THE ASTROPHYSICAL JOURNAL, 193:539–559, 1974 November 1

ON DENSITY WAVES IN GALAXIES. I. SOURCE TERMS AND ACTION CONSERVATION

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Received 1974 March 21

ABSTRACT

Generalized versions of the Lin-Shu dispersion relation and the Toomre-Shu conservation principle of wave action have been derived. These are valid even in the neighborhood of all the Lindblad resonances. Because of the finite radial epicyclic excursions of the resonant stars, each resonance influences a region of the order of one or two epicyclic distances. For our Galaxy, these are annular regions each several kiloparsecs in width. Our generalized dispersion relation gives finite wavenumbers throughout the resonance regions. Provided that there is not also a very rapid temperal decay in amplitude, both leading and trailing short waves are found to decay spatially in the direction of their respective group propagation. This decay process can be described physically in terms of the detailed balance of energy and angular momentum through a generalized conservation principle of wave action. This principle involves a modified form of the flux term. In addition, an important new feature is that it also includes source terms which always result in the damping of the waves at the Lindblad resonances. In particular, we have a detailed source density term of angular momentum which can be compared with the integrated (total) source obtained by Lynden-Bell and Kalnajs. Under certain approximations, we have also rederived this source density by their methods and have shown that it agrees with our formula.

Other implications of this theory include the fact that the surface mass density of the spiral wave in our Galaxy has a peak which correlates well with the peak of ionized hydrogen density. This correlation, first suggested by Lin and Feldman, is now found to be a possible test of wave pattern speed for some galaxies. Our present calculation also elucidates the different behavior of leading and trailing spirals. Within certain conditions, trailing waves are preferred.

Subject headings: galactic structure — stellar dynamics

I. INTRODUCTION

This is the first in a series of papers directed at mechanisms for the generation and maintenance of density waves.

Collective density waves and instabilities have been introduced by Hunter (1963, 1965), Lin and Shu (1964, 1966), Toomre (1964), Kalnajs (1965), and Julian and Toomre (1966) as the basic physical phenomena underlying spiral structure in galaxies. Fruitful comparisons with observations have been made by Lin, Yuan, and Shu (1969), Roberts (1969), Yuan (1969*a*, *b*), Roberts and Yuan (1970), Shu, Stachnik, and Yost (1971), and Shu *et al.* (1972). Their rapid advance has been facilitated by the Lin-Shu hypothesis of "quasi-stationary spiral structure."

Toomre (1969) showed that the Lin-Shu spiral density waves are convective and propagate through the Galaxy in a matter of some 10^9 years. In between the resonances, Toomre (1969) has suggested, and Shu (1970*b*) has shown, that the action density of these waves is a conserved quantity of the wave propagation. Kalnajs (1971)[1] further elucidated the proportionality relations between action density and densities of angular momentum and also of energy in a stationary frame. Thus, the action (angular momentum, energy) density of the wave is established as the fundamental physical measure of the wave concentration in any local region. Toomre (1969) has also suggested that these waves are absorbed in the "vicinity" of the Lindblad resonances, but he ascribed the cause to a process of "phase-mixing." For the Lindblad resonances, Mark (1971) derived an improved version of the Lin-Shu dispersion relation and found that the waves are indeed absorbed by interaction with the resonant stars. A detailed description of the wave length and amplitude was also obtained. Lynden-Bell and Kalnajs (1972, hereinafter referred to as LBK) gave a physical interpretation of this phenomenon in terms of emission and absorption of angular momentum by the resonant stars. They only calculated the total integrated source but were unable to describe the details of this absorption.

Since density waves propagate and are absorbed at the Lindblad resonances, the Lin-Shu hypothesis of "quasi-stationary spiral structure" must be interpreted in terms of mechanisms of regeneration or sources of wave action. Thus, Toomre (1969) stated (p. 909): "neither the damping nor the radial propagation of individual wave packets actually excludes really long-lived spiral wave patterns in a galaxy. If such patterns are to persist, the above simply means that fresh waves (and wave energy) must somehow be created to take the place of older waves that drift away and disappear." He also suggested as viable sources either tidal forcing by external companions and satellites, or forcing by oval distortions and barlike instabilities in a disk. Lin (1969) has suggested that (p. 383): "the reflection of the waves from the central region then stabilizes the wave pattern into a quasi-stationary form by transmitting the signal, via long-range forces, back to the outer regions where the waves originated. Thus, there is

[1] The result in the form of surface densities of angular momentum, etc., was not given here but referred to by Shu (1970*b*, p. 110).

necessarily the co-existence of a very loose spiral structure and a tight spiral structure. Population I objects stand out sharply in the tight pattern while stars with large dispersive motion would primarily participate in the very loose pattern." LBK also suggested that the stars at the corotation resonance may act as a source of wave angular momentum that emits the inward-propagating trailing waves.

Most of the details of these mechanisms have not yet been worked out. Although it is easy to imagine that bars, oval distortions, and companions can excite very open spirals, whether they can also excite more tightly wound waves is a more delicate question. Some early progress has been made by Feldman and Lin (1973), who calculated the response of gaseous cylindrical and disklike systems that are driven by an imposed bar. It is interesting to inquire in detail as to the manner in which bars, companions, and resonant stars act as sources of wave action for these tightly wound waves. For a particularly well-known example of these phenomena, this paper is directed at studying the problem of how source terms for wave action can actually be introduced into a modified action conservation principle: We begin with the modest task of studying the source terms for the case of the wave absorption at Lindblad resonances. We feel that this is a helpful approach because considerable analysis is required to evaluate the sources even in this case where the qualitative physical mechanism is quite well known. In the process of deriving these source terms, it is also necessary to obtain the dispersion relations governing the behavior of density waves at all the Lindblad resonances. In a brief communication (Mark 1971), the dispersion relation at the inner Lindblad resonance has been recorded without proof or detailed examples. Contopoulos (1971) also discussed some aspects of this problem. Our results differ from his because we imposed a self-consistent gravitational potential.

II. THE EQUILIBRIUM AND THE PERTURBATIONS

a) Basic Equations for the Equilibrium and the Waves

As the basic equilibrium to study waves in spiral galaxies, we shall use a disk model (Hunter 1963; Toomre 1963; Lin and Shu 1964; Kalnajs 1965). For the particular spiral galaxy in question, let (ϖ, θ, z) represent cylindrical polar coordinates centered at the galactic center, with the $z = 0$ plane representing the "idealized-disk" of the galaxy. This model is assumed to have the equilibrium surface density of matter $\Sigma(\varpi)$, gravitational potential $\Phi(\varpi, z)$, circular rotation frequency $\Omega(\varpi)$, and epicyclic frequency $\kappa(\varpi)$. Of course,

$$\Omega^2(\varpi)\varpi = \left.\frac{\partial\Phi}{\partial\varpi}\right|_{z=0} ; \quad \kappa^2(\varpi) = (2\Omega)^2\left[1 + \frac{1}{2}\frac{d\ln\Omega}{d\ln\varpi}\right] . \tag{1}$$

If (p_ϖ, p_θ) are the components of momentum (per unit mass) for a star moving in this "disk," we may define its guiding-center circular motion by the radius r of a circular orbit with (Shu 1969)

$$p_\theta = \Omega(r)r^2 . \tag{2}$$

Again relative to unit mass, the orbital energy E, guiding-center circular rotation energy E_c, and epicyclic energy E_e of the orbit are given by

$$E(\varpi, p_\varpi, p_\theta) = \tfrac{1}{2}p_\varpi{}^2 + \frac{p_\theta{}^2}{2\varpi^2} + \Phi(\varpi, 0) , \tag{3}$$

$$E_c(r) = \tfrac{1}{2}\Omega^2(r)r^2 + \Phi(r, 0) = E[r, 0, \Omega(r)r^2] , \tag{4}$$

$$E_e(r, a) \equiv \tfrac{1}{2}\kappa^2(r)r^2a^2 = E - E_c , \tag{5}$$

where a is the dimensionless epicyclic amplitude of the orbit. As equilibrium distribution function for the mass density in phase space, we shall choose

$$F(E_e, r) = P(r)\exp[-E_e/c^2(r)]\left[1 + \frac{E_e}{E_c}\right]^2 , \tag{6}$$

where $c(r)$ represents the dispersion speed of the stars in the radial direction. This distribution differs from that of Shu (1969) only by terms of $O(E_e/E_c)$ and has the additional advantage that both F and $\partial F/\partial E_e$ give finite integrals when integrated over phase space. For the purposes of calculating local wave perturbations, this distribution function still gives the same asymptotic result of Shu, namely,

$$F = P(r)\exp[-E_e/c^2(r)] + O(\varepsilon^2) , \qquad P(r) = \frac{2\Omega(r)\Sigma(r)}{\kappa(r)2\pi c^2(r)} + O(\varepsilon^2) , \tag{7}$$

where

$$\varepsilon(r) \equiv \frac{c(r)}{r\kappa(r)} \tag{8}$$

is a small parameter measuring the typical magnitude of $(E_e/E_c)^{1/2}$ for a given r.

The wave perturbation is characterized by the gravitational potential $\phi(\varpi, \theta, z, t)$ and the surface density of mass $\sigma(\varpi, \theta, t)$:

$$\phi(\varpi, \theta, z, t) = \mathrm{Re}\,[\mathfrak{V}(\varpi, \theta, z, t)] = \mathrm{Re}\,[v(\varpi, z)e^{i(\omega t - m\theta)}]\,, \tag{9}$$

$$\sigma(\varpi, \theta, t) = \mathrm{Re}\,[S(\varpi)e^{i(\omega t - m\theta)}]\,, \tag{10}$$

where ω is the complex wave frequency and $m > 0$ represents the number of spiral arms of the wave. The pattern speed Ω_p of the wave is given as usual by

$$\Omega_p = \omega_{\mathrm{R}}/m\,, \tag{11}$$

where we shall also use the subscripts R and I to denote the real and imaginary parts, respectively, of the quantities involved. At the plane of the disk, we can assume $v(\varpi, z)$ to take the form

$$v(\varpi, 0) = \alpha e^{ip(\varpi)} \equiv V(\varpi)\,, \tag{12}$$

where α is a complex constant and all the spatial variation is included in the complex phase function $p(\varpi)$. The radial wavenumber

$$k(\varpi) = \frac{dp(\varpi)}{d\varpi} \tag{13}$$

is, of course, also a complex function of ϖ. Unlike Lin and Shu (1964) and Shu (1970*b*), this complex wavenumber is necessary to describe the galactic density wave at the Lindblad resonances (Mark 1971). We shall also find it convenient to denote the amplitude of the wave potential by $A(\varpi)$, where

$$A(\varpi) = |\alpha| \exp\,[-p_{\mathrm{I}}(\varpi)]\,. \tag{14}$$

If $|k\varpi| \gg 1$, and $|k_{\mathrm{R}}\varpi| > 1$, then, even though k is a complex number, we can still show as in Shu (1970*b*) that a simple algebraic relation exists[2] between $S(\varpi)$ and $V(\varpi)$; namely, as a consequence of Poisson's equation, we have

$$S(\varpi) = -\frac{s_k V(\varpi)}{2\pi G}\left[k(\varpi) - \frac{i}{2\varpi}\right], \tag{15}$$

where G is the gravitational constant and

$$s_k = \mathrm{sgn}\,[k_{\mathrm{R}}]\,. \tag{16}$$

Now Shu (1970*b*) has shown that the density wave of Lin and Shu (1964) can be discussed by a systematic expansion where

$$\varepsilon \sim a \sim |k\varpi|^{-1} \ll 1\,. \tag{17}$$

We shall call this the "epicyclic approximation" even though, strictly speaking, this should refer only to the smallness of ε and a. The surface density response $S(\varpi)$ can also be calculated from the Liouville equation in $(\varpi, p_\varpi, \theta, p_\theta, t)$, and we have, to two orders in epicyclic approximation (Shu 1970*b*),

$$S(\varpi) = \mathfrak{L}[q]\,, \tag{18}$$

where

$$\mathfrak{L}[q] = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\Sigma(r)V(\varpi)}{\varpi^2\varepsilon^2(r)\kappa^2(r)} \exp\left[-\frac{\xi^2 + \eta^2}{2\varepsilon^2(r)}\right] q[\nu(r), \varpi]\,\frac{d\xi d\eta}{2\pi\varepsilon^2(r)}\,, \tag{19}$$

$$\nu(r) = m\,\frac{[\Omega_p - \Omega(r)]}{\kappa(r)}\,, \qquad \xi^2 + \eta^2 = a^2\,, \qquad \eta = 1 - r/\varpi\,, \tag{20}$$

$$q[\nu(r), \varpi] = 1 - \frac{\nu(r)\pi}{\sin\,[\nu(r)\pi]}\,\mathfrak{Z}(\nu, \varpi)\,, \tag{21}$$

$$\mathfrak{Z}(\nu, \varpi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\,[i\nu s - ik(\varpi)\varpi R_a]\left\{1 + i\varpi k(\varpi)\left[\frac{1}{2}R_a^2\,\frac{d\ln k}{d\ln\varpi} - R_b\right] + im\,\frac{2\Omega(\varpi)}{\kappa(\varpi)}\left[\frac{\partial R_a}{\partial s} + \xi\right]\right\}ds\,, \tag{22}$$

$$R_a = \eta(1 + \cos s) + \xi \sin s\,, \qquad R_b = B_2(\varpi)R_a^2 - [1 + 2B_2(\varpi)]\eta R_a\,, \tag{23}$$

$$B_2(\varpi) = \frac{1}{2}\left[1 - \frac{2}{3}\frac{d\ln\kappa(\varpi)}{d\ln\varpi}\right]. \tag{24}$$

[2] We are thankful to Dr. D. Baldwin for first suggesting that this generalization of Shu's relationship should be possible.

Since at this point we use the complex wavenumber $k(\varpi)$ to give both the phase and the amplitude variations in the wave, the expression for $\mathfrak{Z}(\nu, \varpi)$ does not contain a term involving the derivative of the wave amplitude $A(\varpi)$.

b) Expansion in Terms of the Resonances

The quantity $q(\nu, \varpi)$ is a meromorphic function of ν and tends uniformly to zero as $|\nu| \to \infty$ on the square contour with corners at $(j + \frac{1}{2}, j + \frac{1}{2})$, $(-j - \frac{1}{2}, j + \frac{1}{2})$, $(-j - \frac{1}{2}, -j - \frac{1}{2})$, $(j + \frac{1}{2}, -j - \frac{1}{2})$, where $j = 1, 2, 3, \ldots$. Therefore, we may express

$$q(\nu, \varpi) = \sum_{n=-\infty}^{\infty} \frac{n(-1)^{n+1}}{\nu - n} \mathfrak{Z}(n, \varpi) . \tag{25}$$

Each one of these poles at $\nu = n$ represents the contribution due to one of the Lindblad or corotation resonances, with $n = -1, +1, 0$, corresponding to the inner and outer Lindblad and the corotation resonance, respectively. Since we are concentrating our attention on the region close to one of the Lindblad resonances, say $\nu(r) \simeq n_L \neq 0$, we shall call the $n = n_L$ term "resonant," and the rest we shall collectively denote as "nonresonant" terms. The following consideration is the same for all but the corotation resonance $n = 0$. We shall write

$$S(\varpi) = S^{(L)}(\varpi) + S^{(c)}(\varpi) , \tag{26}$$

where $S^{(L)}(\varpi)$ is the resonant term and the complementary term $S^{(c)}(\varpi)$ is nonresonant.

For each $n \neq n_L$ term we may proceed as in Shu's (1970*b*) analysis to show in Appendix A that

$$S^{(c)} = S_0^{(c)} + S_1^{(c)} , \tag{27}$$

$$S_0^{(c)}(\varpi) = -\frac{k^2(\varpi)}{\kappa^2(\varpi)} V(\varpi)\Sigma(\varpi)\mathfrak{F}^{(c)}(\nu, x) , \tag{28}$$

$$S_1^{(c)}(\varpi) = \frac{iV(\varpi)}{2\pi G\varpi} \frac{d}{d\varpi} \left[\varpi \frac{k(\varpi)}{k_T(\varpi)} \mathfrak{F}^{(c)}(\nu, x)\mathfrak{D}^{(c)}(\nu, x) \right] , \tag{29}$$

where $S_0^{(c)}$ and $S_1^{(c)}$ are the first two terms in the epicyclic expansion (17) for $S^{(c)}$; ν now has argument ϖ; and also

$$x = [k(\varpi)\varepsilon(\varpi)\varpi]^2 , \qquad k_T(\varpi) = \frac{\kappa^2(\varpi)}{2\pi G\Sigma(\varpi)} , \tag{30}$$

$$\mathfrak{F}^{(c)}(\nu, x) = -\frac{1}{x} \sum_{n \neq n_L} \frac{nI_n(x)e^{-x}}{\nu - n} , \tag{31}$$

$$\mathfrak{D}^{(c)}(\nu, x) = \frac{\partial}{\partial \ln x} \ln [x\mathfrak{F}^{(c)}(\nu, x)] , \tag{32}$$

where $I_n(x)$ is the modified Bessel function of the first kind, of order n.

c) The Resonant Terms

In evaluating the resonant contribution to the perturbed surface density, we must evaluate

$$S^{(L)}(\varpi) = \mathfrak{L}\left[\frac{(-1)^{n_L+1} n_L}{\nu(r) - n_L} \mathfrak{Z}(n_L, \varpi) \right] . \tag{33}$$

The operator $\mathfrak{L}$ and the function $\mathfrak{Z}$ are defined in equations (19) and (22). All functions of r such as $\nu(r)$ depend implicitly on the variable of integration η through equation (20). Since $\eta = O(\varepsilon)$, it is customary in the calculations of Lin and Shu (1966) and Shu (1970*b*) to expand all functions of r as a Taylor series around $r = \varpi$. For example,

$$\frac{1}{\nu(r) - n_L} = \frac{1}{\nu(\varpi) - n_L} - \varpi\eta \frac{d}{d\varpi} \left[\frac{1}{\nu(\varpi) - n_L} \right] + \cdots . \tag{34}$$

However, this expansion is valid for all terms except this one. If we did expand $[\nu(r) - n_L]^{-1}$ in this manner, the dispersion relation would have a singularity as $\nu(\varpi) \to n_L$, or $\varpi \to r_L$ (r_L is the radius of Lindblad resonance and ν is assumed to be real in the density waves of Lin and Shu). This resulted in a singularity in their wavenumber $k(\varpi)$ and in the perturbed surface density $S(\varpi)$.

This singularity can be removed and a self-consistent solution obtained (Mark 1971) through the Lindblad resonance if we allow a complex wavenumber[3] $k(\varpi)$ and also the following approximation for the term $[\nu(r) - n_L]^{-1}$:

$$\nu(r) - n_L \simeq \frac{s_\nu}{L}(r - r_L) + i\nu_I(r) , \tag{35}$$

where

$$s_\nu = \text{sgn}\left[\frac{d\nu_R}{dr}\bigg|_{r=r_L}\right] , \tag{36}$$

and L is some positive constant with dimensions of length. The quantities ω and $\nu(r)$ are in general complex. In order to obtain the dispersion relation for purely propagating waves where ω and ν are real, we follow the discussion of Landau (1946) and first calculate the dispersion relation for exponentially growing ($\omega_I < 0$) waves. The initial-valued approach then requires that the dispersion relation for real ω or ν be obtained by analytical continuation $\omega_I \to 0-$, $\nu_I \to 0-$. Towards the end of Appendix A, we will briefly discuss the accuracy of this approximation (35).

Using this approximation for $(\nu - n_L)^{-1}$, we find, to two orders in the epicyclic expansion (17), that the amplitude for the resonant part of the perturbed surface density is

$$S^{(L)}(\varpi) = S_0^{(L)}(\varpi) + S_1^{(L)}(\varpi) , \tag{37}$$

$$S_0^{(L)}(\varpi) = -\frac{k^2(\varpi)}{\kappa^2(\varpi)} V(\varpi)\Sigma(\varpi)\mathfrak{F}^{(L)}[\nu(\varpi), y, \varpi] , \tag{38}$$

$$S_1^{(L)}(\varpi) = \frac{iV(\varpi)}{2\pi G\varpi}\frac{d}{d\varpi}\left\{\varpi\frac{k(\varpi)}{k_T(\varpi)}\mathfrak{F}^{(L)}[\nu(\varpi), y, \varpi]\mathfrak{D}^{(L)}[\nu(\varpi), y, \varpi]\right\} + s_k\frac{V(\varpi)k(\varpi)}{2\pi G}T[\omega, k(\varpi), \varpi] , \tag{39}$$

where $S_0^{(L)}$ and $S_1^{(L)}$ are the first two epicyclic approximations to $S^{(L)}$ and also

$$\mathfrak{F}^{(L)}(\nu, y, \varpi) = -s_\nu n_L \frac{i2^{1/2}L}{\varpi\varepsilon(\varpi)x}\int_{-\infty s_\nu}^{0} \exp\left[-\tau^2 + iu_L\tau - u\right]I_{n_L}(u)d\tau , \tag{40}$$

$$y = k(\varpi)\varpi\varepsilon(\varpi) , \qquad x = y^2 , \tag{41}$$

$$u = x + 2^{1/2}y\tau , \qquad u_L = s_\nu\frac{2^{1/2}L}{\varpi\varepsilon(\varpi)}[\nu(\varpi) - n_L] , \tag{42}$$

$$\mathfrak{D}^{(L)}(\nu, y, \varpi) = \frac{\partial}{\partial \ln x}\ln\left[x\mathfrak{F}^{(L)}(\nu, y, \varpi)\right] . \tag{43}$$

The analysis required to obtain these formulae is straightforward but tedious, and the expression for $T(\omega, k, \varpi)$ is very lengthy. We refer the reader to Appendix A for the details.

III. THE DISPERSION RELATIONS IN THE REGIONS OF LINDBLAD RESONANCES

a) *General Properties of the Dispersion Relations*

If we equate $S = S^{(L)} + S^{(c)}$ from equations (15), (27), and (37), we get, to two orders in the expansion (17), the dispersion relationship

$$0 = D_0(\omega, k, \varpi) + D_1(\omega, k, \varpi) + \cdots , \tag{44}$$

where

$$D_0(\omega, k, \varpi) = 1 - s_k\frac{k}{k_T}\mathfrak{F}(\nu, y, \varpi) , \tag{45}$$

$$D_1(\omega, k, \varpi) = \frac{i}{k\varpi}\left\{-\frac{1}{2} + s_k\frac{d}{d\varpi}\left[\varpi\frac{k}{k_T}\mathfrak{F}(\nu, y, \varpi)\mathfrak{D}(\nu, y, \varpi)\right]\right\} + T(\omega, k, \varpi) , \tag{46}$$

$$\mathfrak{F}(\nu, y, \varpi) = \mathfrak{F}^{(L)}(\nu, y, \varpi) + \mathfrak{F}^{(c)}(\nu, y^2) , \tag{47}$$

$$\mathfrak{D}(\nu, y, \varpi) = \frac{\partial}{\partial \ln x}\ln\left[x\mathfrak{F}(\nu, y, \varpi)\right] , \tag{48}$$

[3] Contopoulos (1971) did not use a complex wavenumber.

where $\mathfrak{F}^{(c)}$, $\mathfrak{D}^{(c)}$, $\mathfrak{F}^{(L)}$, $\mathfrak{D}^{(L)}$ are given in equations (31), (32), (40), and (43), respectively, and T is given in equation (A21) of Appendix A. Given the complex frequency $\omega = \omega_R + i\omega_I$, we can now calculate the complex wavenumber $k(\varpi)$ to two orders in the small parameter $\varepsilon(\varpi)$. This dispersion relation is derived at first for growing waves $\omega_I < 0$; but since the functions $\mathfrak{F}$ and $\mathfrak{D}$ have immediate analytic continuations to $\omega_I \geq 0$, we can apply equation (44) to both neutral waves and damped waves (of course, $\omega_R = m\Omega_p$).

Since D_1 is smaller than D_0 by one order in our small parameter (eq. [17]), it is actually more convenient to expand $k(\varpi)$ in the same small parameter:

$$k(\varpi) = k_0(\varpi) + k_1(\varpi) + \cdots . \tag{49}$$

Although not absolutely necessary, it is convenient also to let $(\omega_I/m\Omega_p) = O(\varepsilon)$, so that from equation (44) we now have two separate dispersion relations determining $k_0(\varpi)$ and $k_1(\varpi)$, namely,

$$D_0(\omega_R, k_0, \varpi) \equiv 1 - s_k \frac{k_0}{k_T(\varpi)} \mathfrak{F}[\nu_R, k_0\varpi\varepsilon(\varpi), \varpi] = 0 , \tag{50}$$

$$i\omega_I \frac{\partial}{\partial\omega_R} D_0(\omega_R, k_0, \varpi) + k_1 \frac{\partial}{\partial k_0} D_0(\omega_R, k_0, \varpi) + D_1(\omega_R, k_0, \varpi) = 0 . \tag{51}$$

We may note that, using relations (20), (48), and (50),

$$\frac{\partial}{\partial\omega_R} D_0(\omega_R, k_0, \varpi) = -\frac{s_k k_0}{\kappa(\varpi)k_T(\varpi)} \frac{\partial}{\partial\nu_R} \mathfrak{F}[\nu_R(\varpi), k_0\varpi\varepsilon(\varpi), \varpi] , \tag{52}$$

$$\frac{\partial}{\partial k_0} D_0(\omega_R, k_0, \varpi) = \frac{1}{k_0}\{1 - 2\mathfrak{D}[\nu_R(\varpi), k_0\varpi\varepsilon(\varpi), \varpi]\} . \tag{53}$$

For the inner Lindblad resonance where $n_L = -1$, $s_\nu = 1$, dispersion relation (50) was already given by Mark (1971). For fixed $\nu_R(\varpi)$ or ω_R it determines $k_0(\varpi)$ the principal part of the complex wavenumber $k(\varpi)$ near the Lindblad resonances. Far away from resonance

$$|u_L| = \left| \frac{2^{1/2}L}{\varpi\varepsilon(\varpi)} [\nu(\varpi) - n_L] \right| \gg 1 , \tag{54}$$

so that on integrations by parts, we find from equation (40) that

$$\begin{aligned}\mathfrak{F}^{(L)} &= -n_L s_\nu \frac{i2^{1/2}L}{\varpi\varepsilon(\varpi)x} \left\{ \frac{e^{-x}}{iu_L} I_{n_L}(x) + \frac{2^{1/2}y}{u_L^2} \frac{d}{dx}[I_{n_L}(x)e^{-x}] + \cdots \right\} \\ &= -\frac{n_L e^{-x}}{x[\nu(\varpi) - n_L]} I_{n_L}(x) - \frac{in_L\varpi\varepsilon(\varpi)}{s_\nu L y[\nu(\varpi) - n_L]^2} \frac{d}{dx}[I_{n_L}(x)e^{-x}] + \cdots . \end{aligned} \tag{55}$$

If only the first term is retained [the corrections are smaller by $O(\varepsilon\varpi/L)$], then $[(1 - \nu^2)(\mathfrak{F}^{(L)} + \mathfrak{F}^{(c)})]$ reduces to $\mathfrak{F}_\nu(x)$, the "reduction factor" as defined by Lin and Shu (1966).

Numerical results for $k_0(\varpi)$ are more conveniently obtained from another form of the dispersion relationship (50). This is obtained by expanding $e^{-u}I_{n_L}(u)$ (where $u = x + 2^{1/2}y\tau$) as a Taylor series in powers of $(2^{1/2}y\tau)$. Then

$$\mathfrak{F}^{(L)}(\nu, y, \varpi) = -n_L \frac{i\pi^{1/2}L}{2^{1/2}\varpi\varepsilon(\varpi)x} \sum_{l=0}^{\infty} \frac{1}{l!} \left[\frac{i}{2^{1/2}} y s_\nu\right]^l w^{(l)}\left(-s_\nu \frac{u_L}{2}\right) \frac{d^l}{dx^l}[I_{n_L}(x)e^{-x}] . \tag{56}$$

where $w(u)$ is the plasma dispersion function (Fried and Conte 1961; Abramowitz and Stegun 1964, p. 297),

$$w(u) = e^{-u^2}\left[1 + \frac{2i}{\pi^{1/2}} \int_0^u e^{s^2} ds\right] = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{-\lambda^2}}{\lambda - u} d\lambda , \tag{57}$$

and

$$w^{(l)}(u) = \frac{d^l w(u)}{du^l} . \tag{58}$$

In the contour integral over λ, if Im $(u) < 0$, it is assumed that the contour is deformed so that the path of integration passes below u. The values of $w(u)$ can be obtained by methods outlined in the above-mentioned references. The function $w^{(l)}(u)$ can then be obtained from the well-known recurrence relations

$$w^{(1)}(u) = -2uw(u) + i\frac{2}{\pi^{1/2}} , \tag{59}$$

$$w^{(l)}(u) = -2[uw^{(l-1)}(u) + (l - 1)w^{(l-2)}(u)] \qquad (\text{for } l \geq 2) . \tag{60}$$

For the inner Lindblad resonance in the Milky Way system, the first term of equation (56) already represents an approximation accurate to 3 percent for trailing waves and 6 percent for leading waves ($\omega_I \ll \omega_R$). Retaining a few more terms gives excellent results. There are two reasons why this approximation is so accurate. First, when $|\varpi - r_L| \leq 2$, we have $|y| \geq 5$; and since also Re $(x) > 0$ remains true, then the succeeding terms of equation (56) are smaller than the preceding ones by $O(1/2^{1/2}y)$. Second, when $|\varpi - r_L| \geq 2$, we have $u_L > 3$ and the succeeding terms are again smaller because $w^{(l)}(-u_L/2) = O(u_L{}^{-l-1})$.

b) Some Numerical Results from the Dispersion Relations

We give here some detailed numerical examples for the short-wavelength waves. Using a Schmidt (1965) model for the Milky Way system, we consider stationary ($\omega_I = \nu_I = 0$) waves of pattern speed $\Omega_p = 13.5$ km s^{-1} kpc^{-1} (cf. Lin *et al.* 1969). For this pattern speed the inner Lindblad resonance ($n_L = -1, s_\nu = 1$) occurs at the radius $r_L = 3.2$ kpc where $[\nu_R(r_L) - n_L] = 0$. Table 1 gives the wavenumbers and wave amplitudes for the short-wavelength trailing wave. Values of the radial distance ϖ (first column) range from 3.2 to 6.5 kpc. The next three columns give some of the parameters that occur in the dispersion relations. We can see how $u_L(\varpi)$ varies rapidly compared with other parameters of the equilibrium, such as represented by ($\varpi\varepsilon$). Also, k_{aR} and k_{aI} are, respectively, the real and imaginary parts of the wavenumber $k_0(\varpi)$ as obtained from the approximate dispersion relation of Mark (1971, eq. [11]). However, in using this relation, we have evaluated his quantity k_L at radius ϖ rather than at radius r_L. We see that the approximate wavenumber $k_a(\varpi) = k_{aR} + ik_{aI}$ has indeed the qualitative behavior discussed in this earlier paper. The accurate form of $k_0(\varpi) = k_{0R} + ik_{0I}$ also exhibits this same behavior, and we see that the approximation $k_a(\varpi)$ gives k_0 to within an accuracy of 30 percent. Most of the errors in this approximation result from omitting the $\mathfrak{F}^{(c)}$ term from equation (50). The quantities k_{1R} and k_{1I} are the real and imaginary parts of $k_1(\varpi)$. Clearly $|k_1| \ll |k_0|$, in accordance with the requirements of expansion (49).

Knowing the imaginary part $k_I = k_{0I} + k_{1I}$ of the wavenumber (to two orders in expansion 49), we may calculate $A(\varpi)$, the amplitude of the wave potential (eqs. [12]–[14]) and also $S(\varpi)$, the corresponding amplitude of the self-consistent surface density response (cf. eq. [15]). These are also given in table 1. As we approach the resonance

TABLE 1

SHORT TRAILING WAVE AT INNER RESONANCE

ϖ (kpc)	ν_R	u_L	$\varpi\varepsilon(\varpi)$ (kpc)	k_{aR} (kpc^{-1})	k_{aI} (kpc^{-1})	k_{0R} (kpc^{-1})	k_{0I} (kpc^{-1})	k_{1R} (kpc^{-1})	k_{1I} (kpc^{-1})	A (arb. un.)	S (arb. un.)	$\mathfrak{D}_I/(\mathfrak{D}_R-\frac{1}{2})$
3.2	−1.000	0.00	.641	−6.24	−6.24	−8.11	−6.18	+.161	−.283	0.009	0.021	+1.5 (−1)
3.3	−0.991	0.32	.654	−6.55	−5.48	−8.38	−5.45	+.131	−.241	0.017	0.038	+1.3 (−1)
3.4	−0.981	0.61	.666	−6.71	−4.74	−8.50	−4.75	+.120	−.216	0.029	0.063	+1.2 (−1)
3.5	−0.973	0.87	.677	−6.73	−4.07	−8.50	−4.10	+.117	−.207	0.046	0.097	+1.1 (−1)
3.6	−0.965	1.12	.688	−6.67	−3.46	−8.41	−3.51	+.112	−.211	0.069	0.140	+9.9 (−2)
3.7	−0.957	1.35	.699	−6.54	−2.93	−8.26	−3.00	+.102	−.223	0.097	0.190	+8.9 (−2)
3.8	−0.949	1.56	.709	−6.37	−2.47	−8.07	−2.55	+.084	−.237	0.131	0.248	+8.0 (−2)
3.9	−0.942	1.75	.719	−6.17	−2.08	−7.85	−2.17	+.060	−.250	0.170	0.310	+7.2 (−2)
4.0	−0.935	1.94	.729	−5.95	−1.75	−7.62	−1.85	+.041	−.257	0.213	0.375	+6.4 (−2)
4.1	−0.929	2.11	.738	−5.74	−1.47	−7.39	−1.57	+.019	−.262	0.260	0.440	+5.7 (−2)
4.2	−0.922	2.27	.747	−5.53	−1.23	−7.17	−1.34	−.014	−.265	0.309	0.506	+5.0 (−2)
4.3	−0.916	2.42	.755	−5.32	−1.04	−6.95	−1.15	−.045	−.262	0.359	0.571	+4.5 (−2)
4.4	−0.910	2.57	.763	−5.12	−0.87	−6.73	−0.98	−.075	−.255	0.410	0.632	+3.9 (−2)
4.5	−0.904	2.70	.771	−4.93	−0.73	−6.53	−0.85	−.103	−.243	0.460	0.690	+3.4 (−2)
4.6	−0.899	2.83	.779	−4.75	−0.62	−6.34	−0.73	−.127	−.229	0.510	0.743	+3.0 (−2)
4.7	−0.893	2.95	.786	−4.59	−0.52	−6.16	−0.63	−.149	−.212	0.558	0.792	+2.6 (−2)
4.8	−0.888	3.07	.794	−4.43	−0.44	−6.00	−0.55	−.168	−.194	0.604	0.835	+2.2 (−2)
4.9	−0.883	3.18	.801	−4.28	−0.37	−5.84	−0.48	−.183	−.174	0.648	0.874	+1.9 (−2)
5.0	−0.878	3.29	.807	−4.15	−0.32	−5.69	−0.42	−.194	−.156	0.689	0.907	+1.6 (−2)
5.1	−0.873	3.39	.814	−4.02	−0.27	−5.55	−0.37	−.203	−.137	0.728	0.936	+1.4 (−2)
5.2	−0.869	3.49	.820	−3.90	−0.23	−5.43	−0.33	−.210	−.118	0.763	0.961	+1.2 (−2)
5.3	−0.864	3.58	.826	−3.79	−0.19	−5.31	−0.29	−.215	−.100	0.796	0.981	+9.8 (−3)
5.4	−0.859	3.68	.832	−3.69	−0.17	−5.19	−0.26	−.218	−.082	0.826	0.997	+8.1 (−3)
5.5	−0.855	3.76	.838	−3.59	−0.14	−5.09	−0.24	−.219	−.065	0.853	1.009	+6.6 (−3)
5.6	−0.851	3.85	.844	−3.50	−0.12	−4.99	−0.21	−.218	−.049	0.878	1.018	+5.3 (−3)
5.7	−0.847	3.93	.849	−3.42	−0.10	−4.89	−0.19	−.217	−.035	0.899	1.024	+4.1 (−3)
5.8	−0.843	4.01	.855	−3.34	−0.09	−4.81	−0.17	−.214	−.021	0.919	1.027	+3.1 (−3)
5.9	−0.839	4.09	.860	−3.27	−0.08	−4.72	−0.16	−.211	−.009	0.936	1.028	+2.3 (−3)
6.0	−0.835	4.16	.865	−3.20	−0.07	−4.65	−0.15	−.207	+.001	0.951	1.027	+1.5 (−3)
6.1	−0.831	4.23	.870	−3.14	−0.06	−4.57	−0.14	−.202	+.011	0.964	1.024	+8.7 (−4)
6.2	−0.827	4.30	.875	−3.07	−0.05	−4.50	−0.13	−.198	+.020	0.975	1.020	+3.2 (−4)
6.3	−0.823	4.37	.880	−3.02	−0.04	−4.44	−0.12	−.192	+.028	0.985	1.014	−1.6 (−4)
6.4	−0.820	4.44	.884	−2.96	−0.04	−4.37	−0.11	−.187	+.036	0.993	1.008	−5.6 (−4)
6.5	−0.816	4.50	.889	−2.91	−0.03	−4.32	−0.10	−.182	+.043	1.000	1.000	−9.1 (−4)

TABLE 2

SHORT LEADING WAVE AT INNER RESONANCE

ϖ (kpc)	k_{0R} (kpc^{-1})	k_{0I} (kpc^{-1})	ϖ (kpc)	k_{0R} (kpc^{-1})	k_{0I} (kpc^{-1})
3.1	8.18	+7.04	5.1	5.66	+0.14
3.2	8.63	+6.26	5.2	5.52	+0.10
3.3	8.89	+5.47	5.3	5.39	+0.07
3.4	8.99	+4.71	5.4	5.27	+0.05
3.5	8.97	+4.01	5.5	5.15	+0.03
3.6	8.85	+3.39	5.6	5.04	+0.01
3.7	8.67	+2.84	5.7	4.94	+0.00
3.8	8.45	+2.37	5.8	4.85	−0.02
3.9	8.20	+1.97	5.9	4.76	−0.02
4.0	7.95	+1.63	6.0	4.68	−0.03
4.1	7.69	+1.34	6.1	4.60	−0.04
4.2	7.44	+1.10	6.2	4.53	−0.04
4.3	7.19	+0.90	6.3	4.46	−0.05
4.4	6.96	+0.73	6.4	4.39	−0.05
4.5	6.74	+0.59	6.5	4.33	−0.05
4.6	6.53	+0.48	6.6	4.27	−0.05
4.7	6.33	+0.38	6.7	4.22	−0.05
4.8	6.14	+0.30	6.8	4.16	−0.05
4.9	5.97	+0.24	6.9	4.11	−0.05
5.0	5.81	+0.18	7.0	4.06	−0.06

radius $r_L = 3.2$ kpc from the outside, both A and S eventually decay monotonically inward. At r_L their values have become negligibly small. This is in marked contrast to the behavior given in Lin and Shu (1964) and Lin *et al.* (1969) and in Shu (1970*b*), where the wavenumber $k_0(\varpi)$ and density amplitude $S(\varpi)$ tend to infinity. These singularities have been removed by our more accurate analysis. We may also note that, even at radii $\varpi \simeq r_L +$ (2 kpc), the wave is already affected by the resonant stars. This is because those stars which have guiding center radii at $r = r_L$ may have excursions due to epicyclic motions which are typically of the order of $|\varpi - r_L| = (c/\kappa)$. Finally, the last column of table 1 gives a quantity which is of interest only to § IV.

The situation for the short-wavelength leading wave is very similar to that of the trailing wave. In particular, the wavenumber k_a of the approximate dispersion relation (Mark 1971, eq. [11]) can be obtained from table 1 (the columns labeled k_{aR} and k_{aI}) by a mere change of sign. But in the solutions of the accurate dispersion relations (50) and (51), the leading and trailing cases differ slightly more than by a mere change of sign. We illustrate this in table 2 by giving k_{0R} and k_{0I} (from eq. [50]) for the leading wave case. Of course, the sign of k_{0R} determines the sense of winding of the waves. On the other hand, the sign of k_{0I} determines the direction of decay of the wave amplitude. Both leading and trailing waves decay in the direction of their respective group velocities (cf. § IV). Thus both of these waves are evanescent.

Near the other Lindblad resonances, these two waves again decay in amplitude in the direction of their group velocities. The various possible configurations are summarized in table 3. The first three columns identify the sense of winding of the waves, and also the type of Lindblad resonance involved. Columns (4)–(6) give the behavior of some physical parameters that identify the wave-propagation side of the resonance. Note that u_{LR} and $(\varpi - r_L)$ are both zero at the resonance radius. The last two columns give the behavior of the wave amplitude and the direction of the group velocity. Since $s_v = -\text{sgn}\,[d(\Omega + n_L\kappa/m)/dr]|_{r_L}$, cases where $s_v = -1$ are rather rare. For galaxies

TABLE 3

CLASSIFICATION OF THE BEHAVIOR OF THE SHORT WAVES AT THE VARIOUS LINDBLAD RESONANCES

	n_L	s_v	$\Omega + n_L(\kappa/m)$	u_{LR}	$\varpi - r_L$	$A(\varpi)$	c_g
T	N	P	D	I	I	I	N
T	N	N	D	D	D	D	N
T	P	P	I	D	D	I	P
T	P	N	I	I	I	D	P
L	N	P	D	I	I	D	P
L	N	N	D	D	D	I	P
L	P	P	I	D	D	D	N
L	P	N	I	I	I	I	N

KEY.—T = trailing wave; L = leading wave; P = positive; N = negative; "I" (or "D") means this quantity increases (decreases) in algebraic value as we proceed from the resonance radius r_L into the wave-propagation region.

where $(\Omega - \kappa/2)$ increases slowly from galactic center, there may be a propagation region between the galactic center and an $n_L = -1$ resonance occurring at the point where the pattern speed $\Omega_p = \Omega - \kappa/2$. At this resonance $s_\nu = -1$.

Dispersion relations (44), (50), and (51) all allow the wave solutions to be growing with time ($\omega_I < 0$) or decaying ($\omega_I > 0$). In this case, the temporally decaying solutions are obtained from dispersion relations which are the analytical continuation of the ones for temporally growing solutions. This is appropriate when we are considering the initial-valued problems in the sense of Landau (1946).

Lest there be any confusion concerning the role of the various dispersion relations, we would like to remind the reader here that equation (44) is the most general form of the dispersion relation for short-wavelength density waves for all spatial regions except at the corotation resonance. For given complex wave frequency ω, it gives the complex wavenumber $k(\varpi)$. However, since D_1 is of $O(\varepsilon)$ smaller than D_0, we have expanded k according to the series (49). Then k_0 is determined by equation (50) and k_1 by equation (51). The quantity k_1 is usually much smaller than k_0, and the main importance of equation (51) is that it contains information concerning the detailed balance of wave action (§ IV). Equation (44) is valid for all complex values of ω, while in relations (50) and (51) we have assumed $(\omega_I/\omega_R) = O(\varepsilon)$. If $|\omega_I/\omega_R| > O(\varepsilon)$, these two latter relations must be modified by replacing ω_R with ω and by omitting the first term of equation (51).

IV. DETAILED BALANCE OF WAVE ACTION

a) Continuity Equation for the Density of Wave Action

In the propagation region in between the Lindblad resonances, Toomre (1969) and Shu (1970*b*) have found that the amplitudes of density waves can be determined by the principle of conservation of wave action:

$$\frac{\partial \mathfrak{A}}{\partial t} + \frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi c_g \mathfrak{A}) = 0\,, \tag{61}$$

where $\mathfrak{A}(\varpi, t)$ is the surface density of wave action and $c_g(\varpi)$ is the kinematic group velocity of the waves ($c_g = -d\omega/dk$). Near the Lindblad resonances, we will show here that we have a continuity equation with a source term, namely,

$$\frac{\partial \mathfrak{A}}{\partial t} + \frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi \mathfrak{P}) = \mathfrak{S}\,, \tag{62}$$

where $\mathfrak{P}(\varpi, t)$ and $\mathfrak{S}(\varpi, t)$ are the flux and source of wave action, respectively. It is no longer possible in general to write $\mathfrak{P} = (-d\omega/dk)\mathfrak{A}$; but as one approaches the propagation region away from the resonances, the quantity $\mathfrak{P}$ reduces to this previous form of the flux.

We will show that equation (62) follows from relation (51). A considerable simplification of the following analysis is possible if we notice first of all that the quantity $1 - 2\mathfrak{D}$ (eq. [53]) is almost a real number even near the resonances. In the propagation region between the resonances, this is obviously true since both k_0 and D_0 are real numbers to a very good approximation ($D_0 = 0$ reduces to the Lin-Shu dispersion relation). Near a Lindblad resonance, if we use the approximate formula

$$D_0(\omega, k, \varpi) \cong 1 + in_L[k_L(\varpi)\varpi\varepsilon(\varpi)]^2 \frac{1}{x}\, w\left[-s_\nu \frac{L(\nu - n_L)}{2^{1/2}\varepsilon\varpi}\right]\,, \tag{63}$$

which corresponds to the approximate dispersion relation of Mark (1971, eq. [11]), then we find that

$$1 - 2\mathfrak{D} = \frac{\partial D_0}{\partial \ln k_0} = 2\frac{\partial D_0}{\partial \ln x} \simeq 2\,, \tag{64}$$

which is real. For the particular example we have (§ III*b*) of a short trailing wave, the last column of table 1 shows that the imaginary part of $(1 - 2\mathfrak{D})$ is less than 15 percent of the corresponding real part.

With the understanding that we keep only the real part of $(1 - 2\mathfrak{D})$, then on multiplying equation (51) with $(-2is_k k_0|\mathfrak{V}|^2{}_{z=0}/8\pi G)$ and using equations (9), (46), (49), (52), and (53), we find

$$\frac{\partial}{\partial t}\left[\frac{k_0{}^2}{8\pi G\kappa k_T}|\mathfrak{V}|^2{}_{z=0}\frac{\partial \mathfrak{F}}{\partial \nu_R}\right] + \frac{1}{\varpi}\frac{\partial}{\partial \varpi}\left[\varpi(1 - 2\mathfrak{D}_R)\left(\frac{-s_k}{8\pi G}|\mathfrak{V}|^2{}_{z=0}\right)\right]$$
$$= 2(k_{0I} + ik_{1R})\left[s_k\frac{(1 - 2\mathfrak{D}_R)}{8\pi G}|\mathfrak{V}|^2{}_{z=0}\right] + s_k\frac{ik_0}{4\pi G}T(\omega_R, k_0, \varpi)|\mathfrak{V}|^2{}_{z=0}\,. \tag{65}$$

The imaginary part of this relation gives a small correction to the phase of the wave corresponding to the wavenumber k_{1R}. The real part of relation (65) gives the continuity equation (62) mentioned earlier. We can now identify

$$\mathfrak{A}(\varpi, t) = \operatorname{Re}\left(\frac{k_0{}^2(\varpi)|\mathfrak{V}|^2{}_{z=0}}{8\pi G\kappa(\varpi)k_T(\varpi)}\frac{\partial}{\partial\nu_R}\mathfrak{F}[\nu_R(\varpi), k_0(\varpi)\varpi\varepsilon(\varpi), \varpi]\right), \tag{66}$$

$$\mathfrak{P}(\varpi, t) = -s_k\frac{|\mathfrak{V}|^2{}_{z=0}}{8\pi G}\operatorname{Re}\{1 - 2\mathfrak{D}[\nu_R(\varpi), k_0(\varpi)\varpi\varepsilon(\varpi), \varpi]\}, \tag{67}$$

$$\mathfrak{S}(\varpi, t) = \mathfrak{S}_0 + \mathfrak{S}_1, \tag{68}$$

$$\mathfrak{S}_0(\varpi, t) = -2\operatorname{Im}[k_0(\varpi)]\mathfrak{P}(\varpi, t), \tag{69}$$

$$\mathfrak{S}_1(\varpi, t) = -\frac{s_k}{4\pi G}|\mathfrak{V}|^2{}_{z=0}\operatorname{Im}\{k_0(\varpi)T(\omega_R, k_0, \varpi)\}. \tag{70}$$

$\mathfrak{F}$, $\mathfrak{D}$, and T are defined in equations (47), (48), and (A21), respectively. $\mathfrak{S}_0$ is the dominant part of $\mathfrak{S}$. For a stationary problem ($\partial\mathfrak{A}/\partial t = 0$), $\mathfrak{S}_0$ corresponds to the dominant spatial decay of the flux due to the term $\exp(-2\int^{\varpi} k_{0I}d\varpi)$. This decay depends on k_{0I} as determined from the dominant dispersion relation (50). Hence the subscript zero in $\mathfrak{S}_0$. Since the flux $\mathfrak{P}$ and especially its divergence must have other more slowly varying terms, these are balanced by the $\mathfrak{S}_1$ term of the source $\mathfrak{S}$.

We have expressed $\mathfrak{A}$ and $\mathfrak{P}$ in terms of the functions $\mathfrak{F}$ and $\mathfrak{D}$ (eqs. [47] and [48]). Far away ($|u_L| \gg 1$; cf. discussion beginning at eq. [55]) from the resonance in question, the function $(1 - \nu_R{}^2)\mathfrak{F}$ reduces to the reduction factor $\mathfrak{F}_\nu(x)$ of Lin and Shu (1966). $\mathfrak{D}$ also reduces to the function $\mathfrak{D}_\nu(x)$ defined by Shu (1970*b*). Thus we can easily see that far away from the resonances, $\mathfrak{A}$ reduces to the wave action density of Toomre (1969) and $\mathfrak{P}$ reduces to $(c_g\mathfrak{A})$. Our generalized conservation relation (62) also reduces to equation (61) of Toomre (1969) and Shu (1970*b*) because $\mathfrak{S}$ becomes very small. This behavior of $\mathfrak{S}$ can easily be seen from equations (68) to (70) and (A21) if we bear in mind the vanishing of $k_{0I}(\varpi)$ and use the same systematic approximations in equation (A21) as we did in equation (55) for $|u_L| \gg 1$. In this limit, the dominant parts of $\mathfrak{S}_0$ and $\mathfrak{S}_1$ actually cancel.

For short trailing waves at the inner Lindblad resonance and the same model as § III*c* ($\Omega_p = 13.5$ km s^{-1} kpc^{-1}, $r_L = 3.2$ kpc), table 4 gives the action density $\mathfrak{A}$, flux $\mathfrak{P}$ and source density $\mathfrak{S}$ as a function of radial distance ϖ. The peak and subsequent rapid inward decay of $|\mathfrak{A}|$ can be explained as follows. As we proceed inward toward the resonance radius, the negative action density $\mathfrak{A}$ first increases in magnitude because the flux $\mathfrak{P}$ decreases. But eventually the magnitude of $\mathfrak{A}$ decays rapidly because of the positive source $\mathfrak{S}$. The small positive values of $\mathfrak{A}$ for $\varpi < 3.7$ kpc may be due to numerical inaccuracy since $\mathfrak{A}$ is already so small in magnitude compared with its peak value. Although $\mathfrak{S}_1$ contributes only about a quarter of the integrated source $\int \mathfrak{S}2\pi\varpi d\varpi$, $\mathfrak{S}_1(\varpi, t)$ can occasionally be about half the magnitude of $\mathfrak{S}_0(\varpi, t)$ at the same radial distance ϖ.

TABLE 4

ACTION DENSITY, FLUX, AND SOURCE

ϖ	$\mathfrak{A}$	$\mathfrak{P}$	$\mathfrak{S}$	ϖ	$\mathfrak{A}$	$\mathfrak{P}$	$\mathfrak{S}$
3.2............	0.0	0.0	0.2	4.9............	−16.7	55.9	80.0
3.3............	0.0	0.0	0.5	5.0............	−18.4	62.7	79.9
3.4............	+ 0.1	0.1	1.3	5.1............	−19.9	69.3	78.7
3.5............	+ 0.1	0.3	2.9	5.2............	−21.3	75.6	76.5
3.6............	+ 0.1	0.7	5.5	5.3............	−22.4	81.6	73.4
3.7............	+ 0.1	1.4	9.4	5.4............	−23.3	87.2	69.8
3.8............	− 0.1	2.6	14.7	5.5............	−23.9	92.3	65.7
3.9............	− 0.5	4.3	21.3	5.6............	−24.4	96.9	61.5
4.0............	− 1.2	6.7	28.9	5.7............	−24.7	101.1	57.0
4.1............	− 2.1	9.8	37.1	5.8............	−24.9	104.8	52.6
4.2............	− 3.4	13.6	45.5	5.9............	−24.9	108.0	48.3
4.3............	− 4.9	18.2	53.7	6.0............	−24.9	110.8	44.3
4.4............	− 6.7	23.5	61.1	6.1............	−24.7	113.1	40.4
4.5............	− 8.7	29.3	67.6	6.2............	−24.4	115.1	36.6
4.6............	−10.7	35.6	72.8	6.3............	−24.1	116.8	33.0
4.7............	−12.8	42.2	76.6	6.4............	−23.8	118.1	29.7
4.8............	−14.8	49.1	79.0	6.5............	−23.4	119.0	26.7

NOTE.—(Mass, length, time) are in units of ($10^6\ M_\odot$, kpc, s kpc km^{-1}).

b) Interpretation in Terms of Wave Angular Momentum and Energy

So far, the derivation of the conservation relation (62) and the forms (66)–(68) of $\mathfrak{A}$, $\mathfrak{P}$, and $\mathfrak{S}$ have proceeded in a formal manner. It would certainly be most interesting if it can be shown, for example, that the quantity $\mathfrak{A}$ which we have formally called wave action density indeed satisfies

$$\mathfrak{H} = m\mathfrak{A}\,, \qquad \mathfrak{E} = \omega_R \mathfrak{A}\,, \tag{71}$$

where $\mathfrak{H}$ and $\mathfrak{E}$ are the densities of wave angular momentum and energy, respectively. It would also be of value to compare $(m\mathfrak{S})$ with the wave angular momentum emitted by the resonant stars (cf. LBK).

Let us first address ourselves to the latter question. For each category of resonant stars (i.e., each n_L) LBK calculated the total rate at which these stars emitted angular momentum to or absorbed it from a density wave. For the particular Lindblad resonance $n = n_L$ in question, their result corresponds to our integrated source

$$\frac{d}{dt}\mathfrak{H}_{n_L} = \int_0^\infty m\mathfrak{S}(\varpi, t) 2\pi\varpi d\varpi\,. \tag{72}$$

Rather than being content with comparing these integrated sources, we shall show that it is possible, under certain approximations, to obtain a source density $\mathfrak{S}_H(\varpi, t)$ using the methods of LBK. They calculated the change in angular momentum experienced by a resonant star in its orbit. The total rate of change is obtained by summing over these individual changes. For each such star, the problem of identifying its contribution to $\mathfrak{S}_H$ reduces to that of identifying what parts of the angular momentum change of this star contributes to $\mathfrak{S}_H$ in the range ϖ to $(\varpi + d\varpi)$. The situation for the Lindblad resonances is greatly simplified by our knowledge (§ III) that the complex wave-number $k(\varpi)$ is large. In particular, the wave amplitude $A(\varpi)$ varies rapidly in space but in a monotonic fashion. Thus we expect a resonant star to interact most strongly with the wave (and thus emit or absorb) at the point along its orbit where it experiences the largest wave amplitude. Since the wave amplitude is monotonically varying in radial distance ϖ, we see that this point on the resonant orbit corresponds to the point of maximum or minimum radial excursion. Let us call this point $\varpi = r_m$. Moreover, at r_m the star is traveling along a wave front because the wave fronts are nearly circular in shape due to the low inclination of the arms (large k_R). Thus we also expect for this resonant star that r_m is a stationary phase point in the wave-star interaction; also confirming the fact that most of the exchanges of angular momentum occur here. Thus we are led to the following approximations listed in order of decreasing importance:

i) A resonant star emits or absorbs angular momentum at that point ($\varpi = r_m$) of its radial excursion which corresponds to the stationary phase point of the wave-star interaction.

ii) The inequality $|k\varpi\varepsilon| > |n_L|$ holds, which turns out to be necessary in order for this stationary phase point ($\varpi = r_m$) to be also the position where the star experiences the largest wave amplitude on its orbit. This makes the stationary phase point a saddlepoint.

iii) The wave-star interaction integral is evaluated by saddlepoint integrations assuming $|k\varpi\varepsilon| \gg 1$.

iv) In the epicyclic approximation (17), only the dominant terms are kept.

The actual reduction of the result of LBK is summarized in Appendix B. For the source density of wave angular momentum, we obtain

$$\mathfrak{S}_H(\varpi, t) = -\frac{mn_L L\Sigma(r_L)}{4\kappa_L{}^2(\varepsilon_L r_L)^4}\frac{|\mathfrak{V}|^2{}_{z=0}}{|k(\varpi)|}\exp\left[-\frac{1}{2}\left(\frac{\varpi - r_L}{\varepsilon_L r_L}\right)^2\right], \tag{73}$$

where the subscripts L imply that the corresponding quantities are evaluated at $r = r_L$. In order to compare with this formula, we must also apply the above-mentioned approximations to $[m\mathfrak{S}(\varpi, t)]$, where $\mathfrak{S}$ is obtained from equation (68). Only approximations (iii) and (iv) are relevant. Approximation (iv) implies that in equation (68) we keep only the first term $\mathfrak{S}_0$ of $\mathfrak{S}$ because it is from the dominant epicyclic approximation. Approximation (iii) implies that we apply similar saddlepoint methods to $\mathfrak{F}$ which determine k_{0I} and $\mathfrak{D}$. Similar methods have been used in Mark (1971) to obtain his approximate dispersion relation (11). If we use this latter relation to determine k_{0I} and $(1 - \mathfrak{D}_R)$, then the first term of $(m\mathfrak{S})$ (eqs. [68] and [69]) gives the same source density of wave angular momentum as $\mathfrak{S}_H(\varpi, t)$ of equation (73) (using eqs. [35] and [42]).

Thus we have shown that *under the aforementioned approximations* (i)–(iv), *we have the same source density of wave angular momentum as that which could be obtained from the results of LBK.* We may note that approximations (iii) and (iv) do not altogether justify the omission of $\mathfrak{F}^{(c)}$ relative to $\mathfrak{F}^{(L)}$, a necessary step in the derivation of the approximate dispersion relation (Mark 1971, eq. [11]). They are omitted because we found in § III*c* that it is a good approximation (30%) to do so. Even though they are not a large contribution, these $\mathfrak{F}^{(c)}$ terms (cf. § II*b*) are interesting because they are probably not contained in the results of LBK. They are from the so-called nonresonant terms of the wave-particle interaction in the resonance region where $\nu(r) \simeq n_L$.

Let us now turn our attention to the interpretation of action density $\mathfrak{A}$ and to equations (71). Toomre's (1969) formula for $\mathfrak{A}$ applied only far away from resonance. Moreover, his energy density is calculated in a rotating frame. Kalnajs (1971) gave formulae for $\mathfrak{A}$, $\mathfrak{H}$, and $\mathfrak{E}$, and also showed that both of equations (71) hold. However, his formulae are difficult[4] to apply to our present context. We prefer then to give in Appendix C a simple physical derivation which gives the results in a form similar to that of Toomre (1969), but for an inertial frame and more general in that it applies even at the Lindblad resonances. We show in this appendix that the wave angular-momentum density $\mathfrak{H}$ and energy density $\mathfrak{E}$ indeed satisfy equations (71) provided $\mathfrak{A}$ is of the form given by equation (66). Note that we actually showed that the integrals of $\mathfrak{H}$ and $\mathfrak{E}$ over the disk are the total wave angular momentum and energy, respectively. There may be some who would doubt that we should identify the integrands $\mathfrak{H}$ and $\mathfrak{E}$ as the respective densities because, in principle, a divergence term could well be added on to $\mathfrak{H}$ and $\mathfrak{E}$. Quite apart from the fact that several workers (Toomre 1969; Shu 1970*b*) in the field identified $\mathfrak{H}$ and $\mathfrak{E}$ in this way, a similar derivation was used in the well-known case of electrodynamics. There, one (Landau and Lifshitz 1971, p. 78; Jackson 1962, pp. 21, 189) often simply takes the integrand of the integral for the energy of the fields and identifies this as energy density. Poynting's theorem confirms this formula for the energy density since it is there involved in a conservation equation with a known rate of dissipation of energy acting as a sink (Joule heat term). In dispersive but only slightly dissipative systems (Landau and Lifshitz 1960, § 61; Stix 1962, § 3-2), the energy density is identified through an averaged form of Poynting's theorem much like our conservation relation (62). Of course, in the absence of dissipation (e.g., between the resonance regions), straightforward characterizations of action density, etc., are possible through the use of variational methods involving the Lagrangian of the whole system. Far away from the resonances, our results reduce to those (Dewar 1972) obtained by such variational methods.

For the Lindblad resonance regions, further arguments can be given which lend additional support to this interpretation of wave action density in terms of densities of wave angular momentum and energy. These will not be presented here because they involve detailed discussions of the functional forms of the action density, flux, and source, and also of the formalism that generated these quantities.

V. SUMMARY AND CONCLUSIONS

For density waves in the vicinity of the Lindblad resonances, we have obtained the corresponding dispersion relations (§ III) and the modified principle of conservation of wave action (§ IV). In the notation of § II, the density wave in question consists of density and potential perturbations of spiral form which have number of arms m and a pattern speed of Ω_p. This wave interacts with stars which form the basic axisymmetric equilibrium disk. In the epicyclic approximation, an individual equilibrium stellar orbit is characterized first of all by the motion of the guiding center in a circle of radius r, with angular frequency $\Omega(r)$ about the center of the galaxy. The deviations from circular orbit are characterized by the epicyclic frequency $\kappa(r)$ (eq. [1]) and also by the small parameter a (eq. [5]) measuring amplitude of deviation. In the guiding center frame of such an orbit, the wave peaks pass by at the frequency $m[\Omega_p - \Omega(r)]$. If this is a nonzero harmonic of the epicyclic frequency $\kappa(r)$, then the wave is in Lindblad resonance with the stars having this value of r $(= r_L$, say); i.e., resonance occurs when $\nu(r)$ (eq. [20]) takes on the integer value n_L $(\neq 0)$. The quantities $n_L = -1$ and $n_L = +1$ are the inner and outer Lindblad resonances. Throughout this paper, we used the Schmidt (1965) model as the basic equilibrium for our Galaxy. In this case, a two-armed wave with a pattern speed of $\Omega_p = 10.7$ or 13.5 km s^{-1} kpc^{-1} has the inner resonance at $r_L = 4.0$ or 3.2 kpc, respectively. For the particular equilibrium distribution function used (eq. [7]), the mean epicyclic radial excursion of the stars is determined by $c(\varpi)/\kappa(\varpi)$, where $c(\varpi)$ is the radial velocity dispersion about circular motion. This radial excursion is a characteristic also of the resonant stars so that the actual influence of the Lindblad resonance extends into an annular region about radius $\varpi = r_L$ of half-width $c/\kappa \sim 1$–2 kpc.

a) The Modified Dispersion Relations

In the earlier discussions of Lin and Shu (1964), the wavenumber is singular as the resonance radius is approached. But, by a more careful discussion which takes into account the finite width of the resonance region, we can obtain a finite wavenumber at the radius r_L. However, for waves that are nearly stationary, the wavenumber $k(\varpi)$ is now a complex function of radial position ϖ. In the present analysis, it is convenient to expand $k = k_0 + k_1 + \cdots$ in orders of the epicyclic amplitude. The quantities k_0 and k_1 are determined by equations (50) and (51), respectively. In a brief communication (Mark 1971) the dispersion relation for k_0 at the inner Lindblad resonance has been recorded without proof. The properties of both k_0 and k_1 are now derived in § III.

The discussion of § III applies to all the Lindblad resonances and for both leading and trailing waves. In general, we obtain a spatial decay of the density waves in the direction of their respective group velocities. A typical set of values for $(-k) = -(k_0 + k_1)$ is illustrated in figure 1 for short trailing waves. We used $\Omega_p = 10.7$ km s^{-1} kpc^{-1}, $r_L = 4$ kpc. The dispersive velocities are obtained from

$$c(\varpi) = \left[\beta \frac{G\Sigma(\varpi)}{\kappa(\varpi)}\right] Q(\varpi) , \tag{74}$$

where $\Sigma(\varpi)$ is the surface mass density of the axisymmetric equilibrium and the quantity inside square brackets

[4] In particular, they contain singularities due to resonant stars. We have used instead a calculation patterned after a plasma result of Coppi, Rosenbluth, and Sudan (1969).

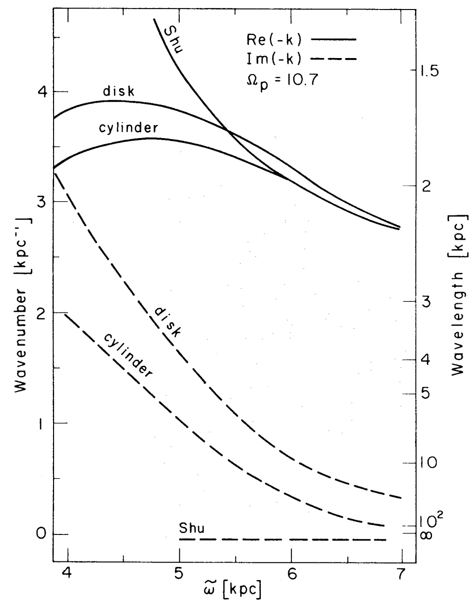

FIG. 1.—For short trailing spiral waves in the inner resonance region of our Galaxy, this figure compares the various wavenumbers k. The real part of $-k$ (*solid curves*) and the corresponding imaginary part (*broken curves*) are plotted versus radial distance from galactic center. Those labeled "disk" or "cylinder" are our present results for the corresponding geometries. Shu's previous calculations are also given for comparison (the real part was really due to Lin and Shu). His wavenumber has a small positive imaginary part which is plotted as a negative number Im $(-k)$.

represents the critical radial velocity dispersion (Toomre 1964) required to suppress all axisymmetric Jeans instabilities. The dimensionless constant $\beta = 3.36$, but this value changes somewhat if effects such as thickness corrections are included (Toomre 1973). $Q(\varpi)$ is another dimensionless parameter introduced by Toomre (1969) to describe deviations of $c(\varpi)$ from the critical value. We shall use $Q \equiv 1$ unless otherwise stated. As we can see from the curves labeled "disk" in figure 1, $k(\varpi)$ is now not divergent as $\varpi \to r_L = 4$ kpc. The sizable imaginary part represents a spatial damping of the wave as its group velocity carries it inward. This resonance damping effect was omitted in previous calculations. For comparison, the corresponding real and imaginary parts of k as taken[5] from Shu's (1970*b*) earlier analysis are also plotted (the real part was actually first derived by Lin and Shu 1964). In his result, the imaginary part of $(-k)$ is negative and also very small so that his $(-k_I)$ curve lies just below the $k = 0$ line. In §§ III and IV, we showed that our dispersion relations (50) and (51) reduce asymptotically to the results of Lin and Shu (1966) and Shu (1970*b*) when $|\varpi - r_L| \gg c/\kappa$. In figure 1, this "matching" of solutions is more evident for the real part of k. However, in obtaining values for the illustrations given here, we have considered it to be sufficient to use a simple numerical procedure. By using the methods described in § III*a*, a better agreement can be obtained. The detailed numbers discussed in table 2 for another case are obtained by this more accurate method.

In figure 1, the other pair of curves labeled "cylinder" is the analogy of the simplified dispersion relation (Mark 1971, eq. [11]) in a cylindrical model. It gives a crude estimate on the geometrical aspects of the correction to finite thickness. The thickness correction that corresponds to the change in $c(\varpi)$ is much more important and is considered in §§ V*b* and V*c*.

b) The Modified Principles of Conservation of Wave Action, Angular Momentum, and Energy

The mechanism for the wave absorption process is best seen in terms of the detailed action-conservation relation (62) which is the generalization of that of Toomre (1969) and Shu (1970*b*). In this equation, the process of absorption is described in terms of the wave action density $\mathfrak{A}(\varpi, t)$, flux $\mathfrak{P}(\varpi, t)$, and source density $\mathfrak{S}(\varpi, t)$. The corresponding quantities for wave angular momentum and energy are obtained (cf. § IV*b*) on multiplication by m and by ω_R, respectively. In the propagation region far ($|\varpi - r_L| \gg c/\kappa$) from the resonances, the flux $\mathfrak{P}$ reduces to the previously known form of $\mathfrak{P} = c_g \mathfrak{A}$, where c_g is the kinematic group velocity ($c_g = -d\omega/dk$). Close to the resonance this reduction is no longer possible if we insist on defining the group velocity (Toomre 1969) kinematically in terms of $(-d\omega/dk)$. In fact, this latter expression develops an imaginary part comparable to its real part. However, if we wish, we could still define a modified group velocity by the relation

$$c_g = \mathfrak{P}/\mathfrak{A} . \tag{75}$$

[5] These numbers from Shu's original expressions were kindly made available to us by Dr. S. Feldman.

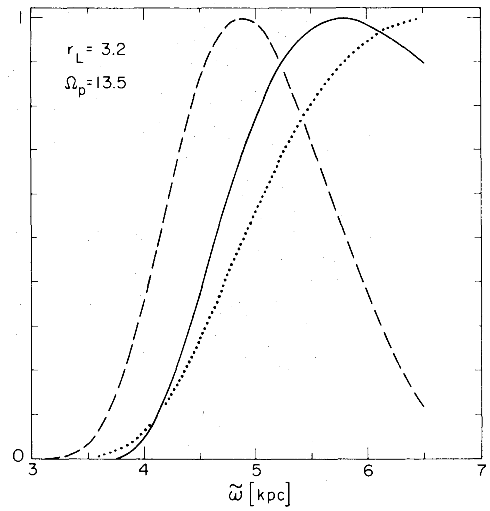

FIG. 2.—For short trailing waves at the inner resonance region of our Galaxy, we have plotted the negative of the wave action density $(-\mathfrak{A})$ (*solid curve*) versus radial distance from galactic center. Also given are the flux $\mathfrak{P}$ (*dotted curve*) and the source density $\mathfrak{S}$ (*dashed curve*) of wave action. The vertical scale is in arbitrary units (more accurate detailed numbers are given in table 3). For this inward propagating wave, the peak in the magnitude of the action density is due to both the inward decrease of the flux and the inward increase of the source. The source density also has a peak due to the competition between the outward decreasing influence of the resonant stars as against the inward decreasing amplitude of the wave. Since wave angular momentum and energy are proportional to wave action, their corresponding densities, fluxes, and sources are also represented by this diagram.

In general, the sense of group propagation as obtained from this formula applied to the Lindblad resonance regions is the same as that obtained using the kinematic definition $(-\partial\omega_R/\partial k_0)_\varpi$ for the same wave far away from the resonances.

The source density $\mathfrak{S}(\varpi, t)$ is a new term which was not present in the previous work of Toomre (1969) and Shu (1970*b*). LBK calculated the integrated source in a physical interpretation of the wave-particle interactions at resonance. Under some approximations discussed in § IV*b* we were able to derive a source density using the methods of LBK. This agreed with ours to lowest order. It is possible that at higher orders we have some additional effects. As predicted by LBK, the sign of the source density at the Lindblad resonances is always such that it damps the wave. The manner in which it occurs in detail is illustrated in figure 2 for the short trailing wave at the inner Lindblad resonance. This wave carries negative action density. The solid curve in this figure is $(-\mathfrak{A})$. The flux $\mathfrak{P}$ (*dotted curve*) and the source density $\mathfrak{S}$ (*dashed curve*) are both nonnegative. The positive flux indicates that the negative action of the wave is carried inward in accordance with a negative group velocity if definition (75) is used. As the wave propagates inward, the magnitude of the action density increases because the flux decreases. But eventually the source density becomes of sufficient magnitude that the wave action decreases rapidly in amplitude inward. Thus a peak in $|\mathfrak{A}|$ occurs at about 5.7 kpc from galactic center. We may note from equations (66)–(70) that all the quantities $\mathfrak{A}$, $\mathfrak{P}$, and $\mathfrak{S}$ are proportional to the square of the wave amplitude. These all become very small at $\varpi = r_L$ because the wave amplitude is small. Thus the near vanishing of $\mathfrak{S}$ at $\varpi = r_L$ is not due to any intrinsic inability to absorb the waves at this radius.

In our figure 2, we have used $\Omega_p = 13.5$ km s^{-1} kpc^{-1}, $r_L = 3.2$ kpc, and $\beta = 2.35$ (cf. eq. [74]). These latter values are in accordance with the thickness correction first given by Shu (1968). Other possible corrections and a discussion of possible values of β have been given recently by Toomre (1973).

c) Density-Wave Amplitude and Distribution of Ionized Hydrogen

Using the parameters mentioned in the preceding paragraph, the solid curve in figure 3 represents the product of the surface mass density $S(\varpi)$ of the wave multiplied by the geometrical factor $|\text{Re}\,(k)|$. This curve has a peak which correlates very well with the location of a similar peak in the observed ionized hydrogen concentration—in line with the proposal of Lin and Feldman (1970). The exact position of this peak was not known to them. In the theory of Lin *et al.* (cf. Lin 1970), star formation results from gas compression by the density wave. It is thus reasonable to assume that the peak in wave amplitude should be associated with a corresponding peak in the density of ionized hydrogen produced by the young stars. In plotting the wave density amplitude in figure 3, the additional factor $|\text{Re}\,(k)|$ was first introduced by Lin and Feldman to account for the crowding of wave crests. As in their work, the observational curve in this figure is the ratio of the densities of ionized to neutral hydrogen. It is necessary to take the ratio in order to remove the effects of H I concentration. Mezger's (1969) number density of H II regions is used for the ionized hydrogen (H II), while the neutral hydrogen (H I) density is that obtained

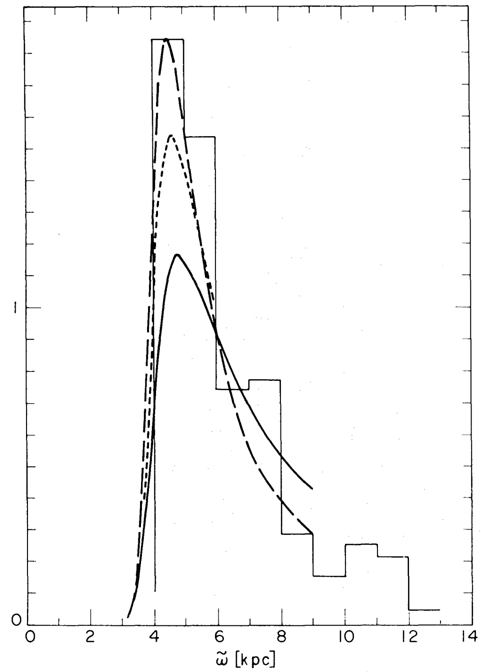

FIG. 3.—The plot of observed H II/H I gas density ratio versus radial distance is compared with three theoretical curves of $S \cdot |\text{Re}\,(k)|$. The solid curve is obtained from a disk model with pattern speed $\Omega_p = 13.5$ km s^{-1} kpc^{-1} and radial stellar dispersive velocity $c(\varpi)$ corrected for thickness. The other two curves differ from the solid one only in the value of $c(\varpi)$ [see text].

by Van Woerden (cf. Oort 1965). The sharp decline in the number of H II regions[6] inside of 4 kpc is a good indication that resonance damping is still operative in our Galaxy. The current theoretical estimate (Contopoulos 1970) of the saturation time is not complete, because it calculates only the response of a few stars under an imposed field. We have reasons to believe that this method gives only a lower bound to the saturation time.

We have presumed that the spiral wave strongly compresses the gas which condenses into stars. Some of the stars in turn create H II regions. Because the relation between the various quantities is most probably nonlinear,[7] we should not expect the strengths of the theoretical and observed peaks in figure 3 to match exactly. In our present incomplete knowledge concerning the details of this chain of events, it may well be useful to explore how much change can be made on the height of the theoretical peak by reasonable adjustments of the parameters available to us. If we assume that Q can be less than unity in some regions, the broken curve in figure 3 has a monotonically changing $Q(\varpi)$ which for $\varpi \ll 6$ kpc has $Q \simeq 0.86$ and for $\varpi \gg 6$ kpc has $Q \simeq 1.14$. In contrast, the dotted curve has $Q = 0.86$. Lower values of Q simulate one possible effect of a galactic halo.

d) *Distinction between Leading and Trailing Spirals*

Shu (1970*a*) has suggested that the resonances distinguish between leading and trailing spirals, but he did not give the actual differences between these senses of windings. In § V*b* we have seen that even near the Lindblad resonances the sense of winding determines the group velocity, and that the waves decay spatially in the direction of this velocity. Since the leading and trailing waves have opposite directions of group propagation, they also decay spatially in opposite directions (at the same resonance). This difference characterizes the difference in behavior of leading and trailing waves at the Lindblad resonances.

In many of the proposed mechanisms for maintenance of spiral density waves, such a difference is already sufficient to suppress the short leading waves relative to the trailing ones. For example, in Lin's (1969) mechanism the waves are initiated in the outer regions near the corotation resonance; the short waves that propagate inward are trailing. Subsequent nonlinear evolution results in a barlike distortion of the inner regions. If an inner resonance exists, then the bar cannot drive the outward-propagating leading waves because these waves are immediately absorbed in the inner resonance region before reaching the propagation region. On the other hand, the short trailing waves propagate inward through the propagation region (which covers most of the galaxy) before they are finally absorbed at the inner resonance.

[6] In Mezger's original presentation of the data, there were five giant H II regions inside or near 4 kpc. Four of these may well be associated with the center of the galaxy. The fifth is in the 3-kpc arm (cf. Lin 1970).

[7] In our discussions with Dr. F. H. Shu, one plausible tentative conclusion is that the H II/H I density ratio is proportional to $[S \cdot \text{Re}\,(k)/\Omega^2\varpi]^2$ near the inner resonance.

The author wishes to thank Jesus the Messiah for his grace and help. He is grateful to Drs. C. C. Lin, A. Toomre, F. H. Shu, and Y. Y. Lau for discussions and encouragement. He is also thankful to Drs. D. Baldwin and R. Kulsrud for discussions on the possibility of "nonlocal reflection." In the initial stages of these calculations, he was greatly helped by discussing certain analogous plasma problems with Drs. J. Callen and B. Coppi. This work is supported by the National Science Foundation under grant GP-34323X1.

APPENDIX A

CALCULATION OF THE SURFACE DENSITY RESPONSE

According to equations (18), (25), and (26), we may write the nonresonant parts of the surface density response as

$$S^{(c)}(\varpi) = \sum_{n \neq n_L} S^{(n)}(\varpi) = \sum_{n \neq n_L} \mathfrak{L}\left[\frac{n(-1)^{n+1}}{\nu(r) - n} \mathfrak{Z}(n, \varpi)\right], \tag{A1}$$

where $\mathfrak{Z}(n, \varpi)$ is defined in equation (22). The $S^{(n)}(\varpi)$ involve integrals over ξ, η, and s. The parameter r depends implicitly on η. To two orders in the epicyclic approximation (17),

$$r = \varpi(1 - \eta), \tag{A2}$$

where $\eta = O(\varepsilon)$. Following Shu (1970*b*), we may expand any functional dependence on r as a Taylor series in $(\varpi\eta)$, keeping only terms up to $O(\varpi\eta)$. Using equations (19), (22), and (A1), we then find that

$$S^{(n)}(\varpi) = \frac{V(\varpi)\Sigma(\varpi)}{[\varepsilon(\varpi)\varpi\kappa(\varpi)]^2} \frac{n(-1)^n}{\nu(\varpi) - n} \left\{\mathfrak{G}^{(n)}(x) + \mathfrak{U}[g^{(n)}] + \langle g^{(n)}|\eta\rangle \frac{d \ln [\nu(\varpi) - n]}{d \ln \varpi}\right\}, \tag{A3}$$

where x is defined in equation (30), and

$$g^{(n)} = \frac{1}{2\pi\varepsilon^2(\varpi)} \exp\left[ins - ik\varpi R_a - \frac{\xi^2 + \eta^2}{2\varepsilon^2(\varpi)}\right], \tag{A4}$$

$$\mathfrak{G}^{(n)}(x) = \langle g^{(n)}|1\rangle = (-1)^n I_n(x) e^{-x}. \tag{A5}$$

Here $I_n(x)$ is a modified Bessel function of the first kind, of index n. The inner product $\langle h|g\rangle$ is

$$\langle h|g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} hg d\xi d\eta ds; \tag{A6}$$

and in terms of it, the linear operator

$$\mathfrak{U}[g] = -\langle g|\eta\rangle \frac{d \ln}{d \ln \varpi}\left[\frac{\langle g|\eta\rangle\Sigma(\varpi)}{\varepsilon^2(\varpi)\kappa^2(\varpi)}\right] + \frac{i2m\Omega(\varpi)}{\kappa(\varpi)}\left\langle g\left|\frac{\partial R_a}{\partial s} + \xi\right.\right\rangle$$
$$+ ik(\varpi)\varpi\left\langle g\left|(\tfrac{1}{2}R_a^2 - \eta R_a)\frac{d \ln k}{d \ln \varpi} - (R_b + \eta R_a)\right.\right\rangle. \tag{A7}$$

If we note that, according to equations (23) and (A4)–(A6), we can show

$$\langle g^{(n)}|\eta\rangle = \frac{ix}{\varpi k(\varpi)} \frac{d}{dx}\mathfrak{G}^{(n)}(x), \qquad \left\langle g^{(n)}\left|\frac{\partial R_a}{\partial s} + \xi\right.\right\rangle = 0, \tag{A8}$$

$$\langle g^{(n)}|(\tfrac{1}{2}R_a^2 - \eta R_a)\rangle = \langle g^{(n)}|(R_b + \eta R_a)\rangle = 0; \tag{A9}$$

then we find

$$\mathfrak{U}[g^{(n)}] = -\frac{ix}{\varpi k(\varpi)} \frac{d\mathfrak{G}^{(n)}(x)}{dx} \frac{d \ln}{d \ln \varpi}\left[\varpi \frac{k(\varpi)}{\kappa^2(\varpi)} \frac{\Sigma(\varpi)}{\nu(\varpi) - n} \frac{d\mathfrak{G}^{(n)}(x)}{dx}\right]. \tag{A10}$$

Using this result and summing equation (A3) over all $n \neq n_L$, we obtain equations (27)–(32). We may note that in equation (A3) the $\mathfrak{G}^{(n)}(x)$ term contributes to $S_0^{(c)}$, and the other terms to $S_1^{(c)}$.

We now evaluate the resonant part of the surface density response $S^{(L)}(\varpi)$ of equation (33). According to the approximation (35) and using equation (A2), we may write

$$\frac{1}{\nu(r) - n_L} = is_\nu \frac{L}{\varpi} \int_{-s_\nu \infty}^{0} \exp\left[-i(\eta - \eta_L)\lambda + \gamma\lambda\right] d\lambda\,, \tag{A11}$$

where

$$\eta_L = 1 - \frac{r_L}{\varpi}\,, \qquad \gamma = -s_\nu \frac{L}{\varpi} \nu_{\rm I}(\varpi)\,. \tag{A12}$$

We assumed $|\nu_{\rm I}(r)| \equiv |\omega_{\rm I}/\kappa(r)| \ll 1$ already, so we need only replace r by ϖ in $\nu_{\rm I}(r)$. Using equation (A11) in relation (33), we may similarly expand functions of r in powers of $(\varpi\eta)$. Up to $O(\varpi\eta)$, we get in analogy to equation (A3) the result

$$S^{(L)}(\varpi) = i \frac{V(\varpi)\Sigma(\varpi)(-1)^{n_L} n_L L}{s_\nu \varepsilon^2(\varpi)\kappa^2(\varpi)\varpi^3} \int_{-s_\nu \infty}^{0} \exp\left[(i\eta_L + \gamma)\lambda\right]\{\mathfrak{G}_L(u) + \mathfrak{U}[g_L]\} d\lambda\,. \tag{A13}$$

In terms of x, y, $g^{(n)}$, and $\mathfrak{G}^{(n)}$ of equations (41), (A4), and (A5), we have

$$u = x + \varepsilon y \lambda\,, \qquad g_L = g^{(n_L)} \exp\left[-i\eta\lambda\right]\,, \tag{A14}$$

$$\mathfrak{G}_L(u) = \langle 1 | g_L \rangle = \mathfrak{G}^{(n_L)}(u) \exp\left[-\tfrac{1}{2}\varepsilon^2(\varpi)\lambda^2\right]\,. \tag{A15}$$

$\mathfrak{U}[g]$ is the same operator defined in equation (A7); and $\mathfrak{U}[g_L]$ can be evaluated if we note from equations (23), (A4)–(A6), (A14), and (A15) that we can show

$$\langle g_L | \eta \rangle = -i\lambda\varepsilon^2(\varpi)\mathfrak{G}_L(u) + iy\varepsilon(\varpi)\mathfrak{G}_L'(u)\,, \tag{A16}$$

$$\left\langle g_L \left| \frac{\partial R_a}{\partial s} + \xi \right.\right\rangle = n_L \varepsilon^2(\varpi) \frac{\lambda}{u} \mathfrak{G}_L(u)\,, \tag{A17}$$

$$\langle g_L | (\tfrac{1}{2} R_a^2 - \eta R_a) \rangle = \frac{1}{2B_2(\varpi)} \langle g_L | (R_b + \eta R_a) \rangle = -\lambda\varepsilon^3(\varpi)(2y + \lambda\varepsilon)[\mathfrak{G}_L'(u) + \tfrac{1}{2}\mathfrak{G}_L''(u)]\,. \tag{A18}$$

The primes on the function $\mathfrak{G}_L(u)$ denote derivatives with respect to u.

The lowest approximation $S_0^{(L)}$ to the resonant density response corresponds to the $\mathfrak{G}_L(u)$ term of equation (A13). It can be put into the form of equation (38) by noting equation (A15), the transformation of the variable of integration,

$$\tau = \varepsilon(\varpi)(\lambda/2^{1/2})\,, \tag{A19}$$

and by replacing $(\eta_L - i\gamma)$ with $(u_L \varepsilon 2^{-1/2})$. This latter replacement almost eliminates approximation (35). It is discussed in more detail at the end of this Appendix. In order to divide $S_1^{(L)}$ into the two terms indicated in equation (39), it is more straightforward to proceed deductively by determining $T(\omega, k, \varpi)$ from equations (39) and (A13). Let us note that $(2\pi G)/(s_k k V)$ times the first term of $S_1^{(L)}$ (eq. [39]) is

$$\frac{s_k n_L (-1)^{n_L}}{k\varpi^2} \frac{d}{d\ln\varpi} \left\{ \frac{L s_\nu}{2k_T \varpi \varepsilon} \int_{-s_\nu\infty}^{0} \exp\left[(i\eta_L + \gamma)\lambda\right](2y + \varepsilon\lambda)\mathfrak{G}_L'(u) d\lambda \right\}\cdot \tag{A20}$$

Using this and equations (39), (41), (42), (A4)–(A7), and (A13)–(A19), we obtain, after much algebra, the formula

$$\begin{aligned} T(\omega, k, \varpi) = {} & \frac{2n_L(-1)^{n_L} s_k s_\nu L}{k_T(\varpi)\varpi^2 \varepsilon(\varpi) y(\varpi)} \\ & \times \int_{-s_\nu\infty}^{0} \exp(iu_L\tau) \left\{ \tfrac{1}{2}(2^{1/2}y + \tau)\left[1 - \frac{2^{1/2}\tau}{\varepsilon(\varpi)} \frac{d}{d\ln\varpi}(iu_L\varepsilon 2^{-1/2})\right]\mathfrak{G}_L'(u) + \left[\tau \frac{d\ln k_T}{d\ln\varpi} + \tau^3 \frac{d\ln\varepsilon^2}{d\ln\varpi}\right] \right. \\ & \times [\mathfrak{G}_L(u) + \tfrac{1}{2}\mathfrak{G}_L'(u)] - 2n_L \frac{m\Omega(\varpi)}{\kappa(\varpi)} \frac{\tau}{u} \mathfrak{G}_L(u) - \tau\left[u \frac{d\ln\varepsilon^2}{d\ln\varpi} + (1 + 2B_2)(u + x)\right] \\ & \left. \times [\mathfrak{G}_L'(u) + \tfrac{1}{2}\mathfrak{G}_L''(u)] \right\} d\tau\,. \end{aligned} \tag{A21}$$

In practice, the integrals in this formula are of the same type as that occurring in the function $\mathfrak{F}^{(L)}$ of equation (40). Thus the manipulations that lead to the alternate form (56) of $\mathfrak{F}^{(L)}$ can also be used to evaluate the function T.

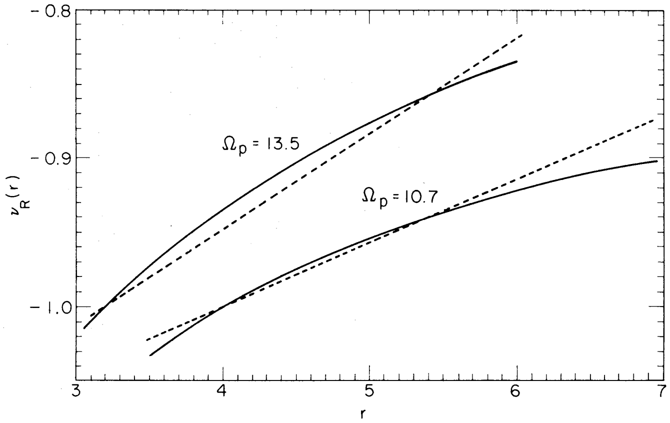

FIG. 4.—The plot of the dimensionless frequency $\nu_R(r)$ (*solid curves*) versus radial distance r for two pattern frequencies Ω_p. The dimensionless frequency $\nu_R = -1$ at the radius of the inner Lindblad resonance. The dashed curves are the corresponding linear approximation (eq. [35]) for each case. This approximation to $\nu_R(r)$ is used only for the region about 1–2 kpc outward of the inner resonance.

Let us also briefly discuss the accuracy of the approximation (35). The function $\nu(r)$ involves the angular frequency of rotation $\Omega(r)$ and the epicyclic frequency $\kappa(r)$ through relation (20). These quantities $\Omega(r)$ and $\kappa(r)$ are observationally not very accurately determined. Thus we deem it sufficient to use approximation (35) so long as it gives $\nu(r)$ to within an error of a few percent. This is possible if we do not force (s_ν/L) to be exactly equal to $(d\nu/dr)_L$, where the subscript L means "evaluated at $r = r_L$." Rather, we allow L to vary slightly to obtain a good fit for $\nu(r)$ in the resonance region in question. For example, consider the inner Lindblad resonance in the Milky Way system, assuming separately two cases where we have waves of pattern speeds $\Omega_p = 10.7$ and 13.5 km s^{-1} kpc^{-1}. The corresponding resonance radii are $r_L = 3.2$ and 4.0 kpc, respectively; and the influence of the inner resonance is felt in the region $3 \leq \varpi \leq 6$ kpc. In figure 4, the solid curves represent the function $\nu_R(r)$ for these pattern speeds (using the 1965 Schmidt model). The dotted lines are from the linear approximation (35) with $L^{-1} = 0.043$ and 0.065 kpc^{-1}, respectively. It is clear that the accuracy of a few percent is maintained. Moreover, in our final formulae (40) and (A21), we replace $(\eta_L - i\gamma)$ (eq. [A12]) by $(u_L \varepsilon 2^{-1/2})$, where u_L is defined in equation (42). For $\varpi \simeq r_L$, these two terms are almost identical. But for $|2^{1/2}L(\nu - n_L)/\varpi\varepsilon| \gg 1$, the use of u_L has the effect of removing the approximation (35) just when it begins to break down. Thus a uniform approximation is obtained so that in fact, far away from the resonances where $|2^{1/2}L(\nu - n_L)/\varpi\varepsilon| \gg 1$, our dispersion relation (50) and action-conservation equation (62) reduce to the corresponding relations of Lin and Shu (1966) and Shu (1970*b*) (cf. §§ III*a* and IV*b* for a demonstration of this point). For exactly the same reasons, equation (A2) seems to be of sufficient accuracy when we evaluate $[\nu(r) - n_L]^{-1}$ in equation (A11). Nor does it seem necessary to include the $O(a^2)$ correction to $\nu(r)$. This latter term is important only at the corotation resonance.

APPENDIX B

REDUCTION OF THE LYNDEN-BELL AND KALNAJS FORMULA FOR THE SOURCE OF WAVE ANGULAR MOMENTUM

Consider the group of resonant stars corresponding to the Lindblad resonance n. For the case $\omega_I \to 0-$, LBK showed that the rate of absorption or emission of angular momentum by these stars is given by (LBK, eq. [30])

$$\frac{dH_n}{dt} = \frac{m}{8\pi} \iint \left[n \frac{\partial F}{\partial J_1} + m \frac{\partial F}{\partial J_2} \right] |\psi_n|^2 \delta(\omega - n\Omega_1 - m\Omega_2) dJ_1 dJ_2 \,, \tag{B1}$$

where $\delta(\lambda)$ is the Dirac delta function with argument λ; J_1, J_2 are the action variables of the equilibrium orbit corresponding to the periodicities in the radial and angular directions, respectively; $\Omega_1(J_1, J_2)$ and $\Omega_2(J_1, J_2)$ are the corresponding frequencies of these periodic motions; $F(J_1, J_2)$ is the equilibrium distribution function; and the ψ_n are defined by

$$\psi_n = \int_0^{2\pi} \int_0^{2\pi} V[\varpi_a(w_1)] \exp [i(nw_1 + mw_2 - m\theta)] dw_1 dw_2 \,, \tag{B2}$$

where (w_1, w_2) are the angle variables corresponding to (J_1, J_2). Formula (B1) gives only the total angular momentum emitted (or absorbed) at a resonance and corresponds to our integrated source $\int_0^\infty \mathfrak{S}(\varpi, t)2\pi\varpi d\varpi$. However, under the reasonable approximations (i)–(iv) listed in § IV*b*, we can obtain from the integrated source (B1) an approximate form for $\mathfrak{S}(\varpi, t)$.

In the relevant epicyclic approximation (cf. § II*a* and LBK)

$$J_1 = \tfrac{1}{2}\kappa(r)r^2a^2\,, \qquad J_2 = \Omega(r)r^2\,, \tag{B3}$$

$$\varpi_a = r(1 - a\cos w_1)\,, \qquad w_2 = \theta\,, \tag{B4}$$

$$\Omega_1 = \kappa(r)\,, \qquad \Omega_2 = \Omega(r)\,. \tag{B5}$$

Using also equation (12), we may now write ψ_n as

$$\psi_n = 2\pi\alpha \int_0^{2\pi} \exp\,[inw_1 + ip(r - ra\cos w_1)]dw_1\,. \tag{B6}$$

Let us first consider the case $n < 0$. It is sufficient to give the derivation for trailing waves. The magnitude of the wave potential $V(\varpi)$ increases rapidly outward, a fact independent of $s_\nu = \mathrm{sgn}\,[d\nu/dr|_L]$ (cf. table 1). Thus the integral in ψ_n is dominated by the saddlepoint $w_1 = \pi$, and it can be evaluated by saddlepoint techniques. Noting equation (13), we get

$$|\psi_n|^2 = \frac{(2\pi)^3}{ra}\frac{|V(r + ra)|^2}{|k(r + ra)|}\,. \tag{B7}$$

In equation (B1), the integrals over J_1 and J_2 can be more conveniently expressed in terms of epicyclic variables. Thus

$$dJ_1dJ_2 = \frac{\kappa^3(r)r^3}{2\Omega(r)}\,adadr\,, \qquad \delta(\omega - n\Omega_1 - m\Omega_2) = \frac{\delta(r - r_L)}{\kappa(r)|d\nu/dr|}\,. \tag{B8}$$

Using the form of F given in equation (7), we also find

$$\frac{\partial F}{\partial J_1} = -\frac{F(r, a)}{\kappa(r)r^2\epsilon^2(r)}\,, \qquad \frac{\partial F}{\partial J_2} = \frac{2\Omega}{\kappa^2 r}\frac{\partial F}{\partial r} + \frac{a^2}{\epsilon^2}\frac{\Omega F}{\kappa^2 r^2}\frac{d\ln(\kappa r^2)}{d\ln r}\,. \tag{B9}$$

Thus clearly $\partial F/\partial J_2$ is much smaller than $\partial F/\partial J_1$ by $O(\epsilon^2)$ so that we need only keep $\partial F/\partial J_1$ in equation (B1). Using equations (7), (35), (B7), (B8), and (B9), we may write relation (B1) as

$$\frac{dH_n}{dt} = -\frac{mn\pi L\Sigma(r_L)}{2\kappa_L{}^2 r_L{}^2\epsilon_L{}^4}\int_0^\infty \exp\left[-\frac{a^2}{2\epsilon_L{}^2}\right]\frac{|V(r_L + r_La)|^2}{|k(r_L + r_La)|}\,da\,, \tag{B10}$$

where the subscript L denotes quantities evaluated at $r = r_L$.

If we consider that the quantity $(r_L + r_La)$ is the maximum radial excursion of a star which has epicyclic amplitude a, then approximation (i) of § IV*b* allows us to identify $(r_L + r_La)$ with ϖ, the radius at which that star contributes a fraction of its angular momentum to the wave. In that case we can write, correct to the lowest epicyclic order,

$$\frac{dH_n}{dt} = \int_{r_L}^\infty \mathfrak{S}_H(\varpi)2\pi\varpi d\varpi\,, \tag{B11}$$

where

$$\mathfrak{S}_H(\varpi) = -\frac{Lmn\Sigma(r_L)}{4\kappa_L{}^2(\epsilon_Lr_L)^4}\frac{|V(\varpi)|^2}{|k(\varpi)|}\exp\left[-\frac{1}{2}\left(\frac{\varpi - r_L}{r_L\epsilon_L}\right)^2\right]\,. \tag{B12}$$

The identification of $\mathfrak{S}_H(\varpi)$ as the source of wave action depends crucially on approximation (i) of § IV*b*. This approximation is partially justified by the fact that the contributions to the integral ψ_n (eqs. [B2] and [B6]) come mainly from the regions when the stellar orbit is at its maximum radial distance from galactic center. Formula (B12) evaluated for the limit $\omega_I \to 0$ is also approximately valid for our case where $0 \neq |\omega_I| \ll |\omega_R|$. Equation (73) results when we restore the factor $\exp(-2\omega_I t)$.

We may note that equation (B11) is for $\mathrm{sgn}\,(n) < 0$ and suggests that the source term $\mathfrak{S}_H(\varpi)$ contributes only for $\varpi > r_L$. This restriction shows up the limitations of the approximation (i) of § III*b*. However, it is not a very important restriction because the wave amplitude and source density are very small when $\varpi < r_L$. For $\mathrm{sgn}\,(n) > 0$, formula (B12) still holds but the integral in equation (B11) now has the limits $\varpi = 0$ and $\varpi = r_L$.

APPENDIX C

WAVE ANGULAR MOMENTUM AND ENERGY

We shall now obtain formulae for $\mathfrak{H}$ and $\mathfrak{E}$, the densities of wave angular momentum and energy, respectively. In the process we will have given an independent proof of the relations (71) between these two densities and the density of wave action $\mathfrak{A}$.

Consider the wave with real frequency ω_R and complex wavenumber $k_0(\varpi)$ which satisfies the dispersion relation (50). Let us excite this wave by some system of externally driven masses which were slowly turned on at $t \to -\infty$. We assume then that this external mass system has a component of the form

$$\sigma_e(\varpi, \theta, t)\delta(z) = \text{Re}\,\{S_e(\varpi)e^{i(\omega t - m\theta)}\}\delta(z)\,, \tag{C1}$$

where ω is a complex frequency with $\omega_I < 0$, $|\omega_I| \ll |\omega_R|$, and $S_e(\varpi)$ has the same rapid spatial dependence on ϖ as the wave density amplitude $S(\varpi)$ (eq. [10]). Instead of relation (15), Poisson's equation now implies

$$S_e(\varpi) = -S(\varpi) - s_k \frac{V(\varpi)k_0(\varpi)}{2\pi G} = -s_k \frac{V(\varpi)k_0(\varpi)}{2\pi G} D_0(\omega, k_0, \varpi) \cong -i\omega_I s_k \frac{V(\varpi)k_0(\varpi)}{2\pi G\kappa(\varpi)} \frac{\partial}{\partial \nu_R} D_0(\omega_R, k_0, \varpi)\,, \tag{C2}$$

where D_0 is defined in equation (45), and we have kept only the dominant terms in the epicyclic approximation (17). Also note the fact that $D_0(\omega_R, k_0, \varpi) = 0$, and that $|\omega_I| \ll |\omega_R|$.

Noting that the wave exerts on the external masses a torque per unit area of $[-\sigma_e(\varpi, \theta, t)\partial\phi(\varpi, \theta, 0, t)/\partial\theta]$, we may calculate the total angular momentum of the wave as

$$\begin{aligned}\Delta h &= \int_{-\infty}^{t}\int_0^{2\pi}\int_0^{\infty} \sigma_e(\varpi, \theta, t)\frac{\partial}{\partial\theta}\phi(\varpi, \theta, 0, t')\varpi d\varpi d\theta dt' \\ &= \frac{1}{2}\,\text{Re}\left\{\int_{-\infty}^{t}\int_0^{2\pi}\int_0^{\infty} \text{i}mS_e(\varpi)V^*(\varpi)\exp{(-2\omega_I t')}\varpi d\varpi d\theta dt'\right\}.\end{aligned} \tag{C3}$$

Here V^* is the complex conjugate of V. In reducing Δh we have used equations (9), (12), and (C1). If, in addition, we eliminate $S_e(\varpi)$ by using relation (C2) and evaluate the time integral of equation (C3), then we find

$$\Delta h = \int_0^{2\pi}\int_0^{\infty} \mathfrak{H}\varpi d\varpi d\theta\,, \tag{C4}$$

where the density of wave angular momentum is

$$\mathfrak{H} \equiv \text{Re}\left\{-\frac{s_k m k_0(\varpi)}{8\pi G\kappa(\varpi)}|\mathfrak{V}|^2{}_{z=0}\frac{\partial}{\partial\nu_R}D_0(\omega_R, k_0, \varpi)\right\}. \tag{C5}$$

Comparing this $\mathfrak{H}$ with $\mathfrak{A}$ as given by equation (66), we see clearly that $\mathfrak{H} = m\mathfrak{A}$. Strictly speaking, formula (C5) only determines the angular momentum density up to a term which is a divergence of some arbitrary vector field. This freedom also occurs in the work of previous authors (Toomre 1969; Shu 1970*b*; Kalnajs 1971). However, the fact that formula (C5) gives us a conservation relation (62) confirms to us that formula (C5) is the relevant angular-momentum density (cf. § IV*b* for the discussion of this).

Similarly, we can show that the density of wave energy $\mathfrak{E}$ satisfies relation (71) if we note that the work done by the wave field on the external masses is $[-\boldsymbol{j}_e(\varpi, \theta, t)\cdot\nabla\phi(\varpi, \theta, 0, t)]$. Here $\boldsymbol{j}_e$ is the external mass flux; and mass conservation,

$$\frac{\partial\sigma_e}{\partial t} + \nabla\cdot\boldsymbol{j}_e = 0\,, \tag{C6}$$

is assumed.

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Note added in proof.—Although we have not explicitly considered this case, our work also applies to all the resonances that occur in axisymmetric density waves. Of course, the wave pattern rotation frequency Ω_p (eq. [11]) is meaningless when the number of arms $m = 0$. However, our previous analysis can also be used in this $m = 0$ case provided we now define the dimensionless frequency $\nu(r)$ by $\nu(r) = \omega/\kappa(r)$. Results analogous to our earlier discussions of $m \neq 0$ cases can now be obtained for axisymmetric waves at the resonances where $\nu(r)$ equals a nonzero integer (the analogy of the corotation resonance does not occur in this $m = 0$ situation). The same resonance absorption phenomenon occurs. It is again possible to give a physical description of this process in terms of densities, fluxes, and sources of wave action or wave energy. The corresponding quantities for wave angular momentum vanish identically as expected. In the detailed analysis, almost all the equations of this paper are valid when $m = 0$. Most notable among the few exceptions is the use of pattern speed Ω_p as mentioned above. Any occurrence of the product $(m\Omega_p)$ or $[m(\Omega_p - \Omega)]$ should be replaced by the frequency ω_R for axisymmetric waves. The quantity $(\Omega + n_L\kappa/m)$ that occurs in § III*b* should now be replaced by $(n_L\kappa)$.

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