# STELLAR DYNAMICS

## Exact Solution of the Self-Gravitation Equation

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*Summary*

A method is given for the discovery of models for unrelaxed, self gravitating, axially-symmetrical, steady-state stellar systems. The main problem solved is to find the distribution function corresponding to an observed density variation. The method is applied in a simple case to give a one-parameter family of such exact models. Finally the " relaxation condition ", that there be no star streaming but only rotation, is imposed. The density of the system then specifies it uniquely. We are thus able to give an exact description of models for rotating flattened globular clusters.

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*Introduction and notation.*—Jean's theorem (1, 2), when applied to axially-symmetrical stellar systems with no further symmetries (3), gives the form of a steady-state distribution function as:

$$f = f(E, \varpi_z)$$

where we use the following notations:
$\mathbf{r} = (x, y, z) = (R, \phi, z) = (r, \theta, \phi)$ are the coordinates in Cartesian, cylindrical and spherical coordinates. In all cases $O_z$ is oriented along the axis of symmetry.
$\mathbf{c} = (u, v, w) = (c_R c_\phi w)$ is the velocity (being coordinates in phase space)
$f\, d^3c\, d^3r$ is the mass of those stars within the phase space box:

$$\mathbf{r},\ \mathbf{r} + d\mathbf{r}$$
$$\mathbf{c},\ \mathbf{c} + d\mathbf{c}.$$

$\psi$ is the gravitational potential.
$\gamma$ is the gravitational constant.
$\rho$ is the gravitational mass density.
$\varpi_z = xv - yu = Rc_\phi$ we shall refer to as the angular momentum.
$E = c^2/2 - \psi$ we shall refer to as the energy.

Note that the mass is omitted from these last two quantities.

For self-gravitating systems the equation

$$\nabla^2 \psi = -4\pi\gamma \int_{\frac{c^2}{2} < \psi} f\left(\frac{c^2}{2} - \psi,\ Rc_\phi\right) d^3c \qquad (1)$$

tells us that the gravity field arises from the stars.

* Received in original form 1961 July 17.

If we knew $f$, then (1) would give us a most unprepossessing equation for $\psi(R, z)$. However the data* of unrelaxed stellar dynamics tell us about $\rho$, rather than $f$. We shall assume that it is possible to obtain the space density $\rho$, and to form from it the potential $\psi$ to which it gives rise. We will thus have

$$\nabla^2\psi = -4\pi\gamma\rho. \tag{2}$$

If instead of expressing $\rho$ as $\rho(R, z)$ we express it as a function of $R^2$, $\psi$, then (1) and (2) give

$$\rho(R^2, \psi) = \int_{\frac{c^2}{2}<\psi} f\left(\frac{c^2}{2} - \psi, Rc_\phi\right) d^3c. \tag{3}$$

From a more theoretical standpoint we may try to find possible models by guessing a reasonable functional form for $\psi(R, z)$ containing several parameters. We may then differentiate it, as in (2), and assure ourselves that it does indeed correspond to a reasonable non-negative distribution of mass. We would then have $\rho(R, z)$, and, by expressing $z$ as $z(R^2, \psi)$, we could obtain $\rho$ as a function of $R^2$, $\psi$. The eventual aim of this type of procedure is to find an exact model containing several parameters which we may choose to fit observed systems.

We thus take $\rho(R^2, \psi)$ as known, either from observation or from theory. The crux of the problem now lies in the solution of (3) regarded as an equation not for $\psi$, but for $f$.

*Method of solution.*—If there is one solution to (3) (for given $\rho(R^2, \psi)$), then there will be many, obtained from each other by reversing the sense of the motion of any of the stars. Such a process is equivalent to the addition to the original solution, $f_0$, of an increment, $\Delta f$, antisymmetrical in $\varpi_z$:

$$\Delta f(E, \varpi_z) = -\Delta f(E, -\varpi_z).$$

$f_1 = f_0 + \Delta f$ has the same density as $f_0$ since

$$\int \Delta f\, d^3c \equiv 0 \quad \text{by symmetry.}$$

$f = \frac{1}{2}\{f_0(E, \varpi_z) + f_0(E, -\varpi_z)\}$ is a solution symmetrical in $\varpi_z$ from which the general solution may be obtained by adding any antisymmetrical bit†. Thus we need only study the symmetrical solutions $f(E, \varpi_z^2)$ which will satisfy

$$\rho(R^2, \psi) = \int_{\frac{c^2}{2}<\psi} f\left(\frac{c^2}{2} - \psi, R^2c_\phi^2\right) d^3c. \tag{4}$$

We write $c^2 = c_\perp^2 + c_\phi^2$, $\dfrac{c_\perp^2}{2} - \psi = X$ and obtain

$$\rho(R^2, \psi) = 4\pi \int_{-\psi}^{0} \int_0^{\sqrt{-2X}} f\left(\frac{c_\phi^2}{2} + X, R^2c_\phi^2\right) dc_\phi\, dX.$$

Differentiating with respect to $\psi$,

$$\frac{\partial\rho}{\partial\psi} = 4\pi \int_0^{\sqrt{2\psi}} f\left(\frac{c_\phi^2}{2} - \psi, R^2c_\phi^2\right) dc_\phi. \tag{5}$$

* Brightness distributions and velocity curves for galaxies are the most that one can normally hope for. $\rho$ can be obtained if mass/light ratios and axial symmetry about an axis of known orientation are assumed.

† This must not be so large that the total $f_2 = f + \Delta f$ becomes negative somewhere.

Write $\frac{c_\phi^2}{2} = Y$, and define a new function

$$g\left(-E, \frac{\varpi_z^2}{2}\right) = \frac{4\pi f(E, \varpi_z^2)}{\sqrt{\varpi_z^2}}.$$

Note that the finding of $g$ solves the problem since

$$f = \frac{1}{4\pi} |\varpi_z| g.$$

Using the above definitions, and multiplying by $\frac{1}{R}$ we have

$$\frac{1}{R}\frac{\partial\rho}{\partial\psi} = \int_0^\psi g(\psi - Y, YR^2)\, dY.$$

Linear integral equations with difference kernals can often be solved by Laplace Transformation. The occurrence of $\psi - Y$ suggests that method here.

$$\int_0^\infty e^{-s\psi}\frac{1}{R}\frac{\partial\rho}{\partial\psi}\, d\psi = \int_0^\infty e^{-s\psi}\int_0^\psi g(\psi - Y, YR^2)\, dY d\psi.$$

We may extend the range of the $Y$ integration from 0 to $\infty$, and define $g(-E, \varpi_{z/2}^2)$ to be zero for $E > 0$. We shall then reverse the order of integration and write the result in the form:

$$\int_0^\infty e^{-s\psi}\frac{1}{R}\frac{\partial\rho}{\partial\psi}\, d\psi = \int_0^\infty e^{-sY}\int_0^\infty g(\psi - Y, YR^2)\, e^{-s(\psi - Y)}\, d\psi\, dY.$$

Since $g$ is zero for $\psi < Y$ we may write this in the form

$$\int_0^\infty e^{-s\psi} R\frac{\partial\rho}{\partial\psi}\, d\psi = \int_0^\infty e^{-\frac{s}{R^2}t}\int_0^\infty e^{-sB} g(B, t)\, dB\, dt,$$

where $B$ has been written for $\psi - Y$ and $t$ for $YR^2$. If we now write $R^2 = \frac{s}{u}$ we obtain (remembering $\rho = \rho(R^2, \psi)$)

$$\int_0^\infty e^{-s\psi}\left(\frac{s}{u}\right)^{1/2}\frac{\partial\rho\left(\frac{s}{u}, \psi\right)}{\partial\psi}\, d\psi = \int_0^\infty e^{-ut}\int_0^\infty e^{-sB} g(B, t)\, dB\, dt.$$

The r.h.s. is the Laplace transform of $g$ with respect to each of its variables so we write it $\tilde{\tilde{g}}(s, u)$ Thus:

$$\left.\begin{aligned} \tilde{\tilde{g}}(s, u) &= \int_0^\infty e^{-s\psi}\left(\frac{s}{u}\right)^{1/2}\frac{\partial\rho\left(\frac{s}{u}, \psi\right)}{\partial\psi}\, d\psi \\ &= \int_0^\infty \frac{s^{3/2}}{u^{1/2}} e^{-s\psi}\rho\left(\frac{s}{u}, \psi\right) d\psi \end{aligned}\right\}. \tag{6}$$

We have to reverse our Laplace transformations to obtain $g$ and hence $f$. In practice, rather than rely on one's own contour integration, it is simpler to use the accumulated brilliance of others compiled in a table of inverse Laplace transformations. With a little familiarity such tables can give immense power even in situations of great complexity*.

* I have found tables of integral transforms Vol. I Bateman project, A. Erdelyi, McGraw-Hill, 1954 very useful.

We now give a particularly simple example to illustrate the method. H. C. Plummer (4), investigating density laws for the globular clusters, found the law $\rho \propto (r^2+a^2)^{-5/2}$ gave good agreement with the observations except possibly at the centre. Eddington came to the same conclusions, and attempted to find a satisfactory theoretical basis for this law. It is not our aim here to justify this law, nor yet to consider the detailed dynamics of clusters in which there is considerable interaction. However, we mention the globular clusters to show that the non-spherical modification of Plummer's law is not merely an academic problem. The potential corresponding to Plummer's law is

$$\psi = A(r^2+a^2)^{-1/2}. \tag{7}$$

We generalise this to a flattened system by trying

$$\psi = A\lambda^{-1/4} \quad \text{where} \quad \lambda = (r^2+a^2)^2 - 2b^2r^2\sin^2\theta \tag{8}$$

($\lambda$ is always positive provided $b^2 < 2a^2$).

Direct calculation of $\rho = -\dfrac{1}{4\pi\gamma}\nabla^2\psi$ gives

$$\rho = \frac{A}{\pi\gamma}\lambda^{-9/4}[(3a^2-2b^2)(r^2+a^2)^2+(4a^2-b^2)b^2r^2\sin^2\theta], \tag{9}$$

which is everywhere positive provided $b^2 < \frac{3}{2}a^2$. Contour maps of two such distributions for different values of $b/a$ are given at the end of this paper. Note that on the axis Plummer's law remains exact. Expressing $\rho$ as $\rho(R^2, \psi)$ we obtain

$$\rho = D\psi^5 + CR^2\psi^9,$$

where $D = (3a^2-2b^2/A^4\pi\gamma)$ and $C = (5b^2(2a^2-b^2)/A^8\pi\gamma)$ are both constants. Forming the expression required in equation (6),

$$\tilde{\tilde{g}}(s,u) = \int_0^\infty e^{-s\psi}\left(\frac{s}{u}\right)^{1/2} \frac{\partial\rho\left(\frac{s}{u},\psi\right)}{\partial\psi}\,d\psi = \left(5\,!\,s^{-5}D + 9\,!\,\frac{s^{-8}}{u}C\right)\left(\frac{s}{u}\right)^{1/2}.$$

If we define $(n+\frac{1}{2})! = \Gamma(n+\frac{3}{2}) = (n+\frac{1}{2})(n-\frac{1}{2})(n-\frac{3}{2})\ldots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$. Then we may write the inverse Laplace transform of $u^{-n-1/2}$ as $t^{n-1/2}((n-\frac{1}{2})!)^{-1}$. Thus performing the inverse transformation on $u$ we have

$$\tilde{g}(s,t) = \pi^{-1/2}(5\,!\,s^{-4\frac{1}{2}}t^{-1/2}D + 9\,!\,2\,s^{-7\frac{1}{2}}t^{1/2}C).$$

Performing the inverse transformation on $s$ we obtain

$$g(B,t) = \pi^{-1/2}\left(\frac{5\,!}{(3\frac{1}{2})!}B^{+3\frac{1}{2}}t^{-1/2}D + \frac{9\,!\,2}{(6\frac{1}{2})!}B^{+6\frac{1}{2}}t^{1/2}C\right)$$

whence

$$f(E,\varpi_z^2) = \frac{1}{4\pi}|\varpi_z|g\left(-E,\frac{\varpi_z^2}{2}\right) \tag{10}$$

$$= F_1(\psi - c^2/2)^{3\frac{1}{2}} + F_2\varpi_z^2(\psi - c^2/2)^{6\frac{1}{2}},$$

where $F_1$ and $F_2$ are the constants $F_1 = \frac{\sqrt{2}}{4\pi^{3/2}}\frac{5!}{3\frac{1}{2}!}D \simeq 0{\cdot}65D$

$$F_2 = \frac{\sqrt{2}}{4\pi^{3/2}}\frac{9!}{6\frac{1}{2}!}C \simeq 12{\cdot}3C.$$

Thus (10) gives the distribution function symmetrical in $\varpi_z$ which corresponds to the potential (8) and the density (9).

The general solution of (3) for $f$ is of course

$$f_2 = f + \Delta f, \qquad (11)$$

where $f$ is given by (10) and $\Delta f(E, \varpi_z) = -\Delta f(E, -\varpi_z)$ is otherwise arbitrary provided

$$f + \Delta f \geqslant 0 \text{ everywhere.} \qquad (12)$$

We now list the properties common to all clusters with distribution functions $f_2$ of the form (11). These quantities are plotted graphically at the end of the paper.

*Potential*: $\psi = A\lambda^{-1/4}$ where $\lambda = (r^2+a^2)^2 - 2b^2r^2\sin^2\theta$ mass $A/\gamma$; radius $a$; "flattening" $b/a$. Note $b/a$ increases as the system becomes flatter.

*Density*:
$$\rho = \frac{A}{\pi\gamma}\lambda^{-9/4}\{(3a^2-2b^2)(r^2+a^2)^2 + (4a^2-b^2)b^2r^2\sin^2\theta\}$$
$$= D\psi^5 + CR^2\psi^9,$$

where
$$D = \frac{3a^2-2b^2}{A^4\pi\gamma} \text{ and } C = \frac{5b^2(2a^2-b^2)}{A^8\pi\gamma} \text{ are constants.}$$

*Distribution function*: $f_2 = f + \Delta f$

where
$$f = F_1(\psi - c^2/2)^{7/2} + F_2R^2c_\phi^{\,2}(\psi - c^2/2)^{13/2}$$
$$F_1 \simeq 0{\cdot}65D, \quad F_2 \simeq 12{\cdot}3C$$

and $\Delta f$ is arbitrary but antisymmetric in $\varpi_z$ and subject to $f + \Delta f \geqslant 0$.

*Velocity dispersion*:
$$\sigma_R^{\,2} = \sigma_z^{\,2} = \psi/6\left[1 - \frac{2}{5}\frac{CR^2\psi^9}{\rho}\right] \qquad (13)$$

$$\frac{1}{\rho}\int c_\rho c_\phi f_2 d^3c = \overline{c_\phi c_\phi} = \sigma_\phi^{\,2} + (\overline{c_\phi})^2 = \psi/6\left[1 + \frac{4}{5}\frac{CR^2\psi^9}{\rho}\right]. \qquad (14)$$

*Mean velocity*:
$$\overline{c_R} = \overline{c_z} = 0$$

$$\overline{c_\phi} = \frac{1}{\rho}\int c_\phi f_z d^3c = \frac{1}{\rho}\int c_\phi \Delta f\, d^3c. \qquad (15)$$

This rotational velocity depends on $\Delta f$ and thus can not be determined without some further knowledge or assumption.

*The virial theorem.*—The cluster whose density distribution obeys Plummer's law is the only spherical system which obeys the virial theorem *locally* in the form: "Mean kinetic energy in any small volume is 1/4 of the energy required to remove that material from the cluster (5)*." Our models also have this property.

$$\tfrac{1}{2}\rho(\sigma_R^2+\sigma_z^2+\sigma_\phi^2+(\overline{c_\phi})^2) = \frac{1}{4}\rho\psi .$$

Integration over the whole cluster gives $T=\tfrac{1}{2}V$, the virial theorem. Eddington found no underlying physical significance to the "local virial theorem". He was probably misled in believing that Plummer's law fitted globular clusters better than modifications of the isothermal gas sphere (9).

*The relaxation condition.*—The cluster with distribution function (10) has $\overline{c_\phi}\equiv 0$ by symmetry and hence $\sigma_\phi^2=\psi/6\ [1+(4/5)(CR^2\psi^9/\rho)]$ whereas we still have $\sigma_R^2=\sigma_z^2=\psi/6[1-(2/5)(CR^2\psi^9/\rho)]$. It is clear that this cluster is not flattened by rotation (since it has none) but by circumferential star streaming (2). This seems unnatural, and it is not clear how such a system could be set up cosmogonically. We wish our clusters to be flattened by their rotation, not by their star streaming. We must formulate what we mean by this. The whole concept of a flattening that a rotating system ought to have is meaningless unless some form of relaxation is assumed. We shall take as our minimal requirement the existence of an isotropic pressure. That is, we shall take $\sigma_R^2=\sigma_\phi^2=\sigma_z^2$. From (13) and (14) this implies

$$(\overline{c_\phi})^2=\frac{\psi}{6}\frac{6}{5}\frac{CR^2\psi^9}{\rho}=\frac{C}{5}\frac{R^2\psi^{10}}{\rho}.$$

*The circular velocity.*—Thus assumption of the relaxation condition leads to the velocity law

$$\overline{c_\phi}=R\psi^5\rho^{-1/2}(C/5)^{1/2}. \tag{16}$$

This virtually completes all that we wish to know about the cluster except for one thing: whether or not this velocity law is possible. We require (16), but we also require (15). Thus we must find a solution $\Delta f$ of

$$\frac{1}{\rho}\int\Delta f c_\phi d^3c=R\psi^5\rho^{-1/2}(C/5)^{1/2} \tag{17}$$

which satisfies the conditions (12).

Multiplying (17) by $R\rho$, we obtain

$$\int_{c^2<2\psi}L(E,\varpi_z^2)d^3c=K(R^2,\psi) \tag{18}$$

where $\varpi_z\Delta f(E,\varpi_z)=L(E,\varpi_z^2)$ (note that the antisymmetry of $\Delta f$ implies the symmetry of $L$), and

$$K(R^2,\psi)=(C/5)^{1/2}R^2\psi^5\sqrt{\rho(R^2,\psi)}=(C/5)^{1/2}R^2\psi^5[D\psi^5+CR^2\psi^9]^{1/2}.$$

* Perhaps more suggestively the R.M.S. velocity at any point is one half of the velocity of escape from that point (assuming stars of equal mass).

Thus $K$ is known and $L$ is unknown. (18) is an integral equation for $L$ of exactly the same form as equation (4). We can thus use our general method to solve for $L$, and hence to determine $\Delta f$ uniquely. We have only to verify that $f+\Delta f \geqslant 0$ everywhere to justify our relaxation condition and our velocity law (16). The rather complicated form of the known function $K$ leads to rather messy mathematics for the determination of $\Delta f$. As this is out of keeping with the body of this paper we give it in the appendix.

*The shapes of the clusters.*—$A/\gamma$ and $a$ may be used as units of mass and length respectively thus the densities (9) have only one true "shape" parameter $b/a$. For $0 \leqslant b^2/a^2 \leqslant (9-\sqrt{33}/8) \simeq 0{\cdot}4065$ the system becomes progressively flatter as $b^2/a^2$ increases and the maximum density is attained at the centre. However for $b^2/a^2 > 0{\cdot}4065$ the maximum density occurs on a ring in the plane of symmetry and this ring increases in diameter as $b^2/a^2$ increases. (This is the continued flattening). At $b^2/a^2 = 3/2$ the density on the axis vanishes and no further values can give real systems as the density becomes negative in places. It seems unlikely that these toroidal systems with $\frac{3}{2} > b^2/a^2 > 0{\cdot}4065$ can be stable. Certainly the similar equilibria of the rotating liquid are not (**7**, **8**). We shall therefore restrict our considerations to the range $0 \leqslant b^2/a^2 \leqslant 0{\cdot}4065$. In particular we draw graphs for the cases $b^2/a^2 = \frac{1}{4}$ and $b^2/a^2 = 0{\cdot}4$.

*Discussion of the general method.*—We have shown that, given a density distribution $\rho$, we can find a distribution function $f(E, \varpi_z^2)$ for any axially symmetrical system. However we have no guarantee that the $f$ so obtained will be positive. If it turns out not to be, then the given density distribution is not that of a steady stellar system. There is however an exception. Densities arising from potentials of Eddington's type (**6**, **3**)

$$\psi = \frac{\zeta(\lambda) - \eta(\mu)}{\lambda - \mu}$$

($\zeta$ and $\eta$ arbitrary, $\lambda$ and $\mu$ spheroidal coordinates) have a third isolating (**2**) integral $I_3$. That is a third independent of $E$ and $\varpi_z$. For these systems Jean's theorem is not so restrictive and we may have $f = f(E, \varpi_z, I_3)$. It seems very likely that the equation

$$\left(\frac{-1}{4\pi\gamma}\nabla^2\psi\right) = \rho(R^2, \psi) = \int f(E, \varpi_z^2, I_3)\, d^3c$$

where $\rho$ is given will have an infinity of symmetrical solutions for $f$. This is because we have shown there is a solution even when we restrict $f$ to be independent of $I_3$. This equation has no solutions* when the ellipsoidal hypothesis is assumed (**6**). Without this assumption we find on the contrary that there are an infinity of self-gravitating systems of Eddington's type for any one Eddingtonian density distribution—a somewhat ironical result.

We note that in the absence of these Eddingtonian systems we can find out more about the distribution function of a flattened system than we can for a spherical one. It is necessary to get an almost "edge on" system with known light distribution and known tilt ($< 10°$ from the line of sight say). For the problem to be of interest the velocity dispersions must not be small; thus, unlike most spirals, the system must not be predominantly flat. If, further, we find a

* Except for solutions independent of $I_3$ and for spherical systems.

dense system, in which star–star interactions are not negligible then it is reasonable to assume the relaxation condition. Knowledge of the density distribution then determines the distribution function uniquely except that one further observation is needed to determine the sense of the rotation.

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## APPENDIX

We have to solve (18) for $L$, to find $\Delta f$, and finally to show that $f+\Delta f \geqslant 0$. $K$ can be written $(D/\sqrt{5})\,\alpha^{1/2}R^2\psi^{7\frac{1}{2}}(1+\alpha R^2\psi^4)^{1/2}$ where $\alpha = C/D$. We would like to expand $K$ as an infinite series since we are unable to perform the operations as it stands. We therefore investigate the range of $\alpha R^2\psi^4$ in the hope that it will always remain $<1$.

$R^2\psi^4$ achieves its maximum on $z=0$ and where $R=a$ as may be found from the formula $\psi^4 = A^4((R^2+z^2+a^2)^2-2b^2R^2)^{-1}$. Its maximum is $\dfrac{A^4}{2(2a^2-b^2)}$. Thus using the definitions of $C$ and $D$ we find $\alpha R^2\psi^4 \leqslant \dfrac{5b^2}{2(3a^2-2b^2)} < 1$ provided $b^2/a^2 < 2/3$. In particular for $b^2/a^2 = 0{\cdot}4$, the greatest value of interest to us, we have $\alpha R^2\psi^4 \leqslant \dfrac{1}{2{\cdot}2}$. It is thus safe to expand $K$ which gives (18) as

$$\int_{c^2/2<\psi,} L(c^2/2-\psi,\, R^2c_\phi{}^2)\,d^3c = qR^2\psi^{7\frac{1}{2}}\left(1+\sum_1^\infty \frac{\frac{1}{2}\,.\,-\frac{1}{2}\,.\,-\frac{3}{2}\,.\,\ldots(\frac{3}{2}-n)}{n!}(\alpha R^2\psi^4)^n\right)$$

where $q = D\sqrt{(\alpha/5)}$.

The series on the right is absolutely and uniformly convergent over the range of interest. Defining a new function

$$G\left(-E, \frac{\varpi_z{}^2}{2}\right) = \frac{4\pi L(E, \varpi_z{}^2)}{\sqrt{\varpi_z{}^2}}$$

our general method tells us that, as in (6),

$$\hat{\hat{G}}(s,u) = \frac{s^{3/2}}{u^{1/2}}\int_0^\infty K(s/u,\psi)\,e^{-s\psi}\,d\psi.$$

That is

$$\hat{\hat{G}}(s,u) = q\sqrt{\pi}\left[\begin{array}{l} s^{-6}u^{-3/2}(\frac{1}{2}\,.\,\frac{3}{2}\,.\,\ldots 7\frac{1}{2}) + \\ +\sum_1^\infty \frac{\frac{1}{2}\,.\,-\frac{1}{2}\,.\,-\frac{3}{2}\,.\,\ldots(\frac{3}{2}-n)}{n!}\,\frac{1}{2}\,.\,\frac{3}{2}\,.\,\ldots(4(n+2)-\frac{1}{2})\,.\,\frac{\alpha^n s^{-3(n+2)}}{u^{n+3/2}} \end{array}\right].$$

If we invert the Laplace transformations term by term in both $u$ and $s$ we obtain

$$G(B,t)=qt^{1/2}\left(\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\ \ldots\frac{15}{2}}{5\,!}B^5 + \sum_1^\infty \frac{(\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}\cdot\ \ldots(\frac{3}{2}-n))(\frac{1}{2}\cdot\frac{3}{2}\cdot\ \ldots(4(n+2)-\frac{1}{2}))}{n\,!(\frac{1}{2}\cdot\frac{3}{2}\cdot\ \ldots(n+\frac{1}{2}))(3n+5)\,!}\alpha^n t^n B^{3n+5}\right).$$

From the definitions of $L$ and $G$ we find $\Delta f(E,\varpi_z)=\dfrac{|\varpi_z|}{4\pi\varpi_z}G\left(-E,\dfrac{\varpi_z^2}{2}\right)$.

Thus

$$\Delta f(-B,\varpi_z)=\frac{D}{4\pi\sqrt{5}}B^{7/2}\left(\left(\frac{\alpha}{2}\right)^{1/2}\varpi_z B^{3/2}\right) \times\left[\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\ \ldots\frac{15}{2}}{5\,!}-\sum_1^\infty\frac{\frac{1}{2}\cdot(\frac{1}{2}\cdot\frac{3}{2}\cdot\ \ldots(4(n+2)-\frac{1}{2}))}{n\,!(n^2-\frac{1}{4})(3n+5)\,!}\left(-\frac{\alpha\varpi_z^2B^3}{2}\right)^n\right].$$

*Convergence etc.*—The maximum of $R^2C_\phi^2/2\ (\psi-c^2/2)^3$ occurs where $c^2=c_\phi^2=\psi/2$ and is therefore $27/256\ R^2\psi^4$. But we showed that for $b^2/a^2\leqslant 0{\cdot}4$, $\alpha R^2\psi^4$ achieved a maximum of $\leqslant 1/2{\cdot}2$. Instead of the variable $\frac{1}{2}\alpha\varpi_z^2B^3$ it is sensible to use a variable $p$ whose maximum is 1 or less. We therefore define

$$p=\sqrt{\frac{256}{27}\cdot\frac{2{\cdot}2}{2}\alpha}\ \varpi_z B^{3/2}.$$

We may then write $\Delta f$

$$\Delta f(-B,\varpi_z)=DB^{7/2}\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\ \ldots\frac{15}{2}}{4\pi\sqrt{5}\,5\,!}\left(\frac{27}{256}\frac{1}{2{\cdot}2}\right)^{1/2}p \times\left(1-\frac{\frac{1}{2}\cdot 5\,!}{\frac{3}{2}\cdot\frac{5}{2}\cdot\ \ldots\frac{15}{2}}\sum_1^\infty\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\ \ldots(4(n+2)-\frac{1}{2})}{n\,!(n^2-\frac{1}{4})(3n+5)\,!}\left(\frac{-27}{256}\frac{p^2}{2{\cdot}2}\right)^n\right).$$

By taking the ratio of the $n$th term to the $(n+1)$th term it is now possible to verify that the series is still absolutely and uniformly convergent in the range $p^2\leqslant 1$ and behaves like a geometrical progression with typical term $A(-p^2/2{\cdot}2)^n$. For comparison purposes we write $f$ in the same notation

$$f(-B,\varpi_z^2)=DB^{7/2}\frac{\sqrt{2}\cdot 5\,!}{2\pi^2\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}\left(1+2\frac{9\cdot 8\cdot 7\cdot 6}{\frac{13}{2}\cdot\frac{11}{2}\cdot\frac{9}{2}}\cdot\frac{27}{256}\cdot\frac{p^2}{2{\cdot}2}\right).$$

Calculating the coefficients by slide rule, we find

$$f_2=f+\Delta f=DB^{7/2}(0{\cdot}650+1{\cdot}03p+1{\cdot}17p^2+0{\cdot}316p^3-0{\cdot}58p^5+0{\cdot}013p^7+\ldots).$$

All remaining terms are odd in $p$. They decrease in magnitude and alternate in sign so that their sum is bounded between the $p^5$ term and the $p^7$ term for all $|p|\leqslant 1$. Putting $f_2=DB^{7/2}M(p)$ we find that in the range $|p|\leqslant 1$ $M(p)$ has a minimum near $p=-0{\cdot}55$ where it takes the value $+0{\cdot}37$. (It takes its maximum value of $3{\cdot}11$ at $p=1$.) Thus it remains positive throughout the range.

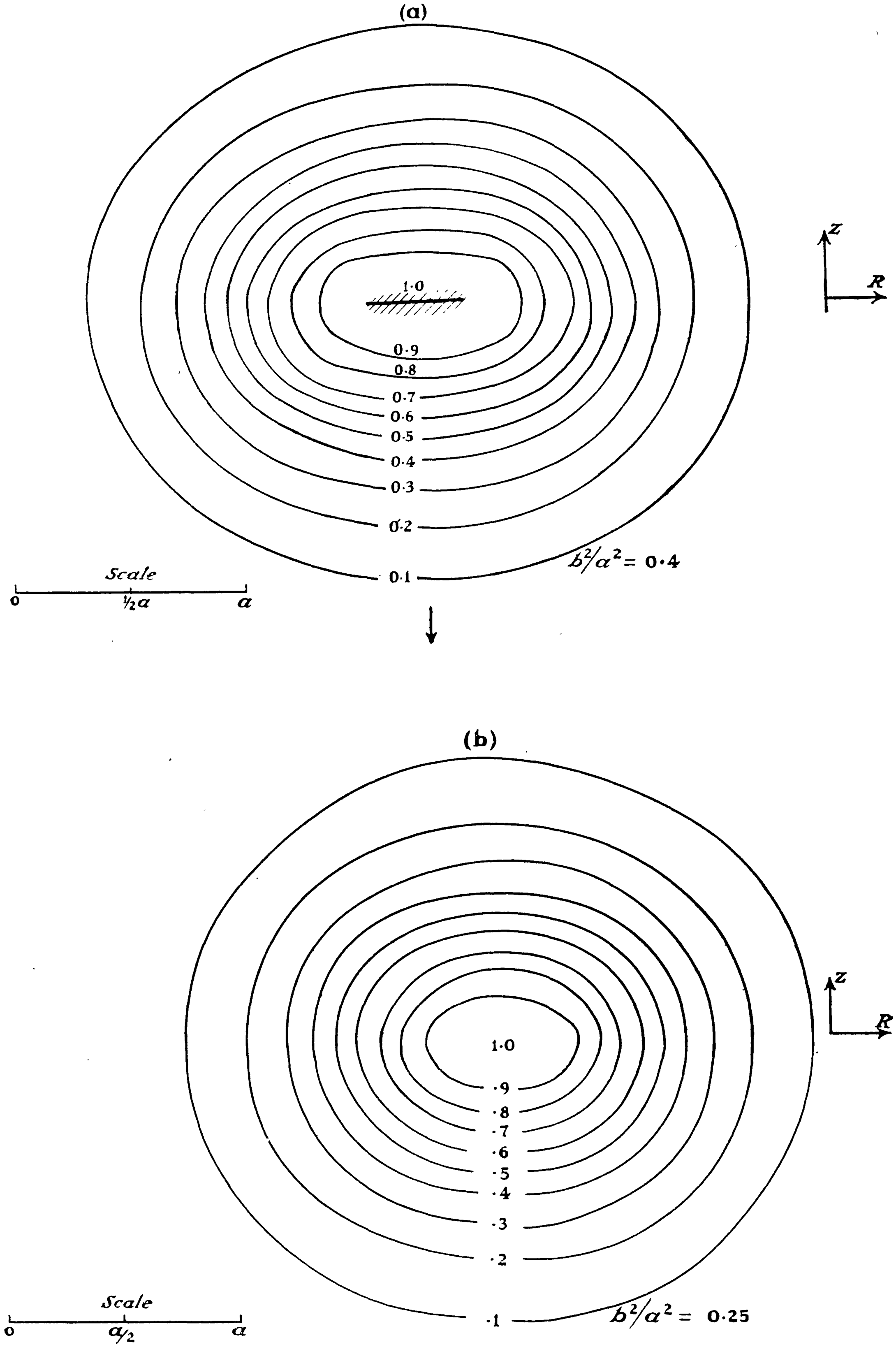


FIG. 1.—*Density (mass per unit volume) in two models for flattened clusters. (Density contours with contour intervals of one-tenth the central density.)*

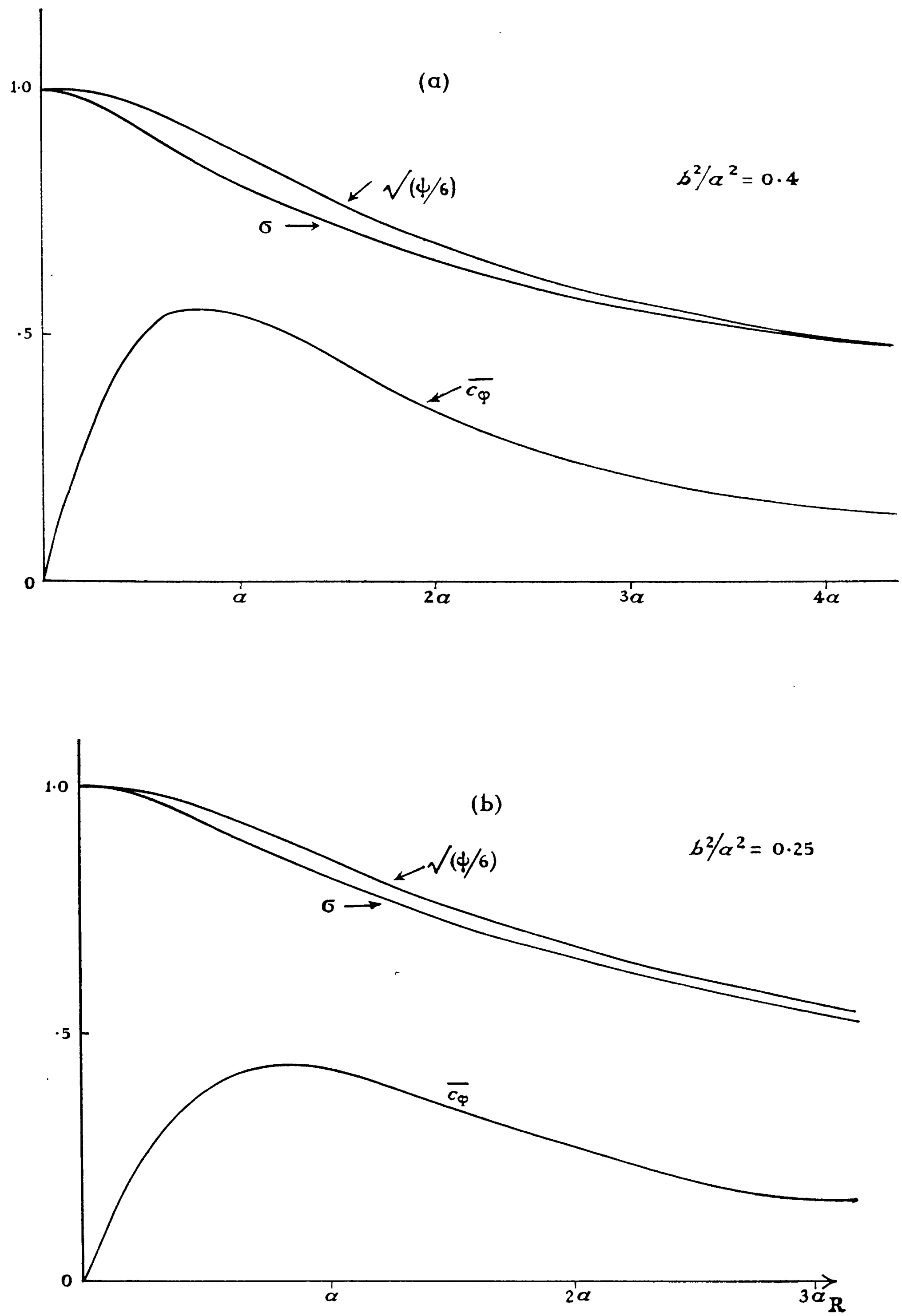


FIG. 2.—*Velocities in the clusters with "flattenings" as indicated, plotted in units of* $(A/6a)^{1/2}$.
$\overline{c_\phi}$, *local mean circumferential velocity on the equatorial plane* $z=0$.
$\sigma=\sigma_R=\sigma_\phi=\sigma_z$, *dispersion of velocities in one of these directions (on* $z=0$).
$\sqrt{(\psi/6)}$ *the value of* $(\sigma^2+(\overline{c_\phi})^2)^{1/2}$ *on* $z=0$ *(a velocity corresponding to the local mean kinetic energy per unit mass).*

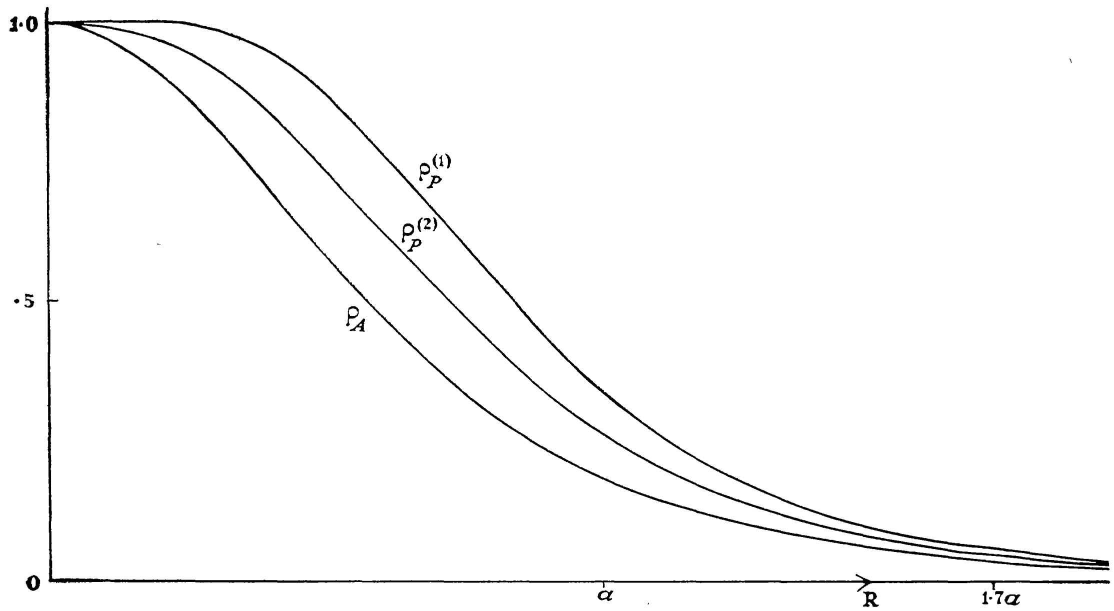


FIG. 3.—*Densities on $z=0$ and $R=0$ respectively, plotted in units of* $(3-2b^2/a^2)A/\pi y a^3$.
$\rho_P{}^{(1)}$, *density on the plane* $z=0$, $b^2/a^2=0\cdot4$.
$\rho_P{}^{(2)}$, *density on the plane* $z=0$, $b^2/a^2=0\cdot25$.
$\rho_A$, *density on the axis (all values of $b/a$) also the density on the plane for $b/a=0$.*

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