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A LAGRANGIAN DERIVATION OF THE ACTION-CONSERVATION THEOREM FOR DENSITY WAVES

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ABSTRACT

A simple, physical derivation of the continuity equation for wave action is obtained from Hamilton's principle and an averaged Lagrangian density. As an example we briefly discuss electrostatic plasma waves, but the primary purpose of the paper is to derive the form of the wave action density for galactic spiral waves in a manner that is both simpler and more easily generalized than that of Shu. The wave action density is expressed in terms of a linear response function that may either be taken from previous work, or found from the single-particle averaged Lagrangian.

I. INTRODUCTION

Toomre (1969) has shown that the higher-order WKB result of Shu (1970) for the propagation of the amplitude of a galactic spiral may be interpreted as a continuity equation for wave action density. This continuity equation for wave action is important in that it gives the variation of wave amplitude with radius, in fact predicting a rapid rise near the inner Lindblad resonance, a prediction which appears to be qualitatively correct (Mark 1971). The fact that waves originating in the outer parts of the galaxy can produce a large disturbance in the inner parts may also have a bearing on the problem of the origin and maintenance of spiral waves. The existence of an action-conservation result is a property shared with many other waves in physics, yet Shu's derivation required rather formidable analysis and did not bring out the general nature of the result, being derived specifically for a Schwarzschild velocity distribution. The primary purpose of this paper is to confirm a suggestion by Lin (1970*a*) (see also Toomre 1969) that this continuity equation might follow naturally from the general method of Whitham (1965), based on the use of an averaged Lagrangian.

In § II we review the Whitham method for linear waves. In § III we apply the method to electrostatic plasma waves in a medium with known dielectric constant, and thereby obtain a generalization of the well-known principle of energy conservation for plasma waves (Stix 1962). In § IV a closely analogous method is applied to the case of density waves on an infinitely thin disk. In an Appendix we also use a more explicit method to derive a specific form for the "dielectric constant" of the disk. We show that this form agrees with that of Lin, Yuan, and Shu (1969).

II. REVIEW OF SMALL-AMPLITUDE AVERAGED LAGRANGIAN THEORY

The wave equation for small disturbances in a nondissipative system can usually be derived from Hamilton's principle

$$\delta \int\int d^3x dt \mathfrak{L}_2 = 0 \,, \tag{1}$$

where $\mathfrak{L}_2$ is the Lagrangian density for linear waves, and is quadratic in the wave amplitude. Whitham (1965) has observed that, in cases where the perturbation is about

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a background state that is slowly varying compared with the period and wavelength of the wave (WKB approximation), equation (1) may be replaced by an asymptotically equivalent equation (this is actually true to all orders, Dewar 1970)

$$\delta \int\int d^3x dt \langle \mathfrak{L}_2 \rangle = 0 , \tag{2}$$

where $\langle \mathfrak{L}_2 \rangle$ is the *averaged Lagrangian* density. The averaging may equivalently be regarded as local space, time, or phase averaging. Representing the wave function Ψ (say) in terms of the WKB *Ansatz*

$$\Psi = a \cos \theta , \tag{3}$$

the averaged Lagrangian quite generally assumes the form (to lowest order in the WKB expansion)

$$\langle \mathfrak{L}_2 \rangle = \tfrac{1}{2} D(\boldsymbol{k}, \omega) a^2 , \tag{4}$$

where $D(\boldsymbol{k}, \omega)$ is a function of the wave vector and frequency, which are defined by

$$\boldsymbol{k} \equiv \nabla\theta , \quad \omega \equiv -\theta_t \equiv -\partial\theta/\partial t . \tag{5}$$

The amplitude a and the phase θ may be varied independently, and the corresponding Euler equations are

$$\partial \langle \mathfrak{L}_2 \rangle / \partial a = 0 \tag{6}$$

and

$$\frac{\partial}{\partial t} \frac{\partial \langle \mathfrak{L}_2 \rangle}{\partial \theta_t} + \nabla \cdot \frac{\partial \langle \mathfrak{L}_2 \rangle}{\partial \nabla\theta} = 0 . \tag{7}$$

Substitution of equation (4) in equation (6) leads to the dispersion relation

$$D(\boldsymbol{k}, \omega) = 0 , \quad \text{which implies} \quad \langle \mathfrak{L}_2 \rangle = 0 . \tag{8}$$

Using equations (5) and (9) and defining the action density N by

$$N \equiv \left| \frac{\partial \langle \mathfrak{L}_2 \rangle}{\partial \omega} \right| , \tag{9}$$

we may write equation (7) as a continuity equation:

$$\frac{\partial N}{\partial t} + \nabla \cdot (\boldsymbol{v}_g N) = 0 , \tag{10}$$

where

$$\boldsymbol{v}_g \equiv -\frac{\partial D/\partial \boldsymbol{k}}{\partial D/\partial \omega} = \frac{\partial \omega}{\partial \boldsymbol{k}} \tag{11}$$

is the group velocity of the wave. Only in the case of a time-independent background state does the canonical wave energy density (Dewar 1970),

$$E \equiv \omega \frac{\partial \langle \mathfrak{L}_2 \rangle}{\partial \omega} = N\omega \, \mathrm{sign} \left(\frac{\partial \langle \mathfrak{L}_2 \rangle}{\partial \omega} \right) , \tag{12}$$

obey a continuity equation also, because ω is constant on a characteristic. This time-independent case occurs frequently, however, and the galactic-spiral problem is a case in point. When E is negative, we speak of a negative energy wave, and this is found to have profound implications for linear stability (when dissipation is included) and also for nonlinear stability when coupling to other waves is considered. These considerations are outside the scope of the present paper, however.

III. ELECTROSTATIC PLASMA WAVES

Regarding the plasma simply as a dispersive, linear dielectric, we may write the perturbed charge density ρ as a linear functional of the electric potential ϕ:

$$\rho(\boldsymbol{x}, t) = \int\int d^3x' dt' \Pi[\boldsymbol{x} - \boldsymbol{x}', t - t'; (\boldsymbol{x} + \boldsymbol{x}')/2, (t + t')/2]\phi(\boldsymbol{x}', t') , \quad (13)$$

where the linear response kernel Π is the "dielectric polarizability." If $\Pi(\Delta \boldsymbol{x}, \Delta t; \bar{\boldsymbol{x}}, \bar{t})$ is an even function of $\Delta \boldsymbol{x}$ and Δt (i.e., unchanged by interchange of $\boldsymbol{x}$, t with $\boldsymbol{x}'$, t'), then Poisson's equation may be derived from the Lagrangian

$$\mathfrak{L}_2 = -\frac{\rho\phi}{2} + \frac{(\nabla\phi)^2}{8\pi} ; \quad (14)$$

and if Π is a slowly varying function of $\bar{\boldsymbol{x}}$ and $\bar{t}$, we may use the averaged Lagrangian method outlined in the previous section.

Defining the frequency- and wavelength-dependent dielectric constant by

$$\epsilon(\boldsymbol{k}, \omega; \boldsymbol{x}, t) = 1 - \frac{4\pi}{k^2}\Pi(\boldsymbol{k}, \omega; \boldsymbol{x}, t) , \quad (15)$$

$$\Pi(\boldsymbol{k}, \omega; \boldsymbol{x}, t) \equiv \int\int d^3\Delta \boldsymbol{x} d\Delta t \Pi(\Delta \boldsymbol{x}, \Delta t; \boldsymbol{x}, t) \exp[-i(\boldsymbol{k}\cdot\Delta \boldsymbol{x} - \omega\Delta t)] , \quad (16)$$

we find the averaged Lagrangian to be

$$\langle\mathfrak{L}_2\rangle = \frac{k^2a^2}{16\pi}\epsilon(\boldsymbol{k}, \omega; \boldsymbol{x}, t) , \quad (17)$$

with a, $\boldsymbol{k}$, and ω being defined by equations (3) and (5) (with $\Psi = \phi$). Thus there is an action-density continuity equation for electrostatic plasma waves, and the action density is given by

$$N = \frac{\langle(\nabla\phi)^2\rangle}{8\pi}\left|\frac{\partial\epsilon}{\partial\omega}\right| . \quad (18)$$

This generalizes the energy-conservation principle (Stix 1962) to cases where the background plasma is time dependent. The requirement that $\Pi(\Delta \boldsymbol{x}, \Delta t)$ be an even function of $\Delta \boldsymbol{x}$ and Δt implies that $\Pi(\boldsymbol{k}, \omega)$ is real, thus eliminating the possibility of damping or growth by other than purely convective means. Thus such phenomena as Landau damping are outside the scope of the averaged Lagrangian method.

IV. GALACTIC SPIRAL WAVES

Let ρ_1 and V_1 be respectively the perturbations in mass density and gravitational potential in a galaxy. Then, assuming a linear response formula analogous to equation (13) to apply, the wave Lagrangian density is

$$\mathfrak{L}_2 = -\frac{\rho_1 V_1}{2} - \frac{(\nabla V_1)^2}{8\pi G} , \quad (19)$$

where G is the gravitational constant. We now make the approximation of an infinitely thin galaxy, and adopt cylindrical polar coordinates (r, θ, z) with origin at the galactic center, axis perpendicular to the plane. Adapting equation (3) to geometry, and taking account of Laplace's equation off the plane (Shu 1970), we may write

$$V_1 = A(r)\cos[\Phi(r, t) + m\theta]\exp(-|kz|) ,$$

$$k(r, t) \equiv \frac{\partial\Phi}{\partial r} , \quad \omega(r, t) = -\frac{\partial\Phi}{\partial t} . \quad (20)$$

The "number of arms" m must be an integer to ensure single-valueness of V_1. By the assumption of linear response, ρ_1 must be given by

$$\rho_1 = \delta(z)\Pi(k, \omega; m, r) V_1 , \tag{21}$$

where the frequency- and wave-vector-dependent linear response coefficient Π will be discussed later. Including an integration over z in the averaging process $\langle\ \rangle$, we find an averaged Lagrangian surface density

$$\langle \mathfrak{L}_2 \rangle = -\frac{\langle(\nabla V_1)^2\rangle}{8\pi G}\epsilon(k, \omega; r) , \tag{22}$$

$$\frac{\langle(\nabla V_1)^2\rangle}{8\pi G} = \frac{|k| A^2}{8\pi G} , \tag{23}$$

$$\epsilon(k, \omega; m, r) = 1 + \frac{2\pi G}{|k|}\Pi(k, \omega; m, r) . \tag{24}$$

In this case the averaged Hamilton's principle, equation (2), may be written

$$\delta \int\int 2\pi r dr dt \langle \mathfrak{L}_2 \rangle = 0 . \tag{25}$$

Varying A still leads to equation (8), so that the dispersion relation is just

$$\epsilon(k, \omega) = 0 .$$

Varying Φ leads to the action-conservation equation

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial r}(v_g N) = 0 , \tag{26}$$

$$N \equiv 2\pi r \frac{\partial \langle \mathfrak{L}_2 \rangle}{\partial \omega} , \tag{27}$$

$$v_g \equiv -\frac{\partial\epsilon/\partial k}{\partial\epsilon/\partial\omega} = \frac{\partial\omega}{\partial k} , \tag{28}$$

the action density now being defined so that $N dr$ is the total wave action between r and $r + dr$.

The linear response function $\Pi(k, \omega)$ has been obtained by Lin *et al.* (1969), who used the epicyclic approximation and a Schwarzschild distribution function

$$\Pi(k, \omega) = -\frac{k^2}{\kappa^2}\frac{[\sigma_* \mathfrak{F}_\nu(x) + \sigma_0 \mathfrak{F}_\nu^{(g)}(x_g)]}{1 - \nu^2} , \tag{29}$$

$$\nu \equiv \frac{\omega - m\Omega}{\kappa} , \tag{30}$$

where $\kappa(r)$ and $\Omega(r)$ are respectively the epicyclic and orbital frequencies of the stellar motions, σ_* and σ_0 are the surface mass density of stars and gas, respectively, and $\mathfrak{F}_\nu(x)$ and $\mathfrak{F}_\nu^{(g)}(x_g)$ are reduction factors modifying the cold-disk results to take into account random motions, x and x_g being defined by

$$x = k^2\langle v_r^2\rangle_{\text{av}}/\kappa^2 \tag{31}$$

$$x_g = k^2 c_s^2/\kappa^2 , \tag{32}$$

where $(\langle v_r^2\rangle_{\text{av}})^{1/2}$ and c_s are the rms radial velocity and equivalent sound speed at radius r. With $\sigma_0 = 0$, equations (27) and (29) lead to a wave-action density agreeing with that obtained by Toomre (1969).

Alternatively we may derive the averaged Lagrangian from first principles, starting (in the case of a purely stellar disk) from the Lagrangian density of Low (1958) (see also Dewar 1972; Galloway and Kim 1971) for a collisionless plasma:

$$\langle \mathfrak{L}_2 \rangle = \int d^2 v f(v_r, v_\theta; r) \bar{L}_2 - \frac{\langle (\nabla V_1)^2 \rangle}{8\pi G}, \tag{33}$$

where $\bar{L}_2$ is that part of the single-particle Lagrangian quadratic in A, time-averaged over part of an unperturbed orbit. In the Appendix it is shown that

$$\bar{L}_2 = -\frac{k^2 A^2}{4} \sum_{n=-\infty}^{\infty} \frac{J_n^2(ka)}{(\omega - m\Omega + n\kappa)^2 - \kappa^2}, \tag{34}$$

where a is the amplitude of the radial excursions, defined by

$$\kappa^2 a^2 = v_r^2 + \left(\frac{\kappa}{2\Omega}\right)^2 v_\theta^2 . \tag{35}$$

Comparing equations (33) and (34) with equations (22)–(24), we find

$$\Pi(k, \omega) = k^2 \sum_{n=-\infty}^{\infty} \frac{\int J_n^2(ka) f(a) d^2 v}{(\omega - m\Omega + n\kappa)^2 - \kappa^2} . \tag{36}$$

For a Schwarzschild distribution

$$f(a) = \frac{\kappa \sigma_*}{4\pi\Omega \langle v_r^2 \rangle_{\text{av}}} \exp\left(-\kappa^2 a^2 / 2\langle v_r^2 \rangle_{\text{av}}\right) \tag{37}$$

we find Kalnajs's (1965) form of the response:

$$\Pi(k, \omega) = \frac{2k^2 \sigma_*}{\kappa^2} \frac{e^{-x}}{x} \sum_{n=1}^{\infty} \frac{n^2 I_n(x)}{\nu^2 - n^2} . \tag{38}$$

Lin, Yuan, and Shu (1969) have shown this form to be equivalent to theirs by use of the Mittag-Leffler theorem in their corrected Appendix C. We may also prove equivalence by summing the series before doing the integration over velocities, using a formula used by Coppi, Rosenbluth, and Sudan (1969):

$$\sum_{n=-\infty}^{\infty} \frac{n}{n - \nu} J_n^2(z) = 1 - \frac{\pi\nu}{\sin(\pi\nu)} J_\nu(z) J_{-\nu}(z) . \tag{39}$$

We obtain a new form for the response function, valid for an arbitrary distribution function:

$$\Pi(k, \omega) = \frac{\pi\nu}{\kappa^2 \sin(\pi\nu)} \int d^2 v \frac{f(a)}{a} \frac{\partial}{\partial a} [J_\nu(ka) J_{-\nu}(ka)] . \tag{40}$$

Using the representation (Erdelyi *et al.* 1954)

$$J_\nu(z) J_{-\nu}(z) = \frac{2}{\pi} \int_0^{\pi/2} J_0(2z \cos\theta) \cos(2\nu\theta) d\theta , \tag{41}$$

and the Schwarzschild distribution equation (37), we may easily prove the equivalence of equations (38) and (29).

V. DISCUSSION

We have shown that the action-conservation equation can be derived from a general physical principle without going through a lengthy higher-order WKB analysis. This automatically generalizes Shu's result to include arbitrary background stellar distribution functions, and also the response of the gas. Generalization to finite disk thickness should also be a fairly trivial matter.

Simple though the method is, it has the limitation of not handling dissipation processes. Away from resonances one may generalize the action conservation equation (10) in an ad hoc manner by replacing 0 on the right-hand side with the term $2\gamma N$, where γ is the imaginary part of the frequency as derived from the local dispersion relation with real k. This prescription is no doubt valid when γ/ω is small, but it breaks down near resonances (Mark 1971).

As it has been shown (Toomre 1969; Shu 1970) that the group velocity of tightly wound spirals is directed toward the center of the galaxy, it appears that these waves must originate in the outer parts of the galaxy, in the neighborhood of the corotation radius. Lin (1970*b*) has suggested that "structural irregularities" due to Jeans instability may excite the waves at this radius. Thus it would appear desirable to have a fuller understanding of propagation in this region than we have at present. In the presence of only mild instability it may transpire that the unstable waves connect coherently with the spiral waves. As particle resonance occurs in this neighborhood (although its effect cancels in the lowest order response), the averaged Lagrangian method is probably not suitable for this part of the problem.

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APPENDIX

We wish to find the time average of the single-particle Lagrangian over several wave, epicyclic, and rotation periods. We shall work within the epicyclic approximation, so we let (r, θ) be the position of the guiding center in its circular motion around the galactic center. Added to these coordinates we have the epicyclic motion $(\xi, \eta/r)$, and the perturbation due to the wave $(r_1, s_1/r)$. Consistently with the tight-winding (WKB) approximation, we take k, ξ, and η to be $O(1)$, r to be $O(\epsilon^{-1})$, where ϵ is the smallness parameter. We do not explicitly order the wave amplitude except to say that it is small compared with a wavelength or epicyclic radius, but not as small as $O(\epsilon)$, since we would otherwise have to correct the epicyclic approximation. One does not anticipate that such corrections would produce any qualitative change in the results, except perhaps near the corotation radius, where the method is not suitable anyway.

Expanding the single-particle Lagrangian in powers of ϵ up to $O(\epsilon^2)$, we have

$$L = \sum_{n=-2}^{2} L_n + O(\epsilon^3) ,$$

$$L_{-2} = \tfrac{1}{2}r^2\Omega^2 - V(r) ,$$

$$\Omega \equiv \dot{\theta} , \tag{A1}$$

$$L_0 = \tfrac{1}{2}(\dot{\xi}^2 + \Omega^2\xi + 4\Omega\xi\dot{\eta} + \dot{\eta}^2) - \tfrac{1}{2}V''(r)\xi^2 , \tag{A2}$$

$$L_2 = \tfrac{1}{2}(\dot{r}_1^2 + \Omega^2 r_1^2 + 4\Omega r_1\dot{s}_1 + s_1^2) - \tfrac{1}{2}V''(r)r_1^2 - r_1\frac{\partial V_1(r+\xi, \theta+\eta/r)}{\partial\xi} . \tag{A3}$$

We have omitted L_1 and L_{-1} as they are linear in one or other of the perturbations, and average to zero (we define ξ, η, r_1 and s_1 to average to zero). The dots denote convective derivatives along the guiding center orbits.

We take as epicyclic trial functions

$$\xi = a \sin \Psi \,, \qquad \eta = b \cos \Psi \,, \tag{A4}$$

and define the epicyclic frequency κ by the derivative of the phase function Ψ (which will later be varied),

$$\kappa \equiv \dot{\Psi} \,. \tag{A5}$$

Inserting equation (A4) into equation (20), we find

$$V_1(r + \xi, \theta + \eta/r) = \mathrm{Re} \left\{ A \exp [i(\Phi + m\theta)] \sum_{n=-\infty}^{\infty} J_n(ka) e^{in\Psi} \right\} + O(\epsilon^2) \,. \tag{A6}$$

For consistency with equation (A6) we must adopt as trial functions

$$r_1 = \mathrm{Re} \left\{ \exp [i(\Phi + m\theta)] \sum_{n=-\infty}^{\infty} r_{1n} e^{in\Psi} \right\} ,$$

$$s_1 = \mathrm{Re} \left\{ \exp [i(\Phi + m\theta)] \sum_{n=-\infty}^{\infty} s_{1n} e^{in\Psi} \right\} . \tag{A7}$$

Averaging over Φ, θ, and Ψ we find

$$\bar{L} = \tfrac{1}{2} r^2 \Omega^2 - V(r) + \tfrac{1}{4} a^2 (\kappa^2 - \kappa_0^2) - \tfrac{1}{4} k^2 A^2 \sum_{n=-\infty}^{\infty} \frac{J_n^2(ka)}{(\omega - m\Omega + n\kappa)^2 - \kappa_0^2} \,,$$

$$\kappa_0^2 \equiv V''(r) + 3\Omega(r)^2 \,, \tag{A8}$$

where we have eliminated b, r_{1n} and s_{1n} by use of their respective Lagrange equations. Variation of a leads to the equation for the epicyclic frequency

$$\kappa^2 = \kappa_0^2 + O(A^2) \,. \tag{A9}$$

The nonlinear correction to the epicyclic frequency is not needed here, so we may take $\kappa = \kappa_0$ and obtain equation (34).

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