f}{\partial v_\phi} - \frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0}$$

{ Vlasov equation in cylindrical system

spherical coordinates (spherical systems): $\begin{cases} \overline{V_r} = 0 \\ \overline{V_\theta} = 0 \end{cases}$

$$\Rightarrow \frac{\partial}{\partial t} n\overline{V_r^2} + \frac{n}{r}\left[2\overline{V_r^2} - (\overline{V_\theta^2} + \overline{V_\phi^2})\right] = -n\frac{\partial \Phi}{\partial r}.$$

(b) Virial Equation

Observe

→ in moment equations converted 6-dim. space evolution eqn. for f to 4 pde's in 3-dim space for $n, \overline{V_r}, \overline{V_\phi}, \overline{V_z}$ (i.e. integrate out velocity) → local balance relation

→ now, construct 'virial equation' by

$\int d^3x \; \underline{X} * \{\text{momentum balance}\}$ ⇒ global balance relation (i.e. integrate out position).

e.g. $$\frac{\partial}{\partial t}(n\overline{V_j}) + \frac{\partial}{\partial x_i}(n\overline{V_i V_j}) + n\frac{\partial \Phi}{\partial x_j} = 0$$

$$\Rightarrow \underset{①}{\int d^3x \, x_k \frac{\partial}{\partial t}(\rho \overline{V_j})} = \underset{②}{-\int x_k \frac{\partial}{\partial x_i}(\rho\overline{V_i V_j})\, d^3x} \underset{③}{-\int \rho x_k \frac{\partial \Phi}{\partial x_j} d^3x}$$

Now:

③ $-\int \rho\, x_k \frac{\partial \Phi}{\partial x_j}\, d^3x = \underline{\underline{W}} = W_{jk}$

potential energy tensor

② $K_{jk} = \frac{1}{2}\int d^3x\ \rho\, \overline{V_j V_k}$

kinetic energy tensor

$\left.\begin{array}{l} T_{jk} \\ \Pi_{jk} \\ W_{jk} \end{array}\right\}$ symmetric

write: $\underline{V} = \underline{\overline{V}} + \underline{\tilde{V}} \Rightarrow$

$K_{jk} = T_{jk} + \frac{1}{2}\Pi_{jk}$

$T_{jk} = \frac{1}{2}\int d^3x\ \rho\, \overline{V}_j\, \overline{V}_k \to$ mean / "directed" kinetic energy

$\Pi_{jk} = \int d^3x\ \rho\, \overline{V}_{jk}^2\, d^3x \to$ fluctn. / Pressure energy

<u>so</u>

$$\frac{1}{2}\frac{d}{dt}\int \rho\,(x_k \overline{V}_j + x_j \overline{V}_k) = 2T_{jk} + \Pi_{jk} + W_{jk}$$

symmetry

Now, can further write:

① $I_{jk} \equiv \int \rho\, x_j x_k \, d^3x$

moment of inertia tensor

$\Rightarrow$

$$\frac{1}{2}\frac{dI_{jk}}{dt} = \frac{1}{2}\int d^3x \; x_j x_k \frac{\partial \rho}{\partial t}$$

so

$$\boxed{\frac{1}{2}\frac{d^2 I_{jk}}{dt^2} = 2T_{jk} + \Pi_{jk} + W_{jk}}$$

Tensor Virial Theorem

- relates gross kinematic and morphological structures of galaxies (shape $\rightarrow$ I_{jk}, W_{jk})
- stationary system $\Rightarrow$

$$0 = 2T_{jk} + \Pi_{jk} + W_{jk}$$

- trace relations:

$$\mathrm{tr}\, \underline{\underline{T}} + \frac{1}{2}\,\mathrm{tr}\, \underline{\underline{\Pi}} = K$$

(obvious from defn.)

K: total kinetic energy

and $\mathrm{tr}\,\underline{\underline{W}} \equiv \underset{\substack{\downarrow \\ \text{total} \\ \text{potential energy}}}{W}$ (defn.)

$\Rightarrow$

$$\mathrm{tr}\left[2T_{jk} + \Pi_{jk} + W_{jk} = 0\right] \Rightarrow$$

$$\boxed{2K + W = 0}$$

scalar virial theorem (for $\ddot{I} = 0$)

$\therefore$ $K = \frac{1}{2} M \langle V^2 \rangle$

$\Rightarrow$

$$\langle V^3 \rangle = \frac{|W|}{M} \equiv \frac{GM}{r_g}$$

$\hookrightarrow$ gravitational radius $r_g \equiv GM^2/|W|$

$\Rightarrow$

Now, observe that total energy given by:

$$E = K + W = -\frac{W}{2} + W = \frac{W}{2}$$

$$= -K!$$

Negative specific heat!

E ↑ →

KE ↓

∴ T (KE / [illegible])

This result implies if:

<u>if</u> → system forms by collecting from ∞, at rest (i.e. where $K = W = E = 0$)

<u>and</u>

→ settles into equilibrium

$\therefore$ – 1/2 of released gravitational energy invested in kinetic form

– other 1/2 disposed of

<u>so</u> – $E_{binding} = K$

c. Moment Equations / Jeans Equations $\rightarrow$ $\left\{\begin{array}{l}\text{Macro balance}\\ \text{relations}\\ \text{(local)}\end{array}\right.$

$$n = \int d^3v\, f$$

$$\bar{v} = \frac{1}{n}\int d^3v\, v f$$

where:

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f - \frac{\partial \bar{\Phi}}{\partial \underline{x}}\cdot\frac{\partial f}{\partial \underline{v}} = 0$$

$\Rightarrow$

$$\boxed{\frac{\partial n}{\partial t} + \frac{\partial}{\partial \underline{x}}\cdot(n\underline{\bar{v}}) = 0}$$

(continuity $\rightarrow$ zeroth moment)

and

$$\boxed{\frac{\partial (n\underline{\bar{v}})_j}{\partial t} + \frac{\partial}{\partial x_i}\left(n\overline{v_i v_j}\right) + n\frac{\partial \bar{\Phi}}{\partial x_j} = 0}$$

(momentum balance $\rightarrow$ first moment)

where: $\overline{v_i v_j} = \frac{1}{n}\int v_i v_j f\, d^3v$

Subtract $\{\bar{v}_j *$ continuity$\}$ from $\{$momentum balance$\}$ $\Rightarrow$

$$\boxed{n\frac{\partial \bar{v}_j}{\partial t} - \bar{v}_j\frac{\partial}{\partial x_i}(n\bar{v}_i) + \frac{\partial}{\partial x_i}\left(n\,\overline{v_i v_j}\right) = -n\frac{\partial \bar{\Phi}}{\partial x_j}}$$

Now, define:

$$\sigma_{ij}^2 = \overline{(v_i - \bar{v}_i)(v_j - \bar{v}_j)} = \overline{v_i v_j} - \bar{v}_i\bar{v}_j$$

$n\,\sigma_{ij}^2 \rightarrow$ stress tensor for anisotropic pressure

Euler eqn.

then:

$$\boxed{n \frac{\partial \overline{v_j}}{\partial t} + n\overline{v_i} \frac{\partial \overline{v_j}}{\partial x_i} = -n \frac{\partial \Phi}{\partial x_j} - \frac{\partial (n\overline{v_j^2})}{\partial x_i}}$$

anisotropic pressure force

in cylindrical coords (disk systems):

{ from V.E. in cylinder coords.

$$\frac{\partial n}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rn\overline{v_r}) + \frac{\partial}{\partial z}(n\overline{v_z}) = 0$$

$$\frac{\partial (n\overline{v_r})}{\partial t} + \frac{\partial}{\partial r}\left(n\overline{v_r^2}\right) + \frac{\partial}{\partial z}\left(n\overline{v_r v_z}\right)$$

$$+ n\left(\frac{\overline{v_r^2} - \overline{v_\phi^2}}{r} + \frac{\partial \Phi}{\partial r}\right) = 0$$

$$\frac{\partial}{\partial t}(n\overline{v_\phi}) + \frac{\partial}{\partial r}\left(n\overline{v_r v_\phi}\right) + \frac{\partial}{\partial z}\left(n\overline{v_\phi v_z}\right)$$

$$+ \frac{2n}{r}\overline{v_\phi v_r} = 0$$

$$\frac{\partial}{\partial t}(n\overline{v_z}) + \frac{\partial}{\partial r}\left(n\overline{v_r v_z}\right) + \frac{\partial}{\partial z}\left(n\overline{v_z^2}\right) + n\frac{\overline{v_r v_z}}{r}$$

$$+ n\frac{\partial \Phi}{\partial z} = 0$$

Applications of Moment/Virial Equations:

a.) Mass density in solar neighborhood

- Goal: Compute mass density ρ and column density σ of galaxy in vicinity of sun.

Now, $\overline{V_z}$ equation $\Rightarrow$

$$\underbrace{\cancel{\frac{\partial}{\partial t}(n\overline{V_z})}}_{\text{stationary}} + \frac{\partial}{\partial r}\Big(n\underbrace{\overline{V_r V_z}}_{\text{thin}}\Big) + \frac{\partial}{\partial z}\left(n\overline{V_z^2}\right) + n\,\underbrace{\frac{\overline{V_r V_z}}{r}}_{\text{thin}} + n\frac{\partial \Phi}{\partial z} = 0$$

Now, thin galaxy $\Rightarrow$

$$\frac{\partial \Phi}{\partial z} = -\frac{1}{n}\frac{\partial}{\partial z}\left(n\overline{V_z^2}\right) \quad \text{and} \quad \nabla^2\Phi = 4\pi G\rho$$

$$\Rightarrow \frac{\partial^2 \Phi}{\partial z^2} = 4\pi G\rho$$

so

$$-\frac{\partial^2\Phi}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{1}{n}\frac{\partial}{\partial z}\left(n\overline{V_z^2}\right)\right) = -4\pi G\rho$$

and if:

$\rightarrow$ popln $\rightarrow$ # density

- measure $n(z)$
- " " $\overline{V_z^2}$

} for population $\Rightarrow \rho$!

$\downarrow$

(any) $\rightarrow$ can avg. over pops.

$\Rightarrow$ Oort Limit : $\rho = \rho(r_{sun}, 0) \cong .15\ M_\odot / pc^{+3}$

solar mass

Observe: $\Rightarrow \rho$ tricky, as requires $\underline{\underline{3}}$ derivatives (1 for star-count)

$\Rightarrow$ better, $\quad \sigma = \int_{-z}^{z} \rho(z)\,dz'$

Oort $\Rightarrow$ σ (700 pc.) $\cong$ 90 $M_\odot$ pc^{-2}

$\rightarrow$ numerous extensions, improvements (Bahcall) see B. and T.

(b) Internal Motion (σ^2) of Elliptical Galaxies

Goal: Use tensor virial theorem to relate stress tensor (σ^2) element to mean rotation velocities

Consider: - axi-symmetric system
- rotating about $\hat{z}$
- 'seen' edge-on

$\Rightarrow$ $(\underline{\underline{T}}, \underline{\underline{\Pi}}, \underline{\underline{W}})_{\substack{xx \\ yy}}$ identical $\Leftrightarrow$ off diagonals vanish (symmetry)

$\Rightarrow$

$2T_{xx} + \Pi_{xx} + W_{xx} = 0$, yy identical

$2T_{zz} + \Pi_{zz} + W_{zz} = 0$

$$\Rightarrow \quad \frac{2T_{xx} + \Pi_{xx}}{2T_{zz} + \Pi_{zz}} = \frac{W_{xx}}{W_{zz}}$$

Now,
- if only $\underline{\underline{\text{mean}}}$ motion (i.e. ordered) is rotation

$$\Rightarrow T_{zz} = 0 \ , \quad 2T_{xx} = \frac{1}{2}\int \rho \overline{V_\phi}^2 \, d^3x = M V_0^2/2$$

- $\Pi_{xx} = M \sigma_0^2$
  $\Pi_{zz} = \underset{\substack{\downarrow \\ \text{anisotropy parameter}}}{(1-\delta)} M \sigma_0^2$

$$\frac{M V_0^2/2 + M\sigma_0^2}{(1-\delta) M\sigma_0^2} = \frac{W_{xx}}{W_{zz}}$$

$$\Rightarrow \quad \frac{V_0^2}{\sigma_0^2} = 2(1-\delta)\frac{W_{xx}}{W_{zz}} - 2$$

ratio of V_ϕ^2 (known) to σ_0^2 (sought) δ.

Now, need W_{xx}/W_{zz}. Eqn. 2-134 of B. and T.,
for ellipsoidal distribution $\Rightarrow$

$$W_{jk} = -\pi^2 G \frac{a_2 a_3}{a_1^2} \left(\frac{a_j}{a_1}\right)^2 A_j \delta_{ij} \int_0^\infty \left(\psi(\infty) - \psi(m)\right)^2 dm$$

axial ratios (ellipsoid)

tabulated (e, a's, etc.)

mass distribution $\Rightarrow$ indep. shape

but then,

surface ellipticity

$$\frac{W_{xx}}{W_{zz}} = F(\epsilon) \text{, only}$$

$\underline{\underline{\text{So}}}$

$$\frac{v_0^2}{\sigma_0^2} = 2(1-\delta)\, F(\epsilon) - 2$$

fctn. shape, only

- see B. and T. 4.5 for curves
- can correct for oblique views

(a) Equilibrium Theory $\leftrightarrow$ Solving Poisson's Equation

$\rightarrow$ Concerned with solution of:

$$\nabla^2\Phi = 4\pi G\int d^3v\, f(\text{IOM}[\Phi])$$

$\hookrightarrow$ BGK solution

$\rightarrow$ Specifying f ?:

$$\underline{v}\cdot\frac{\partial f}{\partial \underline{x}} + \nabla\phi\cdot\frac{\partial f}{\partial \underline{v}} = 0 \quad f=f(\text{IOM})$$

- IOM's: E, $\underline{L}$, L_z, third
- symmetry (spatial, Γ); sphere, disk, triaxial
- functional form classes
- (test:) statistical mechanics (Lynden-Bell)
- stability

$\rightarrow$ examples
- sphere: Polytropes; Isothermal
- disk: Mestel
- triaxial

(1) Spherical Systems

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\int d^3v\, f\left(\tfrac{1}{2}v^2+\Phi, |\underline{r}\times\underline{v}|\right)}$$

Basic equilibrium relation - spherical system.

For $f = f(\mathcal{E})$, only

$$\mathcal{E} = -E + \Phi_0 \quad \text{relative energy}$$

$$\Psi = -\Phi + \Phi_0 \quad \text{relative potential}$$

ref. potential

$$\mathcal{E} = \Psi - v^2/2$$

i.e. non-rotating sphere (easy)

spherical model

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) = -16\pi^2 G\int_0^{\sqrt{2\Psi}} f\left(\Psi-\frac{v^2}{2}\right)v^2\,dv}$$

$$= -16\pi^3 G\int_0^{\Psi} f(\mathcal{E})\sqrt{2(\Psi-\mathcal{E})}\,d\mathcal{E}$$

[i.e. $f(\mathcal{E}) \equiv 0$ for $\mathcal{E} < 0$]

view as:

- nonlinear diffntl. eqn. for Ψ, given f
  - Ψ, ρ from f

or

- linear integral equation for $f(\mathcal{E})$, given Ψ
  - f fcn ρ, Ψ

Models:

1) polytrope

$$f(\mathcal{E}) = \begin{cases} F\mathcal{E}^{n-3/2}, & \mathcal{E} > 0 \\ 0, & \mathcal{E} < 0 \end{cases}$$

crank $\Rightarrow$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) + 4\pi G c_n \Psi^n = 0 \qquad \text{Lane-Emden}$$

$$c_n = (2\pi)^{3/2}(n-3/2)!\,F/n! \qquad n > 1/2$$

$n \equiv$ index of $\rho \sim \Psi^n$ proportionality

$\therefore\ s \equiv r/b \quad , \quad \Psi \equiv \Psi/\Psi_0$

$$\Psi_0 = \Psi(0)$$

$$b = \left(4\pi G\, \psi_0^{n-1} c_n\right)^{-1/2}$$

$\Rightarrow$

$$\begin{cases} \dfrac{1}{s^2}\dfrac{d}{ds}\left(s^2 \dfrac{d\Psi}{ds}\right) = \begin{cases} -\Psi^n, & \psi > 0 \\ 0 & \psi < 0 \end{cases} \\ \text{B.C.}: \quad \Psi = 1 \text{ at } s = 0 \\ \qquad\quad d\Psi/ds = 0 \quad \text{"} \quad \text{"} \end{cases}$$

Lane-Emden Equation

(i.e. no gravitational force)

Now, polytropic gases have equation of state:

$$P = K\rho^\gamma$$

$\gamma = 5/3$ for 3D gas

in hydroequilibrium
$\quad$ pressure $\qquad$ gravity

$$\frac{dP}{dr} = -\rho\frac{d\Phi}{dr}$$

$$\Rightarrow \quad K\gamma\rho^{\gamma-2}\frac{d\rho}{dr} = \frac{d\psi}{dr}$$

$\therefore$

$$\rho^{\gamma-1} = \frac{\gamma-1}{K\gamma}\Psi \quad \underset{\text{similar}}{\Longleftrightarrow} \quad \rho = c_n \Psi^n$$

$\Rightarrow$ density distribution of stellar polytrope of index n same as polytropic gas of $\gamma = 1 + 1/n$!

(ii)

What of $n \to \infty$ polytrope? $\to$ Isothermal sphere

mass internal to r

Hydrostatic equilibrium $\Rightarrow$ $\dfrac{dP}{dr} = -\rho \dfrac{GM(r)}{r^2}$

$\Rightarrow$ $\dfrac{k_B T}{m} \dfrac{d\rho}{dr} = -\rho \dfrac{GM(r)}{r^2}$; $\dfrac{dM}{dr} = 4\pi r^2 \rho$

$\Rightarrow$ $\times \left(r^2 m / \rho k_B T \right)$ and $\dfrac{d}{dr}$

$$\Rightarrow \boxed{\frac{d}{dr}\left(r^2 \frac{d \ln \rho}{dr} \right) = -\frac{Gm}{k_B T} 4\pi r^2 \rho}$$

Now, hypothesize:

$\to$ Boltzmann soln./dist. why?

$$f(\mathcal{E}) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \exp\left[\frac{\Phi - v^2/2}{\sigma^2} \right]$$

{dist. fctn.

$\Rightarrow$

$$\rho = \rho_1 e^{\Phi/\sigma^2}$$

Maxwellian, with $v_T^2 \to \sigma^2$

i.e., Poisson's equation $\Rightarrow$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Psi}{dr}\right) = -4\pi G\rho$$

or, using $\rho = \rho_1 e^{\Psi/\sigma^2}$ $\Rightarrow$

$$\frac{d}{dr}\left(r^2 \frac{d}{dr}\ln\rho\right) = -\frac{4\pi G}{\sigma^2} r^2\rho \qquad \ast$$

$\updownarrow$ compare previous

$$\frac{d}{dr}\left(r^2 \frac{d}{dr}\ln\rho\right) = -\frac{Gm}{k_B T} 4\pi r^2\rho$$

so if: $\boxed{\sigma^2 = \frac{k_B T}{m}}$ isothermal gas and spherical, non-rotating Maxwellian system identical!

Obviously, $\overline{v^2} = 3\sigma^2$

For solution $\rho(r)$ to $\ast$,

$$\rho \sim C r^{-\alpha} \Rightarrow -\alpha = -\frac{4\pi G}{\sigma^2} C r^{2-\alpha}$$

so $\alpha = 2$

$C = \sigma^2/2\pi G$

$$\Rightarrow \quad \boxed{\rho(r) = \sigma^2 / 2\pi G r^2}$$ singular isothermal sphere.

- solution singular, mass infinite
- regularize by core models $\left\{ \begin{array}{l} \text{"core fitting"} \\ \text{"Kings method"} \end{array} \right.$

etc.

Aside: → have been using assumed form of distribution function to obtain $\rho(r)$
→ could equally ask, given $\rho(r)$, what is $f(\mathcal{E})$?.

Now,

$$\rho(r) = 4\pi \int_0^{\Psi} f(\mathcal{E}) \left[2(\Psi - \mathcal{E}) \right]^{1/2} d\mathcal{E}$$

so $\rho = \rho(\Psi)$

$$\frac{\rho(\Psi)}{(8\pi^2)^{1/2}} = 2 \int_0^{\Psi} f(\mathcal{E}) (\Psi - \mathcal{E})^{1/2} d\mathcal{E}$$

$$\Rightarrow \frac{1}{\sqrt{8}\pi} \frac{d\rho}{d\Psi} = \int_0^{\Psi} \frac{f(\mathcal{E})\, d\mathcal{E}}{(\Psi - \mathcal{E})^{1/2}}$$ → Abel integral equation

inverting $\Rightarrow$ (standard)

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\,\pi^2}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}} \frac{d\rho}{d\Psi}\frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}}$$

$\boxed{\text{Eddingtons Formula}}$

$$= \frac{1}{8\pi^2}\left[\int_0^{\mathcal{E}} \frac{d^2\rho}{d\Psi^2}\frac{d\Psi}{(\mathcal{E}-\Psi)^{1/2}} + \frac{1}{\sqrt{\mathcal{E}}}\left(\frac{d\rho}{d\Psi}\right)_{\Psi=0}\right]$$

{given model $\Rightarrow f$

i.e. given ρ (spherically symmetric) $\Rightarrow$ recover f.

but, reveals requirement: $\rho(r)$ can be density for spherical system with $f = f(E)$, only iff:

$$\int_0^{\mathcal{E}} \frac{d\rho}{d\Psi}\frac{d\Psi}{(\mathcal{E}-\Psi)^{1/2}}$$

is increasing function of E!

if not $\Rightarrow$ anisotropy approximation invalid.

See also: Mestel Disk, McLaurin Disk, etc. } B. and T.

i.e. { ∞ Zoology of disk / spheroid equilibrium models

iii) Violent Relaxation (D. Lynden-Bell, MNRAS **136** 101 (1967))

→ Preliminaries { – collisionless relaxation (i.e. ~~stat. states~~ elliptical galaxies) ⇒ ? – what is choice among many equilibria? }

→ Recall generic time scale ordering for virialized gravitating system

~~$\tau_{collision}$~~ $\tau_{relaxation}$ (macroscopic) $< \tau_{collision}$!?

∴ cannot use concepts from standard stat-mech. equilibrium theory to describe galactic structure !!? → $\tau_{relaxation} \to \infty$

→ still, SUCH AN APPROACH IS ATTRACTIVE, IN STRUCTURE, etc.

∴ how approach?

⇒ natural to investigate:

- stationary states of Vlasov equation
- selection criterion
- compatibility of criterion with Vlasov equation dynamics.

), or stationary states, recall $f = f(\text{IOMs})$

i.e. in 1D, $\cancel{\frac{\partial f}{\partial t}} + v\frac{\partial f}{\partial x} - \frac{1}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = 0$

$$f = f(mv^2/2 + \phi(x))$$

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi G \int dv\, f\left(\frac{mv^2}{2} + \phi(x)\right)$$

is solution

→ BGK solution (general)

∃ ∞ of such BGK solutions

2) How specify? ⇒ Ask What IS MOST Probable BGK Solution? (Lynden-Bell)

i.e.

- need probability measure / liklihood function ⇒ entropy.

- but in Vlasov system, f conserved along particle orbits ⇔ 'exclusion' effect!?

∴ can expect:

i.) entropy suggested by L-B. resembles Fermi-Dirac, closely

ii.) coarse graining implicit in phase space partition

⇒ what information is lost in coarse graining ?? (i.e. #3).

→ Kelvin Thm. for Vlasov Plasma/Star System

→ recall Kelvin Thm. for incompressible, inviscid fluid :

i.e. $\underline{\nabla} \cdot \underline{V} = 0$, $\rho =$ const.

$$\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla}\,\underline{V}\right) - \cancel{\rho \nu \nabla^2 \underline{V}} = -\underline{\nabla} p$$

Consider circulation: $\Gamma = \oint \underline{V} \cdot d\underline{l}$, so:

$$\frac{d\Gamma}{dt} = \oint \frac{d\underline{V}}{dt} \cdot d\underline{l} + \oint \underline{V} \cdot \frac{d\underline{l}}{dt}$$

$$= \oint \left(-\frac{\underline{\nabla} p}{\rho} \cdot d\underline{l}\right) + \oint \underline{V} \cdot d\underline{V}$$

$$= -\frac{1}{\rho} \oint \cancel{\left(\underline{\nabla} p \cdot d\underline{l}\right)} + \oint d\cancel{\left(\frac{V^2}{2}\right)}$$

$\therefore \quad \dfrac{d\Gamma}{dt} = 0$

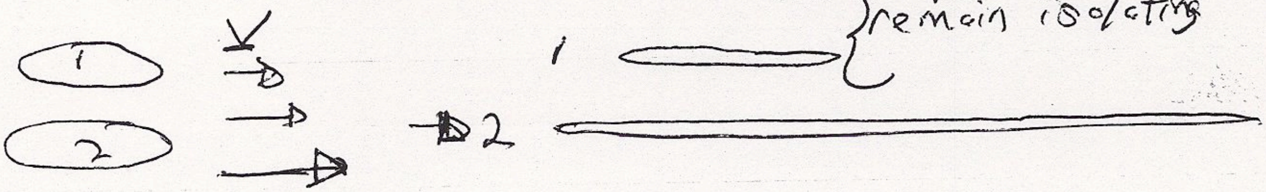
i.e. $\oint \underline{v} \cdot \underline{dl}$ conserved, for all loops in inviscid fluid. $\rightarrow$ $\left\{ \begin{array}{l} \Gamma \text{ is "isolating"} \\ \text{invariant for fluid element} \end{array} \right.$

$\Rightarrow$ <u>no re-connection of loops without viscosity!!</u>

$\Rightarrow$ 'memory' of loops and preclusion of reconnection

$\Rightarrow$ natural to envision evolution of inviscid flow as "<u>stretching</u> of rubber sheets"

i.e.



i.e. {loops 1, 2 remain 'isolating'

$\Rightarrow$ Kelvin Thm. $\Leftrightarrow$ stretching of "fluid shapes" intimately related.

2) for Vlasov Eqn:

- can consider position space projection (parametrized by s) of phase space path

- $\Gamma = \oint \underline{v}(s) \cdot \underline{ds}$

$$\frac{d\Gamma}{dt} = \oint \frac{d}{dt}\underline{V}(s)\cdot d\underline{s} \quad + \oint \underline{V}(s)\cdot \frac{d\underline{s}}{dt}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{V}\cdot\frac{\partial}{\partial \underline{X}}$$

but Vlasov equation $\Rightarrow$

$$\frac{\partial f}{\partial t} + \underline{V}\cdot\frac{\partial f}{\partial \underline{X}} - \frac{\underline{\nabla}\phi}{m}\cdot\frac{\partial f}{\partial \underline{V}} = 0$$

$\Rightarrow$

$$\frac{d\Gamma}{dt} = \oint \frac{\underline{\nabla}\phi}{m}\cdot d\underline{s} + \oint d\left(\underline{V}(s)\right)^2$$

$$= 0$$

$\Rightarrow$ circulation conserved in Vlasov phase space fluid!

$\Rightarrow$ evolution of phase space fluid intrinsically linked to stretching of isolating contours
and
no re-connection of isolating contours without collisions!!

~~Saturday~~ ~~Friday~~, ~~Wed~~ 8/30 ~~Wednesday~~

Thus,

- ∞ of isolating integrals define BGK solution

- phase space partitioning and coarse graining likely looses information re: isolating integrals.

→ The Problem:

- Galactic structure → i.e. solving $\nabla^2\phi =$

$4\pi G \int d\underline{v}\, f(\text{IOM's}) \Leftrightarrow$ specify $\begin{cases} \text{IOM's} \\ \text{form of } f \end{cases}$

- what's "new": $\begin{cases} \text{better appreciation of } f \text{ conservation} \\ \text{awareness of Landau damping and phase mixing} \leftrightarrow \text{development of fine scale structure in } f \end{cases}$

- Concept: regular light distribution, remarkably similar, observed in many elliptical galaxies

⇛ is there a 'natural equilibrium' being relaxed toward

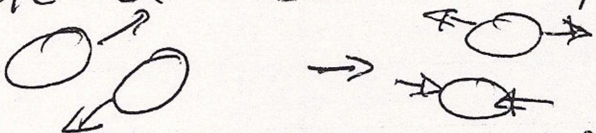
↔ but what is it, $\tau_{relax} \ll \tau_{inter\text{-}collision}$

~~not~~ ~~collect~~ so $\boxed{\text{non-thermal equilibrium}}$ → how/why can f "look like" equilibrium ???

> violent collisionless relaxation !!

Basic Idea:

- during galaxy formation, tidal interaction
  - → collective modes / vibrations excited
  - → expect collective modes to Landau damp (collisionless mechanism) on time scale of few wave periods



- ∴ → expect fine scale structure formation in f

→ $\boxed{\text{need calculate "most probable" of fine scale state}}$

⇒ 2 Questions:

- ①: what is time scale?
- ②: how address 'most probable state'?

○ Time Scale $\to$ Virial Theorem

During formation/interaction, gravitational potential evolves in time (i.e. model Landau Damping) $\Rightarrow$

$$\frac{dE}{dt} = -\frac{\partial \phi}{\partial t} \quad , \quad \left\{ \begin{array}{l} E = m\left(\frac{v^2}{2} - \phi\right) \\ \text{for individual star} \end{array} \right.$$

$$\Rightarrow \quad T_{relax} = \langle (\partial\phi/\partial t)^2 \rangle^{-1/2} / E^2$$

Now, Virial Theorem

$$\frac{1}{2}\ddot{I} = 2T + V$$

at equilibrium: $\ddot{I} = 0$ so $0 = 2T + V$

$E = T + V$

$$\Rightarrow \left\{ \begin{array}{l} T = -E \\ V = 2E \end{array} \right.$$

$$\Rightarrow \quad T \sim \frac{1}{2} E$$

$$\sim \frac{1}{4} m\phi \quad , \text{as } V = \frac{1}{2}\sum_i V_i$$

$\to$ {potential energies mutual

$\therefore$

$$\frac{1}{2}mv^2 \sim \frac{1}{4} m\phi$$

$$\Rightarrow \quad E \sim -3\phi/4$$

$$\Rightarrow \quad T_{relax} \sim \frac{3}{4}\left\langle \left(\frac{\partial}{\partial t} \ln\phi\right)^2 \right\rangle^{-1/2}$$

For estimate of time scale, use Virial Thm:

$$I = \lambda^2 M R^2 \qquad \ddot{I} = 2T + V \qquad V = -\frac{GM^2}{R}$$

$M \equiv$ galactic mass
$R =$ " " radius
$\lambda^2 \equiv$ # (shape)

$$\Rightarrow \quad \lambda^2 \ddot{R} = -\frac{2E}{M} - \frac{GM}{R}$$

(using equilibrium results)

$$\therefore \quad R_0 = \left[GM^2/\cancel{+}2E\right]^{\cancel{1/2}}$$

$\rightarrow$ equilibrium radius
Virial

For small oscillations/perturbations:

$$-\lambda^2 \ddot{(R_0 + \delta R)}^2 = +\frac{2E}{M} + \frac{GM}{R_0(1 + \delta R/R_0)}$$

$$2\lambda^2 R_0 \delta\ddot{R} = +\frac{GM}{R_0^2}\delta R$$

$$\Rightarrow \quad \boxed{\gamma^2 = \frac{GM}{2\lambda^2 R_0^3}}$$

low instability saturates by heating ($\Rightarrow$ phase mixing) in few γ^{-1}. So, taking $\lambda^2 = 1/3$ (spheroid) and $\bar{\rho} = M \big/ \frac{4\pi}{3} R_0^3$ $\Rightarrow$

$$\boxed{\gamma^{-1} \approx \tau_{relax} \approx (2\pi G \bar{\rho})^{-1/2}}$$

i.e. taking
$$\begin{cases} \delta R \sim R_0 \\ \Delta\phi \sim \phi \\ \tau_{relax} = 3/4\gamma \\ \text{(MLTB)} \end{cases}$$

i.e. $\rightarrow$ note relaxation is rapid!!

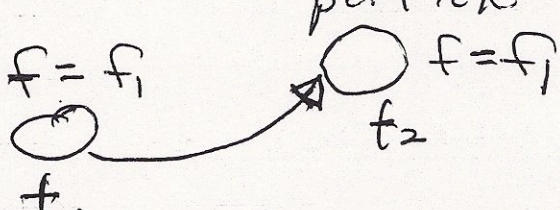
$\rightarrow$ no mass segregation, i.e. as m cancels

$\rightarrow$ global modes eventually damp, via Landau damping/heating etc. Issue: time sufficient to progress toward relaxed state?!

② Predicting the Relaxed State

Recall:

① df/dt $\Rightarrow$ phase space density conserved along particle orbits



$\therefore$ but fine scale structure develops in f

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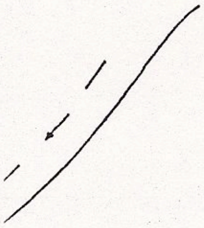
ɔn in phase space. ψ, the
d^3c by way of Poisson's

(12)

ıace conserves its phase-
ıensions rather than $6N$).
s wherever it moves to so
of all those phase elements
put it differently $M(f)$
·eater than f is conserved

ɔnserved quantities. It is
which allows for this
what is meant by an
ussed by Gibbs (5)
lar system was given by
simplified model which

of many non-interacting



s a single particle energy
. Now plot phase space ϵ
higher energy oscillations
ɔn function $f(\epsilon, \phi, t)$. In
circle for illustration.

equilibrium if it is
ges, microscopes of

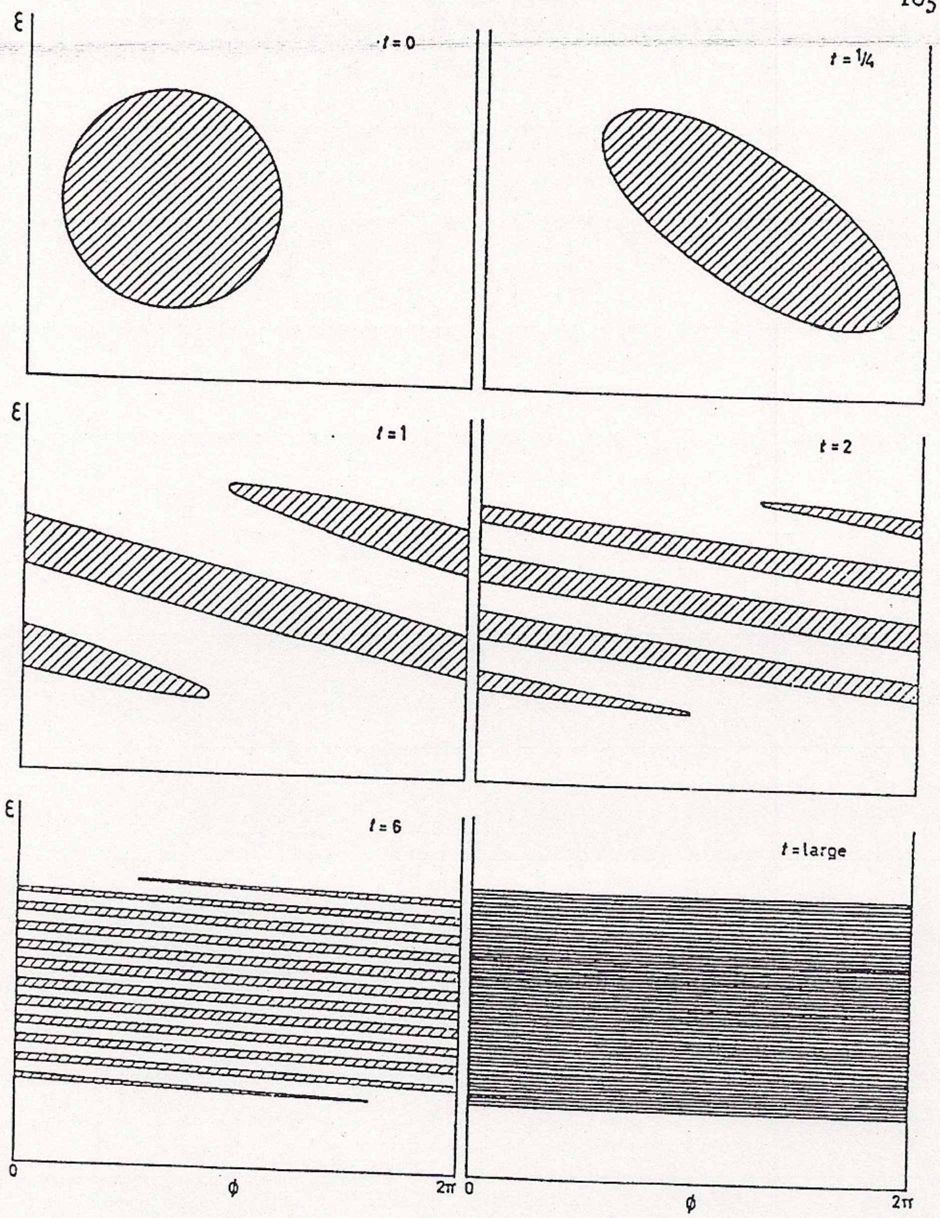
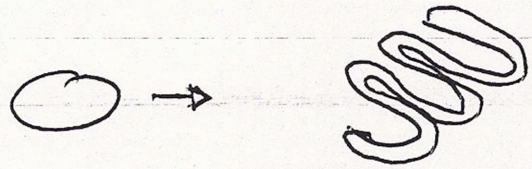


Fig. 2

(ii) If we use only a finite resolution then the distribution function appears to converge to what one would get by averaging $f(\epsilon, \phi, 0)$ over ϕ to obtain $\bar{f}(\epsilon)$.

(iii) In this process of looking with finite resolution we have averaged over regions of different phase densities. In this averaging process $M(f)$ is **not** conserved, so $M(\bar{f})$ is *not* the same function as $M(f)$. $\bar{f}$ must, however, be the result of smoothing f thus there will be restrictions on $M(\bar{f})$. For instance the largest value $\bar{f}$ attains can not be greater than the largest value f attained. Further it can be shown that the mathematical expression of (i) and (ii) is contained in the statement f converges in the mean to $\bar{f}$.

i.e.  (equal area!)

(fineness of scale ↑)

∴

- to cope with fine scale structure must coarse grain (i.e. minimum resolvable scale) for practical calculation

But, must recognize:

- fine structure can develop on scales smaller than resolution scale

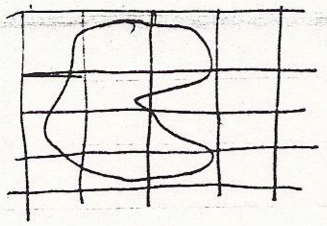
- coarse graining loses information concerning isolating integrals a la' Kelvin ⇒ "cuts up the spaghetti"

→ Counting / Forming Partition

- assume all values of (occupied) f same, i.e. $f = \begin{cases} 0, & \text{empty} \\ 1, & \text{filled} \end{cases}$

- recall $df/dt = 0$, so exclusion ⇒ can't pile phase elements on top of each other

- establish 2 cells $\left\langle \begin{array}{l} \text{micro} \\ \text{macro} \end{array} \right.$



micro → volume ω $\left\{ \begin{array}{l} \text{mass of phase fluid} \\ 0 \\ \eta\omega \end{array} \right.$

↓ → pixel

macro → ν micro-cells volume $\nu\omega$

↓

resolution scale

micro → ala pixel

< scale of fine structure

macro → ala resolution

- to count ⇒ $\boxed{S = \ln \#\text{microstates} = \ln W}$

 - 'know' n_i phase elements in ith macro-cell (i.e. weight)
 - distinguishable phase elements

⇒ # ways to assign micro cells $= \dfrac{\nu!}{(\nu - n_i)!}$

∴ for # microstates → # ways splitting N elements into groups n_i

$$W = \prod_i \frac{\nu!}{(\nu - n_i)!} \frac{N!}{\prod_i n_i!}$$

observe:

	indistinguishable	distinguishable
exclusion	Fermi-Dirac	Lynden-Bell
no exclusion	Bose-Einstein	Maxwell-Boltzman

i.e. $$W_{MB} = \frac{N!}{\prod_i n_i!} \prod_i \nu^{n_i}$$

$$W_{FD} = \prod_i \frac{\nu!}{n_i!\,(\nu - n_i)!}$$

Obviously, Lynden-Bell statistics 'close' to Fermi-Dirac!

– to describe relaxed state:

most probable state [maximize $S = \ln W$]

subject to conservation $\left\{ \begin{array}{l} \text{mass} \\ \text{energy} \\ \text{momentum} \end{array} \right.$ [constraints]

620

→ To calculate $\qquad (\ln n! = n\ln n - n)$

$$\ln W = N(\ln N - 1) - \sum_i \Big\{ n_i(\ln n_i - 1) + (\nu - n_i)\{\ln(\nu - n_i) - 1\} - \nu(\ln \nu - 1) \Big\}$$

if $\bar{f}_i = \dfrac{n_i \eta}{\nu}$ $\equiv$ average phase space density in i^{th} macro cell

$$\ln W = N(\ln N - 1) - \int \frac{d^3x\, d^3v}{\nu\omega} \Big[\frac{\nu}{\eta} \Big\{ \bar{f}\Big[\ln\Big(\frac{\nu\bar{f}}{\eta}\Big) - 1\Big] + (\eta - \bar{f})\Big[\ln\Big\{\frac{\nu}{\eta}(\eta - \bar{f})\Big\} - 1\Big]\Big\} - \nu(\ln\nu - 1)\Big]$$

and constraints:

$$M = \int d^3v\, d^3x\, \bar{f} \quad \rightarrow \quad \text{Lagrange multiplier } \alpha$$

$$E = \int d^3v\, d^3x \left\{ \bar{f}\frac{v^2}{2} - \frac{G}{2}\int d^3v'\, d^3x' \frac{\bar{f}'}{|\underline{x} - \underline{x}'|} \right\} \rightarrow$$

Lagrange multiplier β

63. ~~78~~

$$\delta S = 0 = -\int \frac{d\bar{f}}{\eta\omega}\left\{ \ln \frac{\bar{f}}{\eta - \bar{f}} + \alpha + \beta\left(\frac{v^2}{2} - \phi\right)\right\} d^3x\, d^3v$$

$\Rightarrow$

$$\frac{\bar{f}}{\eta - \bar{f}} = \exp(-\alpha - \beta \epsilon) \qquad \epsilon = \frac{v^2}{2} - \phi$$

$\Rightarrow$

$$\bar{f} = \eta \frac{\exp[-\beta(\epsilon - \mu)]}{1 + \exp[-\beta(\epsilon - \mu)]} \qquad \mu = -\alpha/\beta$$

$\downarrow$ very similar F.D. (cont'd gases)

μ $\downarrow$ "chem potential"

$\therefore$

$$\nabla^2 \phi = -4\pi G \int \bar{f}\, d^3v$$

so if $x^2 \equiv \beta v^2$ $\Rightarrow$

$$\boxed{\nabla^2\phi = -16\pi^2 \eta G \beta^{-3/2} \int_0^\infty x^2 dx \, \frac{\exp[-x^2/2]}{\exp\{-\beta(\phi + \mu)\} + \exp(-x^2/2)}}$$

$\rightarrow$ akin equation for self-gravitating F-D gas!!

$\rightarrow$ reduces to Maxwellian in non-degenerate limit

→ appears to explain/suggest that dynamical system distribution function can resemble Maxwellian

(i.e. recall $\exp[-v^2/\sigma^2]$)

Comments:

→ size of Γ ⇒ must limit relaxation to radius R_0 (~ virial);

→ need "depopulation" factor to eliminate orbits (= particles/phase trajectories) which do not fall inside R_0 relaxation region

i.e. $f = A \exp\left[-\beta\left(\mathcal{E} + \frac{1}{2} h^2/R_0^2\right)\right]$

depop factor ⇒ kills off disk component.

→ does suggest Maxwellian assumption potentially viable (i.e. King Ansatz, $f \sim \exp(-v^2/\sigma^2)$) (with $T \sim$ Mass)

→ careful not to assume Lynden-Bell distribution is true equilibrium, as

Antonov's paradox: Virial Thm $\Rightarrow \begin{cases} 0 = 2T + V \\ E = T + V \end{cases}$

$$\Rightarrow \begin{cases} E = -T \\ V = 2E \end{cases}$$

$\Rightarrow$ grow hotter if energy lost, cooler if energy gained

$\Rightarrow$ basic trend to dis-equilibrium.

Degeneracy? – Is this Necessary?

→ L-B akin to F-D.

→ if $\bar{f} \ll F$ at star formation, then galaxies are non-degenerate

L-B → F-D!

→ so:

① – need estimate phase space density for star formation
② – need estimate phase space density today

for ①

– star formation ⇒ $4\pi G\rho \geq k^2 (\Delta v)^2$
dispersion

$$(G\rho)^{3/2} \ell^3 \geq (\Delta v)^3$$

$$M_* G^{3/2} \geq (\Delta v)^3 / \rho^{1/2}$$

- phase density → formation

$f_{star} \equiv f_{max} \equiv \eta \sim \frac{\rho_{star}}{(\Delta V)^3}$

$\sim \rho_{star} / M_* G^{3/2} \rho_{star}^{1/2}$

$$\boxed{\eta \sim \frac{\rho_{star}^{1/2}}{M_* G^{3/2}}}$$ using $(\Delta V)^3$ above

Now, for today

- $f_{now} = \bar{\rho} / \Delta V^3$

$T = -E$ (virial) → for Galaxy

$M (\Delta V)^2 \sim + G M^2 / R$

$(\Delta V)^2 \sim G R^2 \bar{\rho}$

$(\Delta V)^3 \sim \bar{\rho}^{1/2} G^{3/2} M_{Galaxy}$

$$\boxed{f\Big|_{Galaxy} \sim \bar{\rho}/\Delta v^3 \sim \frac{\bar{\rho}^{1/2}}{G^{3/2} M_{Galaxy}}}$$

Now,

$$f\Big|_{Galaxy} < 1 \Rightarrow \frac{\rho_{star}^{1/2}}{M_* G^{3/2}} > \frac{\bar{\rho}^{1/2}}{G^{3/2} M_{Galaxy}}$$

$$\underbrace{\frac{M_{Galaxy}}{M_*}}_{\substack{\gg 1 \\ (\sim 10^{20})}} \underbrace{\frac{\rho_{star}^{1/2}}{\bar{\rho}^{1/2}}}_{\gg 1} > 1$$

$\therefore$ $f_{Galaxy} < 1$ (f goes to losses due coarse graining)

$\Rightarrow$ Galaxy is non-degenerate

$\Rightarrow$ L-B $\rightarrow$ Maxwell-Boltzmann

$$f \sim \eta \exp[-\beta \epsilon]$$

$$\epsilon = v^2/2 - \Phi$$

∴ "isothermal sphere" type model emerges as visible final state due violent relaxation in collisionless system.

⇒ possible explanation of ~ universal trends in elliptical galaxy (relaxation) suggestive of isothermal sphere (but how → collisionless).

but does it get there? - kinetics!

(see later)