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STELLAR DYNAMICS

ONLY ISOLATING INTEGRALS SHOULD BE USED IN JEANS' THEOREM

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*Summary*

Jeans' theorem that the distribution function of an unrelaxed steady stellar system can be expressed as some (in general multivalued)† function of elementary time-independent integrals of the motion has often been used restrictively. Only obvious integrals such as energy and angular momentum have been considered while there are always five time-independent integrals in all. This paper considers the role of the neglected integrals‡ and shows that certain classes of integrals should indeed be omitted. A criterion is developed for finding some classes of potentials in which further integrals *are* important.

1. *Introduction.*—Several authors (2, 3, 4) have stated that certain integrals are not isolating and have therefore neglected them. We aim to prove that such integrals should not be used in Jeans' theorem (1) and we illustrate the argument by means of the "fifth integral" of the spherical clusters. We show that this integral is non-isolating as was suggested by Kurth (2). We first consider this integral qualitatively.

In a spherical cluster the direction of the angular momentum of a star determines the plane of its orbit. The energy and the magnitude of the angular momentum determine the shape and size of the orbit. The role of the fifth integral is to determine the orbit completely. Its plane, size and shape being already fixed, the only remaining freedom is the orientation of the orbit within its plane. The mathematical specification of this orientation is the last integral. For closed non-circular orbits the orientation is a clear concept but most orbits are not closed. To get a clearer picture in this case we attempt to construct mentally a surface of constant orientation as it occurs in the phase space. To reduce phase space from six dimensions to three we shall only consider a definite angular momentum per unit mass  $\omega$ . We plot  $(r, \phi)$  the space coordinates (in the plane perpendicular to  $\omega$ ) just as if we were describing the space motion alone. However at right angles to this plane of motion we plot the radial momentum which for unit mass is just  $\dot{r}$ . In this space we may now draw the energy surfaces  $\dot{r}^2/2 + \omega^2/2r^2 - \psi(r) = E$ ,  $E < 0$ , where  $\psi(\psi > 0)$  is the gravitational potential. These surfaces are toroids§. At fixed  $r$ ,  $\dot{r}^2$  increases with  $E$ ; thus the toroids

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† The function must be single valued in physical phase space but may take several values in the space of integrals of the motion. See Section 5.

‡ All integrals mentioned in this paper are time-independent.

§ We may write  $\dot{r}^2 = 2(E + \psi) - \omega^2/r^2 \equiv 2\Psi(r)$  say. For a proof that for  $E < 0$ ,  $\omega^2 \neq 0$ ,  $\Psi$  is positive only in a certain range  $r_1 < r < r_2$ , see (2) or (5).

are nested, those belonging to smaller energies lying inside. As we decrease  $E$  we eventually reach the line toroid where  $\dot{r} \equiv 0$ ; this is the circular orbit of the given angular momentum  $\omega$ .

Each orbit lies on an energy toroid. Fixing  $\omega$  and  $E$  determines the orbits except for orientation; thus on any one toroid they are identical except in orientation. Just as in complete phase space no orbits can cross. Each is a spiral lying on its toroid. The pitch of such a spiral may be defined as the angle at the centre swept out between two neighbouring apocentres divided by  $2\pi$ ; this agrees with the primitive idea of pitch as the number of times the spiral goes around the toroid by the short way for each time around by the long way (see Fig. 1). For clarity we have greatly exaggerated the pitches in our diagrams. Actual values lie between one and two (5). The pitch is the same

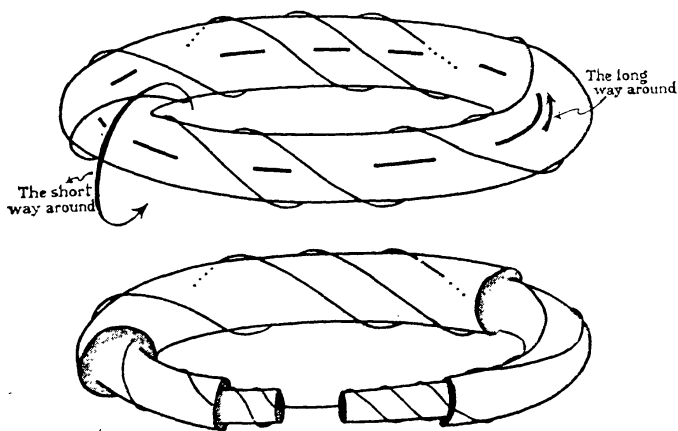


FIG. 1.

for all orbits on one toroid but it varies from toroid to toroid as it is a function of the energy\*. The variation is continuous so almost all our toroids are associated with irrational pitches. Thus their spirals are not closed. Any one such spiral visits the neighbourhood of every point on its energy surface. Such orbits are said to be quasi-ergodic in our reduced phase space. Almost no orbits have rational pitch so we shall neglect the possibility of any star having such an orbit. We now indicate why the distribution function cannot depend on the "orientation". If it did then there must exist  $E, \omega$  which do not belong to the exceptional set of toroids associated with rational pitch, and  $\chi_1, \chi_2, \Delta$  such that

$$f(E, \omega, \chi_1) - f(E, \omega, \chi_2) = \Delta > 0.$$

Here we have used  $\chi$  to label the different spirals on the toroid specified by  $E, \omega$ .  $f$  is the distribution function. Hereafter we talk of  $E, \omega$  as a toroid and  $E, \omega, \chi$  as an orbit, since these numbers completely specify the object concerned. There are points  $P_1$  of  $E, \omega, \chi_1$ , in the neighbourhood of any point of  $E, \omega$ . Choose some point  $P_2$  of  $E, \omega, \chi_2$ . Then  $f$  takes a finite jump  $\Delta$  between  $P_2$  and a point  $P_1$  that can be made arbitrarily close in phase space. Hence  $f$  is discontinuous.

\* When  $\omega$  is not fixed it is a function of both  $E$  and  $\omega$ . The possibility arises that for some potentials the pitch might be rational and independent of  $E, \omega$ . It may be shown that this only occurs for Newtonian  $\alpha'/r$  and simple harmonic  $\alpha r^2$  potentials in both of which cases there are 5 independent isolating integrals.

This contradicts the physical definition of  $f$  as a local average. We therefore deduce that  $f$  must be independent of  $\chi$ . Before leaving this qualitative discussion for a more exact approach it is interesting to consider a surface of constant  $\chi$ . We so define this orientation that all orbits with pericentres at  $\phi=0$  have  $\chi=0$ . Consider the line segment  $\dot{r}=0$ ,  $\dot{\phi}=0$  which lies between  $r=0$  and the line toroid. The surface  $\chi=0$  is generated when we draw all the orbits that intersect this line. We may picture this as follows. Consider the plane  $\phi=\delta\phi$ . Each point of our line segment is moved along its own orbit which remains on its toroid and bends around that toroid (by the short way) by an amount corresponding to the pitch. In this plane our new line segment is curved because of the variation of pitch with energy. If we now look at the plane  $\phi=\alpha$  for increasing  $\alpha$  we find our line segment is wound more and more tightly into a spiral centred on the line toroid. The surface  $\chi=0$  is swept out by this spiral when  $\alpha$  is varied over the complete range  $-\infty$  to  $+\infty$ . Our picture is black throughout showing that  $\chi=0$  is a surface visiting the neighbourhood of all points in our reduced phase space.

2. *Mathematical discussion of the 5th integral.*—Using  $r^2\dot{\phi}=\omega$  we may write the energy equation in the form:

$$\left(\frac{dr}{d\phi}\right)^2 = (2(E+\psi) - \omega^2 r^{-2}) r^4 \omega^{-2}$$

whence

$$\chi = \text{const.}$$

where

$$\chi \equiv \phi - \int_{r_1}^r \frac{\omega dr}{\pm r^2 (2(E+\psi) - \omega^2 r^{-2})^{1/2}} \equiv \phi - K \text{ say}$$

$\chi$  is the integral sought\*.  $r_1$  may be taken arbitrarily; we shall choose it to be the value of  $r$  at pericentre. Kurth (2) has shown that for non-circular, non-rectilinear bound orbits the expression  $\Psi = E + \psi - \omega^2/2r^2$  is positive in a region  $r_1 < r < r_2$ . Also  $\Psi$  has simple zeros at the end points  $r_1, r_2$ . Now the mathematical expression

$$J(r) \equiv \int_{r_1}^r \frac{dr}{\pm (Q(r))^{1/2}} \quad \text{the solution of} \quad \left(\frac{dr}{dJ}\right)^2 = Q(r)$$

is an inverse periodic function, provided  $Q$  is positive in  $r_1 < r < r_2$  with simple zeros at the end points.  $r$  is a periodic function of  $J$  with period

$$2 \int_{r_1}^{r_2} Q^{-1/2} dr$$

$2r^4\omega^{-2}\Psi(r)$  satisfies the conditions on  $Q$  for all bound non-circular non-rectilinear orbits. Thus  $K$  is inverse periodic.

It is therefore multivalued, its values being  $K + 2nP$ , where

$$P = \int_{r_1}^{r_2} (2r^4\omega^{-2}\Psi(r))^{-1/2} dr.$$

\* We have only found this in axes specially oriented for the star in question. For use in Jeans' theorem we would have to convert the integral to cluster coordinates, in which case the mathematical expression is cumbersome. The expression for  $\chi$  has often been written down, e.g. (5), but it has not been discussed as an integral for use in Jeans' theorem.

But  $\phi$  is multivalued with values  $\phi + 2m\pi$ . Thus  $\chi$  is multivalued with values  $\chi - 2nP + 2m\pi$ .  $P/\pi$  is the pitch of the spirals in the previous discussion. If it is irrational, the set of multiple values for  $\chi$  is a dense set. Any orbit must have a certain population independent of our labelling, thus:

$$f(E, \omega, \chi) \equiv f(E, \omega, \chi - 2nP + 2m\pi).$$

We shall now show that if  $f$  is Lebesgue-integrable over almost all the surfaces\* of constant  $E, \omega$ , then it is independent of  $\chi$  almost everywhere (see(7)). We shall consider only bound stars  $E < 0$ , so the surfaces  $E = E_0, \omega = \omega_0$  are all toroids of finite area. Suitable coordinates on the toroids are the  $\phi$  and  $K$  used earlier but with the restrictions  $0 \leq \phi < 2\pi, 0 \leq K < 2P$ .

*Lemma.*—If  $Q, Q'$  are sets of points of positive surface measure on any toroid for which  $P/\pi$  is irrational, then they have a value of  $\chi$  in common. By this we mean that there exist  $R, R'$ , points of  $Q, Q'$  respectively, such that the set of values that  $\chi$  takes at  $R$  is the same as the set it takes at  $R'$ .

*Proof.*—Let the set of all values of  $\phi - K$  attained at points of  $Q$  be called  $A$ . Since  $Q$  has positive surface measure  $A$  is of positive linear measure. Thus, by a theorem on the density of measurable sets, "Given any  $\epsilon > 0$  there exists an interval  $\Delta (\equiv [s - \delta/2, s + \delta/2])$  of length  $\delta(\epsilon)$  such that the average density of  $Q$  in  $\Delta$  is greater than  $1 - \epsilon$ ." That is  $\mu(Q \cap \Delta) > (1 - \epsilon)\mu(\Delta)$  where  $\mu(B)$  is the measure of the set  $B$ .  $Q$  and  $\Delta$  are contained in the interval  $S \equiv [-2P, 2\pi]$ .

The set  $X$  of all values of  $\chi = \phi - K + 2m\pi - 2nP$  (all  $m, n$ ) attained in  $Q$  will be of average density greater than  $1 - \epsilon$  in each of the intervals  $\Delta_m^n$ , where

$$\Delta_m^n \equiv \left[ s + 2m\pi - 2nP - \frac{\delta}{2}, s + 2m\pi - 2nP + \frac{\delta}{2} \right].$$

Since the set  $s + 2m\pi - 2nP$  (all  $m, n$ ) is dense we may place the centre of some set  $\Delta_m^n$  as close as we please to any point. It follows that in all intervals of length  $\delta$ ,  $X$  has average density  $\geq (1 - \epsilon)$ . Whence, dividing off the range  $s$  into intervals of length  $\delta$  and atoning for overlap of 2 intervals at one end of the range, we find  $X$  has average density in  $S \geq 1 - 2\epsilon$ . Since  $\epsilon$  is arbitrarily small  $\mu(X \cap S) = \mu(S)$ . But  $A'$  the set of values of  $\phi - K$  attained in  $Q'$  is contained in  $S$  and has positive measure. Hence  $A'$  and  $X$  have points in common. Thus there are points in  $Q, Q'$  which have at least one value (and hence all multi-values) of  $\chi$  in common. Q.E.D.

Consider any two subdivisions of  $T$  (the toroid surface) into sets of positive measure,  $Q_1, \dots, Q_r, \dots, Q_1', \dots, Q_t'$ . Let  $M_r, (m_r)$  be the upper (lower) bound of the value of  $f$  in  $Q_r$ . Define  $M_t', m_t'$  similarly. Now take  $T$  to be a toroid with irrational pitch  $P/\pi$ . Then by our Lemma  $Q_r, Q_t'$  have values of  $\chi$  in common and hence they have values of  $f$  in common. Thus

$$M_r \geq m_t' \text{ and } M_t' \geq m_r \text{ for all } r, t.$$

Thus  $\sum_t M_r \mu(Q_t') = M_r \sum_t \mu(Q_t') \geq \sum_t m_t' \mu(Q_t')$  or, taking lower bounds over all subdivisions  $Q'$  and assuming  $f$  integrable on  $T$ ,

$$M_r \mu(T) \geq \int_T f d\mu \equiv I \text{ say.}$$

\* Since  $E, \omega$ , both constant, give a differentiable set of surfaces the above condition on  $f$  follows if  $f$  is Lebesgue-integrable in phase space.

Define

$$\bar{f} = \frac{I}{\mu(T)}.$$

Then

$$M_r \geq \bar{f} \quad \text{or} \quad M_r - \bar{f} \geq 0.$$

Similarly

$$m_r \leq \bar{f} \quad \text{or} \quad m_r - \bar{f} \leq 0.$$

Now  $f - \bar{f}$  is clearly integrable over  $T$ . Hence there exists a set of subdivisions  $Q_1 \dots Q_r \dots$  such that

$$\sum_r (M_r - \bar{f}) \mu(Q_r) - \sum_r (m_r - \bar{f}) \mu(Q_r) < \epsilon.$$

Both terms taken with their signs are positive (from above). Thus

$$\epsilon > \sum_r (M_r - \bar{f}) \mu(Q_r) \geq 0 \geq \sum_r (m_r - \bar{f}) \mu(Q_r) > -\epsilon.$$

Now the maximum of  $|f - \bar{f}|$  is either  $M_r - \bar{f}$  or  $-(m_r - \bar{f})$ , whichever is greater. Whichever it is, it is greater than their sum, since both are positive. Thus

$$2\epsilon > \sum_r (\text{Max in } Q_r \{|f - \bar{f}|\}) \mu(Q_r) \geq 0$$

$$\therefore \int_T |f - \bar{f}| d\mu \geq 0$$

since the integral certainly exists when  $f$  is integrable.

But  $\epsilon$  is arbitrarily small. Thus  $\int_T |f - \bar{f}| d\mu = 0$ . Hence  $f$  is the constant  $\bar{f}$  almost everywhere on  $T$ .

Summing up, we have proved that on any toroid  $T$  where  $P/\pi$  is irrational and  $f$  is integrable,  $f$  is constant almost everywhere. Since the two sets of exceptional toroids ( $P/\pi$  rational and  $f$  not integrable) are of measure zero in the set of all toroids,  $f$  is independent of  $\chi$  almost everywhere in phase space.

3. *The general case.*—We use the following definition of a non-isolating integral. Let  $I$  be the integral in question and let  $I, I_1 \dots I_4$  be a complete set of independent integrals. Consider the space  $I_i = a_i \quad i = 1 \dots 4$  where the  $a_i$  are constants.  $I$  is said to isolate in this space if there exists at least one pair of sets  $Q, Q'$  each of positive measure in the space, such that for all  $P \in Q, P' \in Q'$

$$I(P) \neq I(P').$$

If for almost all sets  $(a_1 \dots a_4)$   $I$  does not isolate in the corresponding space,  $I_i = a_i, i = 1 \dots 4$ , then we say  $I$  is non-isolating.

The following theorem is sufficient for our purposes although further demands on the continuity of the  $I_i$  could remove the "almost".

*Theorem.*—If  $f$  is continuous in phase space and  $I$  is non-isolating, then  $f$  is independent of  $I$  almost everywhere.

*Proof.*—Suppose not; then  $f$  is dependent on  $I$  for a set of values of  $(a_1 \dots a_4)$  which are not of zero measure in the 4-dimensional  $a_1 \dots a_4$  space. Then there will exist a  $\Delta > 0$  such that the set  $U$  of values  $(a_1 \dots a_4)$  for which there exist  $a, a^*$ , with

$$f(a_1, \dots, a_4, a) - f(a_1, \dots, a_4, a^*) > \Delta,$$

is not of measure zero. We have written  $f=f(I_1 \dots I_4, I)$ .  $I$  cannot isolate on all the surfaces  $I_i=a_i$ ,  $i=1 \dots 4$ , where  $(a_1, a_2, a_3, a_4) \in U$  since these have non-zero measure. Choose  $(a_1 \dots a_4) \in U$  such that  $I$  does not isolate in  $I_i=a_i$ ,  $i=1 \dots 4$ , and let  $P, P^*$  be points on this surface where  $I=a, I=a^*$  respectively. Since  $f$  is continuous, there is a neighbourhood of  $P$  with positive surface measure within which  $f$  differs from its value at  $P$  by less than  $\Delta/2$ . Similarly for  $P^*$ . Hence  $I$  cannot take the same value at 2 points, one in each of these neighbourhoods. Hence  $I$  isolates on this surface, which contradicts its selection. Thus the theorem is true.

The ergodic hypothesis is that in any sufficiently complicated dynamical system the energy is the only isolating time-independent integral of the motion. The task of proving this conjecture, and of clarifying the vague terms used in the statement of it, is known as the ergodic problem. We are interested in the exceptional cases when the energy is not the only integral for a single particle moving in a potential field. Thus we are faced with the ergodic problem for one particle. We would like to classify all potentials by their isolating integrals. This is not a task to be undertaken lightly.

4. *Ameliorating circumstances for the problem of real interest.*—Consider a class of potentials of the form

$$\psi = \psi(x, y, z, \zeta(\lambda))$$

where  $\lambda = \lambda(x, y, z)$  is a fixed “coordinate” and  $\zeta(\lambda)$  is an arbitrary function.

For a certain  $\zeta$  let time-independent integrals of the equations of motion be  $I_1, I_2, \dots, I_5$ . As we vary  $\zeta$  these integrals will change to some other functions of coordinate and momenta. Thus the integrals are functionals of  $\zeta$ . We say  $I$  is a local integral in the potentials  $\psi(x, y, z, \zeta(\lambda))$  if for all functions  $\zeta(\lambda)$ ,  $I(\mathbf{p}, \mathbf{r}, \zeta(\lambda))$  is an integral of the motion. Thus  $I$  is local if it is a point function of  $\zeta(\lambda)$  rather than the far more general functional\*.

We shall look for all possible local integrals since these are the integrals which maintain their form (as functions of  $\mathbf{p}, \mathbf{r}, \zeta$ ) for a whole class of potentials. We may indeed observe that the Hamiltonian itself, ignorable coordinates, separable coordinates, integrals of Liouville’s type, etc., are all local integrals. We now give intuitive reasons why we should expect all isolating integrals to be local (except for those which only appear when 3 or 4 isolate already).

Let  $H$  be the Hamiltonian. The condition that  $I$  be a constant of the motion is that the Poisson bracket  $[H, I] = 0$ . This is the condition that  $H$  be invariant under the infinitesimal contact transformation generated by  $I$  and that therefore this transformation be a symmetry operation on the Hamiltonian. The nomenclature is somewhat unfortunate as we find ourselves ascribing 5 separate symmetries to an arbitrary Hamiltonian (since it has 5 time-independent integrals of the motion). However if the Hamiltonian possesses isolating integrals besides the energy then it must indeed take a special form, so we shall hereafter use the word symmetry only in connection with *isolating* integrals. Several of the possible symmetries of the Hamiltonian do correspond with symmetries of the system in ordinary space. We take this analogy seriously and postulate that *all* symmetries of the Hamiltonian behave like spatial symmetries.

\* The general form  $I = I(\mathbf{p}, \mathbf{r}, \zeta(\lambda), \zeta'(\lambda), \zeta''(\lambda) \dots \zeta^{(n)}(\lambda))$  was originally used in the definition of local since this is the most general functional which depends on  $\zeta$  only in the neighbourhood of  $\lambda$ . However, this generalization is dull, as it may be proved that all integrals local by such a definition are local in the sense used above.

If we demand of a function in 3 dimensions that it has some particular infinitesimal symmetry operation (e.g. rotation about the  $z$  axis) this does not define the function. It merely demands that it take the form  $f(R, z)$  rather than  $f(x, y, z)$ . This invariance under a symmetry operation reduces the free function from a function of 3 variables to a free function of 2 variables. In our Hamiltonian system the Hamiltonian isolates if the potential is time-independent. We expect that the potential will be free to a function of 2 variables if we demand one other isolating constant of the motion and will be free to a function of one variable if we demand two others. Thus we expect that isolating integrals will maintain their form for a whole class of potentials. The only exceptions will be those isolating integrals which only exist when several others exist too. It may then be that the potential is so shaped by the symmetries required that it has no further functional freedom. This happens for the integral of the line of apses for the central inverse square law of force. However provided one of the other symmetries, necessary for the existence of this one, exists for a whole set of potentials  $\psi(\mathbf{r}, \zeta(\lambda))$  we will still be discussing all the potentials which can have isolating integrals. We may thus make a special search for such awkward exceptions once we have found the other possible integrals.

From the point of view of stellar dynamics an isolated example of a completely defined potential admitting certain integrals is of only passing interest; we would like to consider the equilibrium configurations of stellar dynamics as evolutionary sequences for relaxing stellar systems. The relaxation is slow and thus at any instant the system is near one of the equilibria, but the slow relaxation causes a secular change in this equilibrium. We are thus interested in whole sets of equilibrium states. What is more, if we are looking for a secular series, the distribution function must change slowly along the series. Thus in some sense the gravitational potentials of the members of the series must possess the same isolating integrals. More exactly those dynamical real-symmetries of the Hamiltonian which generate the (constant) isolating integrals of the motion must be preserved along the series. There is little doubt that we are interested in integrals possessed by potentials that are still fairly undetermined since one would expect the shape of a stellar system to be governed at least as much by the Poisson equation as it is by the continuity equation. If we specify our potential very closely before we use the Poisson equation, it is probable that the latter will only be soluble for rather eccentric distribution functions. We try, therefore, to specify our potential as little as possible, consistent with sufficient freedom in the distribution function to fit the observations. It is thus the isolating integrals possessed by all potentials of a given form involving an arbitrary function which are of real interest to us.

5. *Remark on Jeans' theorem.*—In solving the collisionless Boltzmann-Liouville equation by the standard method, what is actually proved is that the gradient of  $f$  is perpendicular to the surfaces on which the integrals  $I_1 \dots I_6$  are constant. Thus  $f=f(I_1 \dots I_6)$ ; but if there is more than one piece to the surface  $I_i=A_i$ ,  $i=1 \dots 6$ , ( $A_i$  constants) then  $f$  may take a different value on each piece (8).

An illustration is provided by the one dimensional problem of particles moving in a gaussian potential well. The only constant of the motion is the energy. The energy surfaces are connected for  $E < 0$  and disconnected for  $E > 0$ . In this latter case the particles on one surface always have  $c > 0$  and those

on the other  $c < 0$ . Hence the sign of the velocity is conserved.  $f$  must have a single value for  $E \leq 0$  but may be double valued  $\pm f$  for  $E > 0$ . Needless to say when there are several isolating integrals it is not merely the energy which distinguishes trapped and untrapped particles but the result still turns on the connectivity of the surface  $I_i = A_i, i = 1 \dots 6$  where  $i$  is the number of isolating integrals.

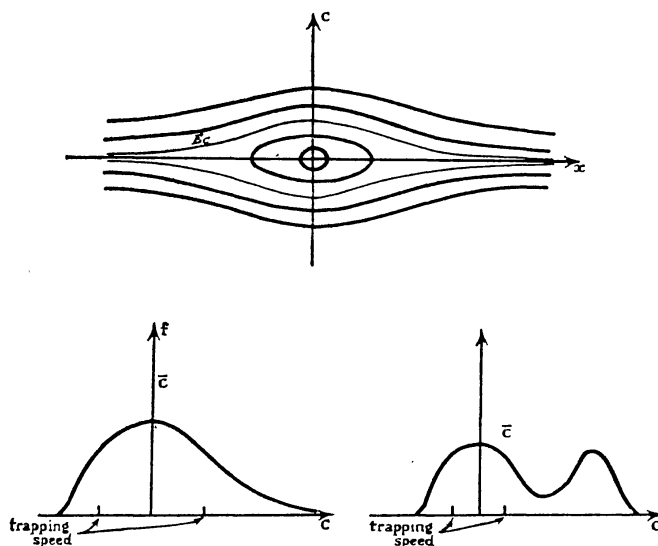


FIG. 2.

*Conclusion.*—The systems of greatest interest for stellar dynamics are the sets of potentials  $\psi(\mathbf{r}, \zeta(\lambda))$  which possess isolating integrals  $I_1 \dots I_i, 1 \leq i \leq 5$ . A large class of these will be local integrals. If the other integrals do not isolate, their neglect is valid when we use Jeans' theorem. Wherever possible this neglect should be fully justified by *showing* that the remaining integrals are non-isolating. For a discussion of the possible existence of isolating but non-local integrals see (3), (10) and (9)\*.

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\* These give power series expansions of unknown convergence for another integral. Even if the series are only asymptotic the first few terms may still give approximate isolating integrals of the motion even where the corresponding exact integral does not isolate. Quasi integrals of this type may well be of great importance in stellar dynamics.



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