Chapter 10

Solitons

Starting in the 19^{th} century, researchers found that certain nonlinear PDEs admit exact solutions in the form of solitary waves, known today as *solitons*. There's a famous story of the Scottish engineer, John Scott Russell, who in 1834 observed a hump-shaped disturbance propagating undiminished down a canal. In 1844, he published this observation[1], writing,

> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

Russell was so taken with this phenomenon that subsequent to his discovery he built a thirty foot wave tank in his garden to reproduce the effect, which was precipitated by an initial sudden displacement of water. Russell found empirically that the velocity obeyed $v \simeq \sqrt{g(h+u_{\rm m})}$, where h is the average depth of the water and $u_{\rm m}$ is the maximum vertical displacement of the wave. He also found that a sufficiently large initial displacement would generate two solitons, and, remarkably, that solitons can pass through one another undisturbed. It was not until 1890 that Korteweg and deVries published a theory of shallow water waves and obtained a mathematical description of Russell's soliton.

[1] J. S. Russell, *Report on Waves*, 14^{th} Meeting of the British Association for the Advancement of Science, pp. 311-390.

Nonlinear PDEs which admit soliton solutions typically contain two important classes of terms which feed off each other to produce the effect:

$$\text{DISPERSION} \quad \rightleftharpoons \quad \text{NONLINEARITY}$$

The effect of dispersion is to spread out pulses, while the effect of nonlinearities is, often, to draw in the disturbances. We saw this in the case of front propagation, where dispersion led to spreading and nonlinearity to steepening.

In the 1970's it was realized that several of these nonlinear PDEs yield entire families of exact solutions, and not just isolated solitons. These families contain solutions with arbitrary numbers of solitons of varying speeds and amplitudes, and undergoing mutual collisions. The three most studied systems have been

- The *Korteweg-deVries equation*,

$$u_t + 6uu_x + u_{xxx} = 0 \ . \tag{10.1}$$

  This is a generic equation for 'long waves' in a dispersive, energy-conserving medium, to lowest order in the nonlinearity.

- The *Sine-Gordon equation*,

$$\phi_{tt} - \phi_{xx} + \sin\phi = 0 \ . \tag{10.2}$$

  The name is a play on the Klein-Gordon equation, $\phi_{tt} - \phi_{xx} + \phi = 0$. Note that the Sine-Gordon equation is periodic under $\phi \to \phi + 2\pi$.

- The *nonlinear Schrödinger equation*,

$$i\psi_t \pm \psi_{xx} + 2|\psi|^2\psi = 0 \ . \tag{10.3}$$

  Here, ψ is a complex scalar field. Depending on the sign of the second term, we denote this equation as either NLS(+) or NLS(−), corresponding to the so-called *focusing* (+) and *defocusing* (−) cases.

Each of these three systems supports soliton solutions, including exact N-soliton solutions, and nonlinear periodic waves.

10.1 The Korteweg-deVries Equation

Let h_0 denote the resting depth of water in a one-dimensional channel, and $y(x,t)$ the vertical displacement of the water's surface. Let L be a typical horizontal scale of the wave. When $|y| \ll h_0$, $h_0^2 \ll L^2$, and $v \approx 0$, the evolution of an x-directed wave is described by the KdV equation,

$$y_t + c_0\, y_x + \frac{3c_0}{2h_0}\, yy_x + \tfrac{1}{6} c_0\, h_0^2\, y_{xxx} = 0 \ , \tag{10.4}$$

where $c_0 = \sqrt{gh_0}$. For small amplitude disturbances, only the first two terms are consequential, and we have

$$y_t + c_0\, y_x \approx 0 \ , \tag{10.5}$$

the solution to which is

$$y(x,t) = f(x - c_0 t) \ , \tag{10.6}$$

where $f(\xi)$ is an *arbitrary* shape; the disturbance propagates with velocity c_0. When the dispersion and nonlinearity are included, only a *particular* pulse shape can propagate in an undistorted manner; this is the soliton.

It is convenient to shift to a moving frame of reference:

$$\tilde{x} = x - c_0 t \qquad , \qquad \tilde{t} = t \ , \tag{10.7}$$

hence

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}} \qquad , \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}} - c_0\, \frac{\partial}{\partial \tilde{x}} \ . \tag{10.8}$$

Thus,

$$y_{\tilde{t}} + c_0\, y_{\tilde{x}} + \frac{3c_0}{2h_0}\, y\, y_{\tilde{x}} + \tfrac{1}{6} c_0\, h_0^2\, y_{\tilde{x}\tilde{x}\tilde{x}} = 0 \ . \tag{10.9}$$

Finally, rescaling position, time, and displacement, we arrive at the KdV equation ,

$$u_t + 6uu_x + u_{xxx} = 0 \ , \tag{10.10}$$

which is a convenient form.

10.1.1 KdV solitons

We seek a solution to the KdV equation of the form $u(x,t) = u(x - Vt)$. Then with $\xi \equiv x - Vt$, we have $\partial_x = \partial_\xi$ and $\partial_t = -V\partial_\xi$ when acting on $u(x,t) = u(\xi)$. Thus, we have

$$-Vu' + 6uu' + u''' = 0 \ . \tag{10.11}$$

Integrating once, we have

$$-Vu + 3u^2 + u'' = A \ , \tag{10.12}$$

where A is a constant. We can integrate once more, obtaining

$$-\tfrac{1}{2}Vu^2 + u^3 + \tfrac{1}{2}(u')^2 = Au + B \ , \tag{10.13}$$

where now both A and B are constants. We assume that u and all its derivatives vanish in the limit $\xi \to \pm\infty$, which entails $A = B = 0$. Thus,

$$\frac{du}{d\xi} = \pm\, u\, \sqrt{V - 2u} \ . \tag{10.14}$$

With the substitution

$$u = \tfrac{1}{2} V \operatorname{sech}^2\theta \ , \tag{10.15}$$

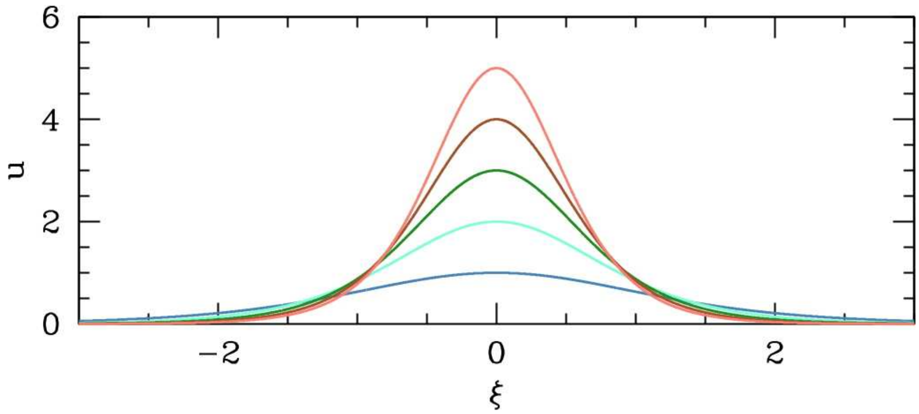

Figure 10.1: Soliton solutions to the KdV equation, with five evenly spaced V values ranging from $V=2$ (blue) to $V=10$ (orange). The greater the speed, the narrower the shape.

we find $d\theta = \mp\frac{1}{2}\sqrt{V}\,d\xi$, hence we have the solution

$$u(x,t) = \tfrac{1}{2}V \operatorname{sech}^2\!\Big(\tfrac{\sqrt{V}}{2}\,(x - Vt - \xi_0)\Big) \ . \tag{10.16}$$

Note that the maximum amplitude of the soliton is $u_{\rm max} = \frac{1}{2}V$, which is proportional to its velocity V. The KdV equation imposes no limitations on V other than $V \geq 0$.

10.1.2 Periodic solutions : soliton trains

If we relax the condition $A = B = 0$, new solutions to the KdV equation arise. Define the cubic

$$\begin{aligned} P(u) &= 2u^3 - Vu^2 - 2Au - 2B \\ &\equiv 2(u-u_1)(u-u_2)(u-u_3) \ , \end{aligned} \tag{10.17}$$

where $u_i = u_i(A,B,V)$. We presume that A, B, and V are such that all three roots $u_{1,2,3}$ are real and nondegenerate. Without further loss of generality, we may then assume $u_1 < u_2 < u_3$. Then

$$\frac{du}{d\xi} = \pm\sqrt{-P(u)} \ . \tag{10.18}$$

Since $P(u) < 0$ for $u_2 < u < u_3$, we conclude $u(\xi)$ must lie within this range. Therefore, we have

$$\begin{aligned} \xi - \xi_0 &= \pm\int\limits_{u_2}^{u} \frac{ds}{\sqrt{-P(s)}} \\ &= \pm\bigg(\frac{2}{u_3-u_1}\bigg)^{1/2} \int\limits_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\!\theta}} \ , \end{aligned} \tag{10.19}$$

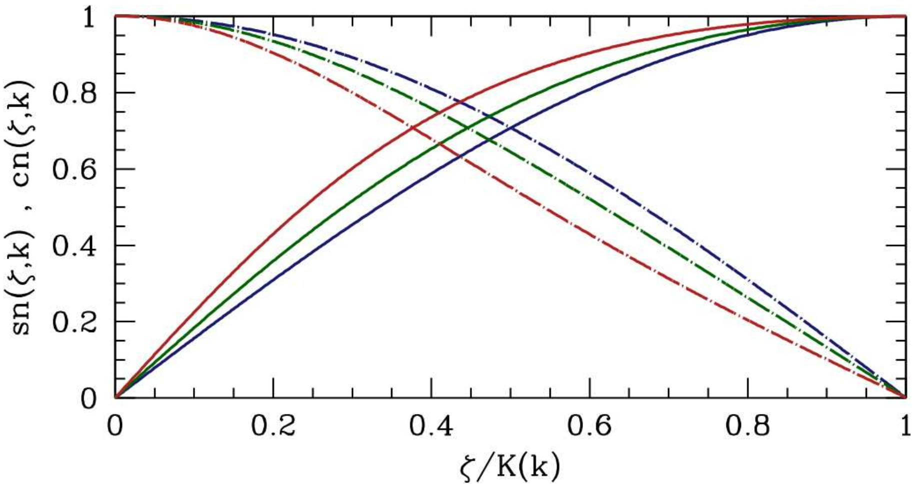

Figure 10.2: The Jacobi elliptic functions $\mathrm{sn}(\zeta, k)$ (solid) and $\mathrm{cn}(\zeta, k)$ (dot-dash) *versus* $\zeta/\mathrm{K}(k)$, for $k = 0$ (blue), $k = \frac{1}{\sqrt{2}}$ (green), and $k = 0.9$ (red).

where

$$u \equiv u_3 - (u_3 - u_2)\,\sin^2\!\phi \tag{10.20}$$

$$k^2 \equiv \frac{u_3 - u_2}{u_3 - u_1}\ . \tag{10.21}$$

The solution for $u(\xi)$ is then

$$u(\xi) = u_3 - (u_3 - u_2)\,\mathrm{sn}^2(\zeta, k)\ , \tag{10.22}$$

where

$$\zeta = \sqrt{\frac{u_3 - u_1}{2}}\,\big(\xi - \xi_0\big) \tag{10.23}$$

and $\mathrm{sn}(\zeta, k)$ is the Jacobi elliptic function.

10.1.3 Interlude: primer on elliptic functions

We assume $0 \le k^2 \le 1$ and we define

$$\zeta(\phi, k) = \int\limits_0^\phi \frac{d\theta}{\sqrt{1 - k^2\sin^2\!\theta}}\ . \tag{10.24}$$

The sn and cn functions are defined by the relations

$$\mathrm{sn}(\zeta, k) = \sin\phi \tag{10.25}$$

$$\mathrm{cn}(\zeta, k) = \cos\phi\ . \tag{10.26}$$

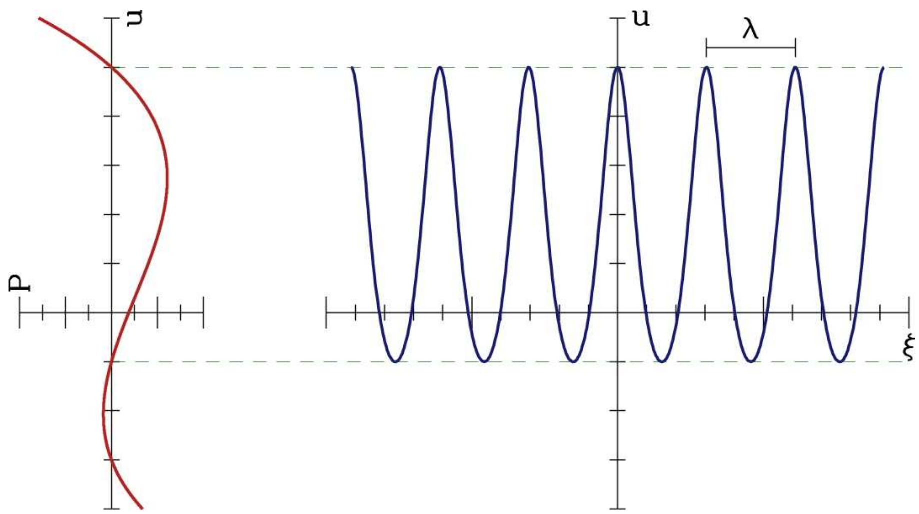

Figure 10.3: The cubic function $P(u)$ (left), and the soliton lattice (right) for the case $u_1 = -1.5$, $u_2 = -0.5$, and $u_3 = 2.5$.

Note that $\mathrm{sn}^2(\zeta, k) + \mathrm{cn}^2(\zeta, k) = 1$. One also defines the function $\mathrm{dn}(\zeta, k)$ from the relation

$$\mathrm{dn}^2(\zeta, k) + k^2\,\mathrm{sn}^2(\zeta, k) = 1 \ . \tag{10.27}$$

When ϕ advances by one period, we have $\Delta\phi = 2\pi$, and therefore $\Delta\zeta = Z$, where

$$Z = \int\limits_0^{2\pi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\!\theta}} = 4\,\mathbb{K}(k) \ , \tag{10.28}$$

where $\mathbb{K}(k)$ is the complete elliptic integral of the first kind. Thus, $\mathrm{sn}(\zeta + Z, k) = \mathrm{sn}(\zeta, k)$, and similarly for the cn function. In fig. 10.2, we sketch the behavior of the elliptic functions over one quarter of a period. Note that for $k = 0$ we have $\mathrm{sn}(\zeta, 0) = \sin\zeta$ and $\mathrm{cn}(\zeta, 0) = \cos\zeta$.

10.1.4 The soliton lattice

Getting back to our solution in eqn. 10.22, we see that the solution describes a *soliton lattice* with a wavelength

$$\lambda = \frac{\sqrt{8}\,\mathrm{K}(k)}{\sqrt{u_3 - u_1}} \ . \tag{10.29}$$

Note that our definition of $P(u)$ entails

$$V = 2(u_1 + u_2 + u_3) \ . \tag{10.30}$$

There is a simple mechanical analogy which merits illumination. Suppose we define

$$W(u) \equiv u^3 - \tfrac{1}{2} V u^2 - A u \ , \tag{10.31}$$

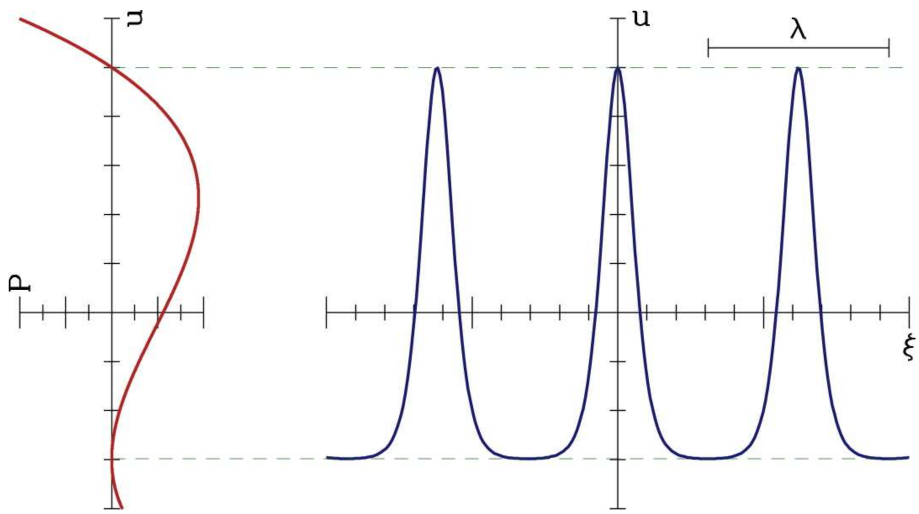

Figure 10.4: The cubic function $P(u)$ (left), and the soliton lattice (right) for the case $u_1 = -1.5$, $u_2 = -1.49$, and $u_3 = 2.50$.

and furthermore $E \equiv B$. Then

$$\frac{1}{2}\left(\frac{du}{d\xi}\right)^2 + W(u) = E \ , \tag{10.32}$$

which takes the form of a one-dimensional Newtonian mechanical system, if we replace $\xi \to t$ and interpret u_ξ as a velocity. The potential is $W(u)$ and the total energy is E. In terms of the polynomial $P(u)$, we have $P = 2(W - E)$. Accordingly, the 'motion' $u(\xi)$ is flattest for the lowest values of u, near $u = u_2$, which is closest to the local maximum of $W(u)$.

Note that specifying $u_{\rm min} = u_2$, $u_{\rm max} = u_3$, and the velocity V specifies all the parameters. Thus, we have a three parameter family of soliton lattice solutions.

10.1.5 N-soliton solutions to KdV

In 1971, Ryogo Hirota[2] showed that exact N-soliton solutions to the KdV equation exist. Here we discuss the Hirota solution, following the discussion in the book by Whitham.

The KdV equation may be written as

$$u_t + \left\{3u^2 + u_{xx}\right\}_x = 0 \ , \tag{10.33}$$

which is in the form of the one-dimensional continuity equation $u_t + j_x = 0$, where the current is $j = 3u^2 + u_{xx}$. Let us define $u = p_x$. Then our continuity equation reads $p_{tx} + j_x = 0$, which can be integrated to yield $p_t + j = C$, where C is a constant. Demanding that u and

[2] R. Hirota, *Phys. Rev. Lett.* **27**, 1192 (1971).

its derivatives vanish at spatial infinity requires $C = 0$. Hence, we have

$$p_t + 3p_x^2 + p_{xxx} = 0 \ . \tag{10.34}$$

Now consider the nonlinear transformation

$$p = 2\,(\ln F)_x = \frac{2F_x}{F} \ . \tag{10.35}$$

We then have

$$p_t = \frac{2F_{xt}}{F} - \frac{2F_x F_t}{F^2} \tag{10.36}$$

$$p_x = \frac{2F_{xx}}{F} - \frac{2F_x^2}{F^2} \tag{10.37}$$

and

$$p_{xx} = \frac{2F_{xxx}}{F} - \frac{6F_x F_{xx}}{F^2} + \frac{4F_x^3}{F^3} \tag{10.38}$$

$$p_{xxx} = \frac{2F_{xxxx}}{F} - \frac{8F_x F_{xxx}}{F^2} - \frac{6F_x x^2}{F^2} + \frac{24F_x^2 F_{xx}}{F^3} - \frac{12F_x^4}{F^4} \ . \tag{10.39}$$

When we add up the combination $p_t + 3p_x^2 + p_{xxx} = 0$, we find, remarkably, that the terms with F^3 and F^4 in the denominator cancel. We are then left with

$$F\big(F_t + F_{xxx}\big)_x - F_x\big(F_t + F_{xxx}\big) + 3\big(F_{xx}^2 - F_x F_{xxx}\big) = 0 \ . \tag{10.40}$$

This equation has the two-parameter family of solutions

$$F(x,t) = 1 + e^{\phi(x,t)} \ . \tag{10.41}$$

where

$$\phi(x,t) = \alpha\,(x - b - \alpha^2\,t) \ , \tag{10.42}$$

with α and b constants. Note that these solutions are all annihilated by the operator $\partial_t + \partial_x^3$, and also by the last term in eqn. 10.40 because of the homogeneity of the derivatives. Converting back to our original field variable $u(x,t)$, we have that these solutions are single solitons:

$$u = p_x = \frac{2(FF_{xx} - F_x^2)}{F^2} = \frac{\alpha^2\,f}{(1+f)^2} = \tfrac{1}{2}\alpha^2\,\text{sech}^2(\tfrac{1}{2}\phi) \ . \tag{10.43}$$

The velocity for these solutions is $V = \alpha^2$.

If eqn. 10.40 were linear, our job would be done, and we could superpose solutions. We will meet up with such a felicitous situation when we discuss the Cole-Hopf transformation for the one-dimensional Burgers' equation. But for KdV the situation is significantly more difficult. We will write

$$F = 1 + F^{(1)} + F^{(2)} + \ldots + F^{(N)} \ , \tag{10.44}$$

with

$$F^{(1)} = f_1 + f_2 + \ldots + f_N \ , \tag{10.45}$$

where

$$f_j(x,t) = e^{\phi_j(x,t)} \tag{10.46}$$

$$\phi_j(x,t) = \alpha_j \, (x - \alpha_j^2 \, t - b_j) \ . \tag{10.47}$$

We may then derive a hierarchy of equations, the first two levels of which are

$$\big(F_t^{(1)} + F_{xxx}^{(1)}\big)_x = 0 \tag{10.48}$$

$$\big(F_t^{(2)} + F_{xxx}^{(2)}\big)_x = -3\big(F_{xx}^{(1)} F_{xx}^{(1)} - F_x^{(1)} F_{xxx}^{(1)}\big) \ . \tag{10.49}$$

Let's explore the case $N = 2$. The equation for $F^{(2)}$ becomes

$$\big(F_t^{(2)} + F_{xxx}^{(2)}\big)_x = 3 \, \alpha_1 \alpha_2 \, (\alpha_2 - \alpha_1)^2 \, f_1 f_2 \ , \tag{10.50}$$

with solution

$$F^{(2)} = \left(\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\right)^2 f_1 f_2 \ . \tag{10.51}$$

Remarkably, this completes the hierarchy for $N = 2$. Thus,

$$\begin{aligned} F &= 1 + f_1 + f_2 + \left(\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\right)^2 f_1 f_2 \\ &= \det \begin{pmatrix} 1 + f_1 & \frac{2\sqrt{\alpha_1\alpha_2}}{\alpha_1+\alpha_2} \, f_1 \\ \frac{2\sqrt{\alpha_1\alpha_2}}{\alpha_1+\alpha_2} \, f_2 & 1 + f_2 \end{pmatrix} \ . \end{aligned} \tag{10.52}$$

What Hirota showed, quite amazingly, is that this result generalizes to the N-soliton case,

$$F = \det \mathsf{Q} \ , \tag{10.53}$$

where Q is the symmetric matrix,

$$\mathsf{Q}_{mn} = \delta_{mn} + \frac{2\sqrt{\alpha_m \alpha_n}}{\alpha_m + \alpha_n} \, f_m \, f_n \ . \tag{10.54}$$

Thus, N-soliton solutions to the KdV equation may be written in the form

$$u(x,t) = 2 \, \frac{\partial^2}{\partial x^2} \, \ln \det \mathsf{Q}(x,t) \ . \tag{10.55}$$

Consider the case $N = 2$. Direct, if tedious, calculations lead to the expression

$$u = 2 \, \frac{\alpha_1^2 f_1 + \alpha_2^2 f_2 + 2 \, (\alpha_1 - \alpha_2)^2 f_1 f_2 + \big(\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\big)^2 \big(\alpha_1^2 f_1 f_2^2 + \alpha_2^2 f_1^2 f_2\big)}{\Big[1 + f_1 + f_2 + \big(\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\big)^2 f_1 f_2\Big]^2} \ . \tag{10.56}$$

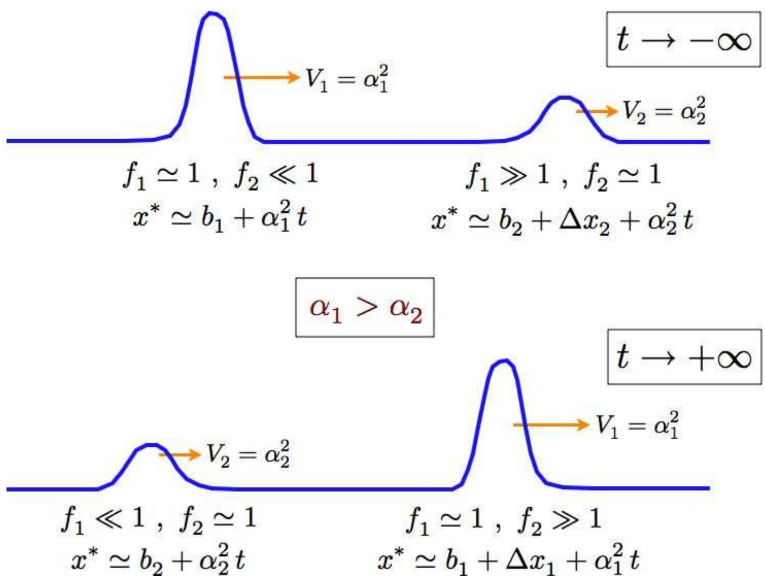

Figure 10.5: Early and late time configuration of the two soliton solution to the KdV equation.

Recall that

$$f_j(x,t) = \exp\big[\alpha_j\,(x_j - \alpha_j^2\, t - b_j)\big]\ . \tag{10.57}$$

Let's consider (x,t) values for which $f_1 \simeq 1$ is neither large nor small, and investigate what happens in the limits $f_2 \ll 1$ and $f_2 \gg 1$. In the former case, we find

$$u \simeq \frac{2\alpha_1^2\, f_1}{(1+f_1)^2} \qquad (f_2 \ll 1) \quad , \tag{10.58}$$

which is identical to the single soliton case of eqn. 10.43. In the opposite limit, we have

$$u \simeq \frac{2\alpha_1^2\, g_1}{(1+g_1)^2} \qquad (f_2 \gg 1) \quad , \tag{10.59}$$

where

$$g_1 = \bigg(\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\bigg)^2 f_1 \tag{10.60}$$

But multiplication of f_j by a constant C is equivalent to a translation:

$$C\, f_j(x,t) = f_j(x + \alpha_j^{-1} \ln C\, , t) \equiv f_j(x - \Delta x_j\, , t)\ . \tag{10.61}$$

Thus, depending on whether f_2 is large or small, the solution either acquires or does not acquire a spatial shift Δx_1, where

$$\Delta x_j = \frac{2}{\alpha_j}\, \ln\bigg|\frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2}\bigg|\ . \tag{10.62}$$

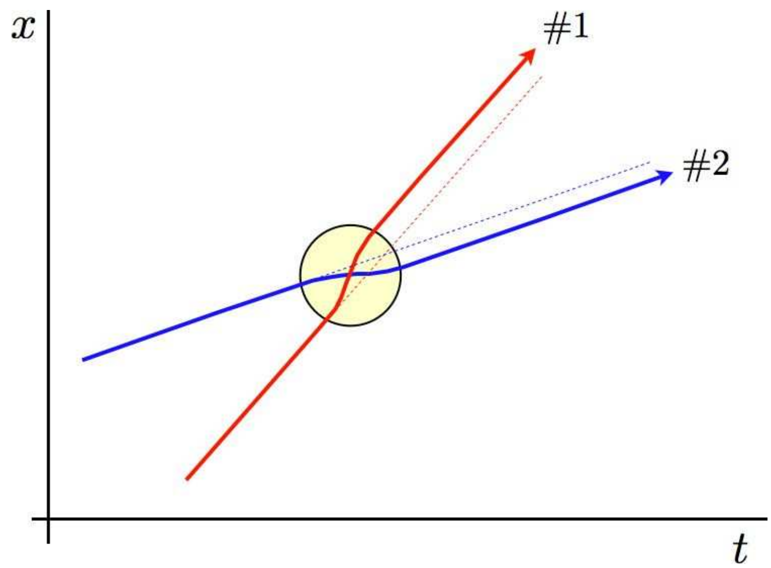

Figure 10.6: Spacetime diagram for the collision of two KdV solitons. The strong, fast soliton (#1) is shifted forward and the weak slow one (#2) is shifted backward. The red and blue lines indicate the centers of the two solitons. The yellow shaded circle is the 'interaction region' where the solution is not simply a sum of the two single soliton waveforms.

The function $f(x,t) = \exp\big[\alpha(x - \alpha^2 t - b)\big]$ is monotonically increasing in x (assuming $\alpha > 0$). Thus, if at fixed t the spatial coordinate x is such that $f \ll 1$, this means that the soliton lies to the right. Conversely, if $f \gg 1$ the soliton lies to the left. Suppose $\alpha_1 > \alpha_2$, in which case soliton #1 is stronger (*i.e.* greater amplitude) and faster than soliton #2. The situation is as depicted in figs. 10.5 and 10.6. Starting at early times, the strong soliton lies to the left of the weak soliton. It moves faster, hence it eventually overtakes the weak soliton. As the strong soliton passes through the weak one, it is shifted *forward*, and the weak soliton is shifted *backward*. It hardly seems fair that the strong fast soliton gets pushed even further ahead at the expense of the weak slow one, but sometimes life is just like that.

10.1.6 Bäcklund transformations

For certain nonlinear PDEs, a given solution may be used as a 'seed' to generate an entire hierarchy of solutions. This is familiar from the case of Riccati equations, which are nonlinear and nonautonomous ODEs, but for PDEs it is even more special. The general form of the Bäcklund transformation (BT) is

$$u_{1,t} = P(u_1\,,\,u_0\,,\,u_{0,t}\,,\,u_{0,x}) \tag{10.63}$$

$$u_{1,x} = Q(u_1\,,\,u_0\,,\,u_{0,t}\,,\,u_{0,x})\ , \tag{10.64}$$

where $u_0(x,t)$ is the known solution.

Figure 10.7: "Wrestling's living legend" Bob Backlund, in a 1983 match, is subjected to a devastating *camel clutch* by the Iron Sheikh. The American Bob Backlund has nothing to do with the Bäcklund transformations discussed in the text, which are named for the 19$^{\rm th}$ century Swedish mathematician Albert Bäcklund. Note that Bob Backlund's manager has thrown in the towel (lower right).

A Bäcklund transformation for the KdV equation was first obtained in 1973[3]. This provided a better understanding of the Hirota hierarchy. To proceed, following the discussion in the book by Scott, we consider the earlier (1968) result of Miura[4], who showed that if $v(x,t)$ satisfies the modified KdV (MKdV) equation[5],

$$v_t - 6v^2 v_x + v_{xxx} = 0 \ , \tag{10.65}$$

then

$$u = -(v^2 + v_x) \tag{10.66}$$

solves KdV:

$$\begin{aligned} u_t + 6uu_x + u_{xxx} &= -(v^2+v_x)_t + 6(v^2+v_x)(v^2+v_x)_x - (v^2+v_x)_{xxx} \\ &= -\big(2v+\partial_x\big)\big(v_t - 6v^2 v_x + v_{xxx}\big) = 0 \ . \end{aligned} \tag{10.67}$$

From this result, we have that if

$$v_t - 6(v^2+\lambda)\, v_x + v_{xxx} = 0 \ , \tag{10.68}$$

then

$$u = -(v^2 + v_x + \lambda) \tag{10.69}$$

solves KdV. The MKdV equation, however, is symmetric under $v \to -v$, hence

$$u_0 = -v_x - v^2 - \lambda \tag{10.70}$$

$$u_1 = +v_x - v^2 - \lambda \tag{10.71}$$

[3]H. D. Wahlquist and F. B. Eastabrook, *Phys. Rev. Lett.* **31**, 1386 (1973).
[4]R. M. Miura, *J. Math. Phys.* **9**, 1202 (1968).
[5]Note that the second term in the MKdV equation is proportional to $v^2 v_x$, as opposed to uu_x which appears in KdV.

both solve KdV. Now define $u_0 \equiv -w_{0,x}$ and $u_1 \equiv -w_{1,x}$. Subtracting the above two equations, we find

$$u_0 - u_1 = -2v_x \quad \Rightarrow \quad w_0 - w_1 = 2v \ . \tag{10.72}$$

Adding the equations instead gives

$$\begin{aligned} w_{0,x} + w_{1,x} &= 2(v^2 + \lambda) \\ &= \tfrac{1}{2}(w_0 - w_1)^2 + 2\lambda \ . \end{aligned} \tag{10.73}$$

Substituting for $v = \frac{1}{2}(w_0 - w_1)$ and $v^2 + \lambda - \frac{1}{2}(w_{0,x} + w_{1,x})$ into the MKdV equation, we have

$$(w_0 - w_1)_t - 3\big(w_{0,x}^2 - w_{1,x}^2\big) + (w_0 - w_1)_{xxx} = 0 \ . \tag{10.74}$$

This last equation is a Bäcklund transformation (BT), although in a somewhat nonstandard form, since the RHS of eqn. 10.63, for example, involves only the 'new' solution u_1 and the 'old' solution u_0 and its first derivatives. Our equation here involves third derivatives. However, we can use eqn. 10.73 to express $w_{1,x}$ in terms of w_0, $w_{0,x}$, and w_1.

Starting from the trivial solution $w_0 = 0$, eqn. 10.73 gives

$$w_{1,x} = \tfrac{1}{2} w_1^2 + \lambda \ . \tag{10.75}$$

With the choice $\lambda = -\frac{1}{4}\alpha^2 < 0$, we integrate and obtain

$$w_1(x,t) = -2\alpha \, \tanh\big(\tfrac{1}{2}\alpha x + \varphi(t)\big) \ , \tag{10.76}$$

were $\varphi(t)$ is at this point arbitrary. We fix $\varphi(t)$ by invoking eqn. 10.74, which says

$$w_{1,t} = 3 w_{1,x}^2 - w_{1,xxx} = 0 \ . \tag{10.77}$$

Invoking $w_{1,x} = \frac{1}{2} w_1^2 + \lambda$ and differentiating twice to obtain $w_{1,xxx}$, we obtain an expression for the RHS of the above equation. The result is $w_{1,t} + \alpha^2 w_{1,x} = 0$, hence

$$w_1(x,t) = -\alpha \, \tanh\Big[\tfrac{1}{2}\alpha\,(x - \alpha^2 t - b)\Big] \tag{10.78}$$

$$u_1(x,t) = \tfrac{1}{2}\alpha^2 \, \text{sech}^2\Big[\tfrac{1}{2}\alpha\,(x - \alpha^2 t - b)\Big] \ , \tag{10.79}$$

which recapitulates our earlier result. Of course we would like to do better, so let's try to insert this solution into the BT and the turn the crank and see what comes out. This is unfortunately a rather difficult procedure. It becomes tractable if we assume that successive Bäcklund transformations commute, which is the case, but which we certainly have not yet proven. That is, we assume that $w_{12} = w_{21}$, where

$$w_0 \xrightarrow{\lambda_1} w_1 \xrightarrow{\lambda_2} w_{12} \tag{10.80}$$

$$w_0 \xrightarrow{\lambda_2} w_2 \xrightarrow{\lambda_1} w_{21} \ . \tag{10.81}$$

Invoking this result, we find that the Bäcklund transformation gives

$$w_{12} = w_{21} = w_0 - \frac{4(\lambda_1 - \lambda_2)}{w_1 - w_2} \ . \tag{10.82}$$

Successive applications of the BT yield Hirota's multiple soliton solutions:

$$w_0 \xrightarrow{\lambda_1} w_1 \xrightarrow{\lambda_2} w_{12} \xrightarrow{\lambda_3} w_{123} \xrightarrow{\lambda_4} \cdots \ . \tag{10.83}$$

10.2 Sine-Gordon Model

Consider transverse electromagnetic waves propagating down a superconducting transmission line, shown in fig. 10.8. The transmission line is modeled by a set of inductors, capacitors, and Josephson junctions such that for a length dx of the transmission line, the capacitance is $dC = \mathcal{C}\,dx$, the inductance is $dL = \mathcal{L}\,dx$, and the critical current is $dI_0 = \mathcal{I}_0\,dx$. Dividing the differential voltage drop dV and shunt current dI by dx, we obtain

$$\frac{\partial V}{\partial x} = -\mathcal{L}\,\frac{\partial I}{\partial t} \tag{10.84}$$

$$\frac{\partial I}{\partial x} = -\mathcal{C}\,\frac{\partial V}{\partial t} - \mathcal{I}_0\,\sin\phi\ , \tag{10.85}$$

where ϕ is the difference $\phi = \phi_{\rm upper} - \phi_{\rm lower}$ in the superconducting phases. The voltage is related to the rate of change of ϕ through the Josephson equation,

$$\frac{\partial\phi}{\partial t} = \frac{2eV}{\hbar}\ , \tag{10.86}$$

and therefore

$$\frac{\partial\phi}{\partial x} = -\frac{2e\mathcal{L}}{\hbar}\,I\ . \tag{10.87}$$

Thus, we arrive at the equation

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{1}{\lambda_{\rm J}^2}\,\sin\phi = 0\ , \tag{10.88}$$

where $c = (\mathcal{L}\,\mathcal{C})^{-1/2}$ is the *Swihart velocity* and $\lambda_{\rm J} = (\hbar/2e\mathcal{L}\mathcal{I}_0)^{1/2}$ is the *Josephson length*. We may now rescale lengths by $\lambda_{\rm J}$ and times by $\lambda_{\rm J}/c$ to arrive at the *sine-Gordon equation*,

$$\phi_{tt} - \phi_{xx} + \sin\phi = 0\ . \tag{10.89}$$

This equation of motion may be derived from the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\phi_t^2 - \tfrac{1}{2}\phi_x^2 - U(\phi)\ . \tag{10.90}$$

We then obtain

$$\phi_{tt} - \phi_{xx} = -\frac{\partial U}{\partial\phi}\ , \tag{10.91}$$

and the sine-Gordon equation follows for $U(\phi) = 1 - \cos\phi$.

Assuming $\phi(x,t) = \phi(x - Vt)$ we arrive at $(1 - V^2)\,\phi_{\xi\xi} = U'(\xi)$, and integrating once we obtain

$$\tfrac{1}{2}(1 - V^2)\,\phi_\xi^2 - U(\phi) = E\ . \tag{10.92}$$

This describes a particle of mass $M = 1 - V^2$ moving in the *inverted potential* $-U(\phi)$. Assuming $V^2 < 1$, we may solve for ϕ_ξ:

$$\phi_\xi^2 = \frac{2(E + U(\phi))}{1 - V^2}\ , \tag{10.93}$$

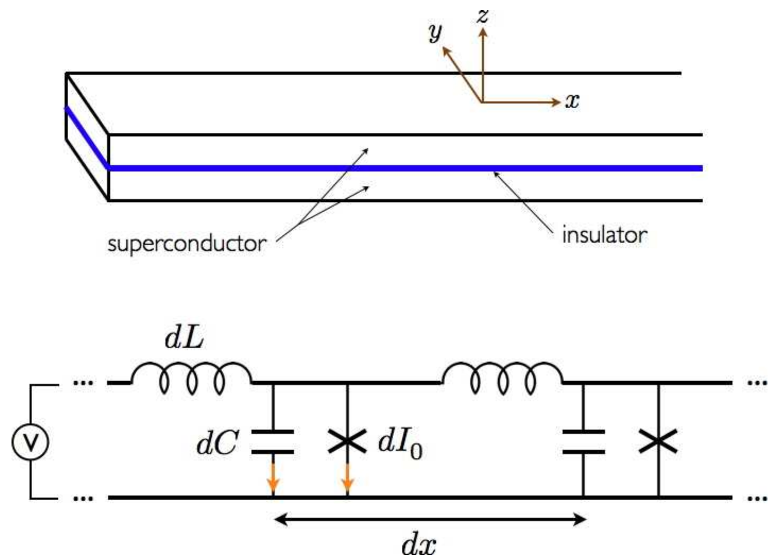

Figure 10.8: A superconducting transmission line is described by a capacitance per unit length $\mathcal{C}$, an inductance per unit length $\mathcal{L}$, and a critical current per unit length $\mathcal{I}_0$. Based on fig. 3.4 of A. Scott, *Nonlinear Science*.

which requires $E \geq -U_{\rm max}$ in order for a solution to exist. For $-U_{\rm max} < E < -U_{\rm min}$, the motion is bounded by turning points. The situation for the sine-Gordon (SG) model is sketched in fig. 10.9. For the SG model, the turning points are at $\phi^*_\pm = \pm\cos^{-1}(E+1)$, with $-2 < E < 0$. We can write

$$\phi^*_\pm = \pi \pm \delta \qquad , \qquad \delta = 2\cos^{-1}\sqrt{-\frac{E}{2}}\ . \tag{10.94}$$

This class of solutions describes periodic waves. From

$$\frac{\sqrt{2}\,d\xi}{\sqrt{1-V^2}} = \frac{d\phi}{\sqrt{E+U(\phi)}}\ , \tag{10.95}$$

we have that the spatial period λ is given by

$$\lambda = \sqrt{2\,(1-V^2)}\int\limits_{\phi^*_-}^{\phi^*_+} \frac{d\phi}{\sqrt{E+U(\phi)}}\ . \tag{10.96}$$

If $E > -U_{\rm min}$, then ϕ_ξ is always of the same sign, and $\phi(\xi)$ is a monotonic function of ξ. If $U(\phi) = U(\phi+2\pi)$ is periodic, then the solution is a 'soliton lattice' where the spatial period of ϕ mod 2π is

$$\widetilde{\lambda} = \sqrt{\frac{1-V^2}{2}}\int\limits_0^{2\pi} \frac{d\phi}{\sqrt{E+U(\phi)}}\ . \tag{10.97}$$

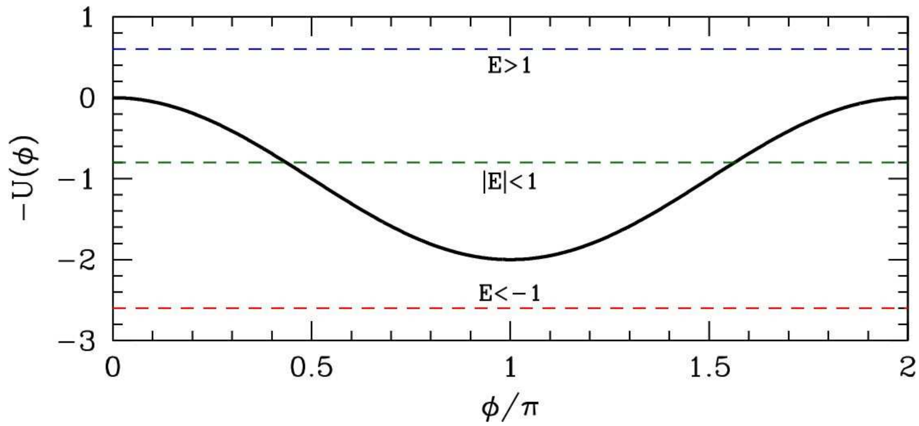

Figure 10.9: The inverted potential $-U(\phi)=\cos\phi-1$ for the sine-Gordon problem.

For the sine-Gordon model, with $U(\phi)=1-\cos\phi$, one finds

$$\lambda=\sqrt{-\frac{\pi\,(1-V^2)}{E(E+2)}} \qquad , \qquad \widetilde{\lambda}=\sqrt{\frac{8\,(1-V^2)}{E+2}}\;\mathbb{K}\left(\sqrt{\frac{2}{E+2}}\right) \; . \tag{10.98}$$

10.2.1 Tachyon solutions

When $V^2>1$, we have

$$\tfrac{1}{2}(V^2-1)\,\phi_\xi^2+U(\phi)=-E \; . \tag{10.99}$$

Such solutions are called *tachyonic*. There are again three possibilities:

- $E>U_{\rm min}$: no solution.
- $-U_{\rm max}<E<-U_{\rm min}$: periodic $\phi(\xi)$ with oscillations about $\phi=0$.
- $E<-U_{\rm max}$: tachyon lattice with monotonic $\phi(\xi)$.

It turns out that the tachyon solution is unstable.

10.2.2 Hamiltonian formulation

The Hamiltonian density is

$$\mathcal{H}=\pi\,\phi_t-\mathcal{L} \; , \tag{10.100}$$

where

$$\pi=\frac{\partial\mathcal{L}}{\partial\phi_t}=\phi_t \tag{10.101}$$

is the momentum density conjugate to the field ϕ. Then

$$\mathcal{H}(\pi,\phi) = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}\phi_x^2 + U(\phi) \ . \tag{10.102}$$

Note that the total momentum in the field is

$$\begin{aligned} P &= \int\limits_{-\infty}^{\infty} dx\,\pi = \int\limits_{-\infty}^{\infty} dx\,\phi_t = -V \int\limits_{-\infty}^{\infty} dx\,\phi_x \\ &= -V\Big[\phi(\infty)-\phi(-\infty)\Big] = -2\pi n V \ , \end{aligned} \tag{10.103}$$

where $n = \big[\phi(\infty)-\phi(-\infty)\big]/2\pi$ is the *winding number*.

10.2.3 Phonons

The Hamiltonian density for the SG system is minimized when $U(\phi)=0$ everywhere. The ground states are then classified by an integer $n\in\mathbb{Z}$, where $\phi(x,t)=2\pi n$ for ground state n. Suppose we linearize the SG equation about one of these ground states, writing

$$\phi(x,t) = 2\pi n + \eta(x,t) \ , \tag{10.104}$$

and retaining only the first order term in η from the nonlinearity. The result is the Klein-Gordon (KG) equation,

$$\eta_{tt} - \eta_{xx} + \eta = 0 \ . \tag{10.105}$$

This is a linear equation, whose solutions may then be superposed. Fourier transforming from (x,t) to (k,ω), we obtain the equation

$$\big(-\omega^2 + k^2 + 1\big)\,\hat\eta(k,\omega) = 0 \ , \tag{10.106}$$

which entails the dispersion relation $\omega = \pm\omega(k)$, where

$$\omega(k) = \sqrt{1+k^2} \ . \tag{10.107}$$

Thus, the most general solution to the $(1+1)$-dimensional KG equation is

$$\eta(x,t) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ A(k)\,e^{ikx}\,e^{-i\sqrt{1+k^2}\,t} + B(k)\,e^{ikx}\,e^{i\sqrt{1+k^2}\,t} \right\} \ . \tag{10.108}$$

For the Helmholtz equation $\eta_{tt}-\eta_{xx}=0$, the dispersion is $\omega(k)=|k|$, and the solution may be written as $\eta(x,t)=f(x-t)+g(x+t)$, which is the sum of arbitrary right-moving and left-moving components. The fact that the Helmholtz equation preserves the shape of the wave is a consequence of the absence of dispersion, *i.e.* the phase velocity $v_{\rm p}(k)=\frac{\omega}{k}$ is the same as the group velocity $v_{\rm g}(k)=\frac{\partial\omega}{\partial k}$. This is not the case for the KG equation, obviously, since

$$v_{\rm p}(k) = \frac{\omega}{k} = \frac{\sqrt{1+k^2}}{k} \qquad , \qquad v_{\rm g}(k) = \frac{\partial\omega}{\partial k} = \frac{k}{\sqrt{1+k^2}} \ , \tag{10.109}$$

hence $v_{\rm p} v_{\rm g} = 1$ for KG.

10.2.4 Mechanical realization

The sine-Gordon model can be realized mechanically by a set of pendula elastically coupled. The kinetic energy T and potential energy U are given by

$$T = \sum_n \tfrac{1}{2} m\ell^2 \dot{\phi}_n^2 \tag{10.110}$$

$$U = \sum_n \Big[\tfrac{1}{2}\kappa \big(\phi_{n+1} - \phi_n\big)^2 + mg\ell\big(1 - \cos\phi_n\big)\Big] \ . \tag{10.111}$$

Here ℓ is the distance from the hinge to the center-of-mass of the pendulum, and κ is the torsional coupling. From the Euler-Lagrange equations we obtain

$$m\ell^2 \ddot{\phi}_n = -\kappa\big(\phi_{n+1} + \phi_{n-1} - 2\phi_n\big) - mg\ell \sin\phi_n \ . \tag{10.112}$$

Let a be the horizontal spacing between the pendula. Then we can write the above equation as

$$\ddot{\phi}_n = \overbrace{\frac{\kappa a^2}{m\ell^2}}^{\equiv c^2} \cdot \overbrace{\frac{1}{a}\Bigg[\bigg(\frac{\phi_{n+1} - \phi_n}{a}\bigg) - \bigg(\frac{\phi_n - \phi_{n-1}}{a}\bigg)\Bigg]}^{\approx \phi_n''} - \frac{g}{\ell}\sin\phi_n \ . \tag{10.113}$$

The continuum limit of these coupled ODEs yields the PDE

$$\frac{1}{c^2}\,\phi_{tt} - \phi_{xx} + \frac{1}{\lambda^2}\,\sin\phi = 0 \ , \tag{10.114}$$

which is the sine-Gordon equation, with $\lambda = (\kappa a^2/mg\ell)^{1/2}$.

10.2.5 Kinks and antikinks

Let us return to eqn. 10.92 and this time set $E = -U_{\rm min}$. With $U(\phi) = 1 - \cos\phi$, we have $U_{\rm min} = 0$, and thus

$$\frac{d\phi}{d\xi} = \pm \frac{2}{\sqrt{1 - V^2}}\,\sin\big(\tfrac{1}{2}\phi\big) \ . \tag{10.115}$$

This equation may be integrated:

$$\pm \frac{d\xi}{\sqrt{1 - V^2}} = \frac{d\phi}{2\sin\frac{1}{2}\phi} = d\ln\tan\tfrac{1}{4}\phi \ . \tag{10.116}$$

Thus, the solution is

$$\phi(x,t) = 4\tan^{-1}\exp\bigg(\pm\frac{x - Vt - x_0}{\sqrt{1 - V^2}}\bigg) \ . \tag{10.117}$$

where ξ_0 is a constant of integration. This describes either a kink (with $d\phi/dx > 0$) or an antikink (with $d\phi/dx < 0$) propagating with velocity V, instantaneously centered at $x = x_0 + Vt$. Unlike the KdV soliton, the amplitude of the SG soliton is independent of its

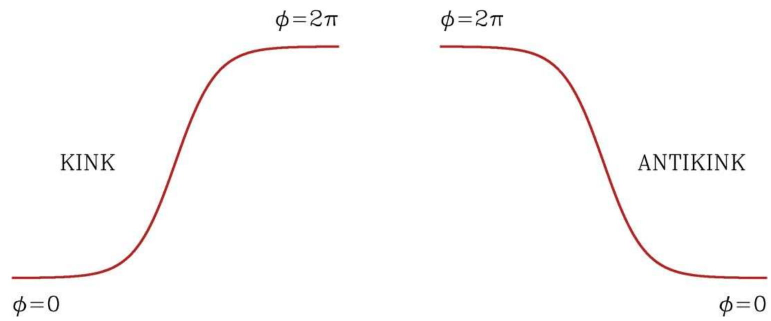

Figure 10.10: Kink and antikink solutions to the sine-Gordon equation.

velocity. The SG soliton is *topological*, interpolating between two symmetry-related vacuum states, namely $\phi = 0$ and $\phi = 2\pi$.

Note that the width of the kink and antikink solutions decreases as V increases. This is a Lorentz contraction, and should have been expected since the SG equation possesses a Lorentz invariance under transformations

$$x = \frac{x' + vt'}{\sqrt{1 - v^2}} \tag{10.118}$$

$$t = \frac{t' + vx'}{\sqrt{1 - v^2}} \ . \tag{10.119}$$

One then readily finds

$$\partial_t^2 - \partial_x^2 = \partial_{t'}^2 - \partial_{x'}^2 \ . \tag{10.120}$$

The moving soliton solutions may then be obtained by a Lorentz transformation of the stationary solution,

$$\phi(x, t) = 4 \tan^{-1} e^{\pm(x - x_0)} \ . \tag{10.121}$$

The field ϕ itself is a Lorentz scalar, and hence does not change in magnitude under a Lorentz transformation.

10.2.6 Bäcklund transformation for the sine-Gordon system

Recall D'Alembert's method of solving the Helmholtz equation, by switching to variables

$$\zeta = \tfrac{1}{2}(x - t) \qquad \partial_x = \tfrac{1}{2}\partial_\zeta + \tfrac{1}{2}\partial_t \tag{10.122}$$

$$\tau = \tfrac{1}{2}(x + t) \qquad \partial_t = -\tfrac{1}{2}\partial_\zeta + \tfrac{1}{2}\partial_t \ . \tag{10.123}$$

The D'Alembertian operator then becomes

$$\partial_t^2 - \partial_x^2 = -\,\partial_\zeta\,\partial_\tau \ . \tag{10.124}$$

This transforms the Helmholtz equation $\phi_{tt} - \phi_{xx} = 0$ to $\phi_{\zeta\tau} = 0$, with solutions $\phi(\zeta, \tau) = f(\zeta) + g(\tau)$, with f and g arbitrary functions of their arguments. As applied to the SG equation, we have

$$\phi_{\zeta\tau} = \sin\phi \ . \tag{10.125}$$

Suppose we have a solution ϕ_0 to this equation. Suppose further that ϕ_1 satisfies the pair of equations,

$$\phi_{1,\zeta} = 2\lambda\,\sin\left(\frac{\phi_1 + \phi_0}{2}\right) + \phi_{0,\zeta} \tag{10.126}$$

$$\phi_{1,\tau} = \frac{2}{\lambda}\,\sin\left(\frac{\phi_1 - \phi_0}{2}\right) - \phi_{0,\tau}\ . \tag{10.127}$$

Thus,

$$\begin{aligned}\phi_{1,\zeta\tau} - \phi_{0,\zeta\tau} &= \lambda\cos\left(\frac{\phi_1 + \phi_0}{2}\right)\left(\phi_{1,\tau} - \phi_{0,\tau}\right) \\ &= 2\cos\left(\frac{\phi_1 + \phi_0}{2}\right)\,\sin\left(\frac{\phi_1 - \phi_0}{2}\right) \\ &= \sin\phi_1 - \sin\phi_0\ .\end{aligned} \tag{10.128}$$

Thus, if $\phi_{0,\zeta\tau} = \sin\phi_0$, then $\phi_{1,\zeta\tau} = \sin\phi_1$ as well, and ϕ_1 satisfies the SG equation. Eqns. 10.126 and 10.127 constitute a Bäcklund transformation for the SG system.

Let's give the 'Bäcklund crank' one turn, starting with the trivial solution $\phi_0 = 0$. We then have

$$\phi_{1,\zeta} = 2\lambda\,\sin\tfrac{1}{2}\phi_1 \tag{10.129}$$

$$\phi_{1,\tau} = 2\lambda^{-1}\sin\tfrac{1}{2}\phi_1\ . \tag{10.130}$$

The solution to these equations is easily found by direct integration:

$$\phi(\zeta,\tau) = 4\tan^{-1} e^{\lambda\zeta}\,e^{\tau/\lambda}\ . \tag{10.131}$$

In terms of our original independent variables (x,t), we have

$$\lambda\zeta + \lambda^{-1}\tau = \tfrac{1}{2}\big(\lambda + \lambda^{-1}\big)\,x - \tfrac{1}{2}\big(\lambda - \lambda^{-1}\big)\,t = \pm\,\frac{x - vt}{\sqrt{1-v^2}}\ , \tag{10.132}$$

where

$$v \equiv \frac{\lambda^2 - 1}{\lambda^2 + 1} \qquad \Longleftrightarrow \qquad \lambda = \pm\left(\frac{1+v}{1-v}\right)^{1/2}\ . \tag{10.133}$$

Thus, we generate the kink/antikink solution

$$\phi_1(x,t) = 4\tan^{-1}\exp\left(\pm\,\frac{x - vt}{\sqrt{1-v^2}}\right)\ . \tag{10.134}$$

As was the case with the KdV system, successive Bäcklund transformations commute. Thus,

$$\phi_0 \xrightarrow{\lambda_1} \phi_1 \xrightarrow{\lambda_2} \phi_{12} \tag{10.135}$$

$$\phi_0 \xrightarrow{\lambda_2} \phi_2 \xrightarrow{\lambda_1} \phi_{21}\ , \tag{10.136}$$

with $\phi_{12} = \phi_{21}$. This allows one to eliminate the derivatives and write

$$\tan\left(\frac{\phi_{12} - \phi_0}{4}\right) = \left(\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\right) \tan\left(\frac{\phi_2 - \phi_1}{4}\right) . \tag{10.137}$$

We can now create new solutions from individual kink pairs (KK), or kink-antikink pairs ($\mathrm{K\overline{K}}$). For the KK system, taking $v_1 = v_2 = v$ yields

$$\phi_{\mathrm{KK}}(x,t) = 4\tan^{-1}\left(\frac{v\sinh(\gamma x)}{\cosh(\gamma v t)}\right) , \tag{10.138}$$

where γ is the Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1 - v^2}} . \tag{10.139}$$

Note that $\phi_{\mathrm{KK}}(\pm\infty, t) = \pm 2\pi$ and $\phi_{\mathrm{KK}}(0,t) = 0$, so there is a phase increase of 2π on each of the negative and positive half-lines for x, and an overall phase change of $+4\pi$. For the $\mathrm{K\overline{K}}$ system, if we take $v_1 = -v_2 = v$, we obtain the solution

$$\phi_{\mathrm{K\overline{K}}}(x,t) = 4\tan^{-1}\left(\frac{\sinh(\gamma v t)}{v\cosh(\gamma x)}\right) . \tag{10.140}$$

In this case, analytically continuing to imaginary v with

$$v = \frac{i\omega}{\sqrt{1 - \omega^2}} \qquad \Longrightarrow \qquad \gamma = \sqrt{1 - \omega^2} \tag{10.141}$$

yields the *stationary breather* solution,

$$\phi_{\mathrm{B}}(x,t) = 4\tan^{-1}\left(\frac{\sqrt{1-\omega^2}\,\sin(\omega t)}{\omega\cosh(\sqrt{1-\omega^2}\,x)}\right) . \tag{10.142}$$

The breather is a localized solution to the SG system which oscillates in time. By applying a Lorentz transformation of the spacetime coordinates, one can generate a moving breather solution as well.

Please note, in contrast to the individual kink/antikink solutions, the solutions ϕ_{KK}, $\phi_{\mathrm{K\overline{K}}}$, and ϕ_{B} are not functions of a single variable $\xi = x - Vt$. Indeed, a given multisoliton solution may involve components moving at several different velocities. Therefore the total momentum P in the field is no longer given by the simple expression $P = V\big(\phi(-\infty) - \phi(+\infty)\big)$. However, in cases where the multikink solutions evolve into well-separated solitions, as happens when the individual kink velocities are distinct, the situation simplifies, as we may consider each isolated soliton as linearly independent. We then have

$$P = -2\pi \sum_i n_i\, V_i , \tag{10.143}$$

where $n_i = +1$ for kinks and $n_i = -1$ for antikinks.

10.3 Nonlinear Schrödinger Equation

The Nonlinear Schrödinger (NLS) equation arises in several physical contexts. One is the Gross-Pitaevskii action for an interacting bosonic field,

$$S[\psi,\psi^*] = \int\! dt \int\! d^d\!x \bigg\{ i\psi^* \, \frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\, \boldsymbol{\nabla}\psi^* \!\cdot\! \boldsymbol{\nabla}\psi - U\big(\psi^*\psi\big) + \mu\,\psi^*\psi \bigg\} \ , \tag{10.144}$$

where $\psi(\boldsymbol{x},t)$ is a complex scalar field. The local interaction $U\big(|\psi|^2\big)$ is taken to be quartic,

$$U\big(|\psi|^2\big) = \tfrac{1}{2} g\, |\psi|^4 \ . \tag{10.145}$$

Note that

$$U\big(|\psi|^2\big) - \mu\, |\psi|^2 = \tfrac{1}{2} g \bigg(|\psi|^2 - \frac{\mu}{g}\bigg)^{\!2} - \frac{\mu^2}{2g} \ . \tag{10.146}$$

Extremizing the action with respect to ψ^* yields the equation

$$\frac{\delta S}{\delta\psi^*} = 0 = i\,\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\, \boldsymbol{\nabla}^2\psi - U'\big(\psi^*\psi\big)\,\psi + \mu\,\psi \ . \tag{10.147}$$

Extremization with respect to ψ yields the complex conjugate equation. In $d=1$, we have

$$i\psi_t = -\frac{\hbar^2}{2m}\,\psi_{xx} + U'\big(\psi^*\psi\big)\,\psi - \mu\,\psi \ . \tag{10.148}$$

We can absorb the chemical potential by making a time-dependent gauge transformation

$$\psi(x,t) \longrightarrow e^{i\mu t}\,\psi(x,t) \ . \tag{10.149}$$

Further rescalings of the field and independent variables yield the generic form

$$i\psi_t \pm \psi_{xx} + 2\,|\psi|^2\,\psi = 0 \ , \tag{10.150}$$

where the $+$ sign pertains for the case $g<0$ (attactive interaction), and the $-$ sign for the case $g>0$ (repulsive interaction). These cases are known as *focusing*, or NLS($+$), and *defocusing*, or NLS($-$), respectively.

10.3.1 Amplitude-phase representation

We can decompose the complex scalar ψ into its amplitude and phase:

$$\psi = A\,e^{i\phi} \ . \tag{10.151}$$

We then find

$$\psi_t = \big(A_t + iA\phi_t\big)\,e^{i\phi} \tag{10.152}$$

$$\psi_x = \big(A_x + iA\phi_x\big)\,e^{i\phi} \tag{10.153}$$

$$\psi_{xx} = \big(A_{xx} - A\phi_x^2 + 2iA_x\phi_x + iA\phi_{xx}\big)\,e^{i\phi} \ . \tag{10.154}$$

Multiplying the NLS($\pm$) equations by $e^{-i\phi}$ and taking real and imaginary parts, we obtain the coupled nonlinear PDEs,

$$-A\phi_t \pm \big(A_{xx} - A\phi_x^2\big) + 2A^3 = 0 \tag{10.155}$$

$$A_t \pm \big(2A_x\phi_x + A\phi_{xx}\big) = 0 \ . \tag{10.156}$$

Note that the second of these equations may be written in the form of a continuity equation,

$$\rho_t + j_x = 0 \ , \tag{10.157}$$

where

$$\rho = A^2 \tag{10.158}$$

$$j = \pm 2A^2\phi_x \ . \tag{10.159}$$

10.3.2 Phonons

One class of solutions to NLS($\pm$) is the spatially uniform case

$$\psi_0(x,t) = A_0\, e^{2iA_0^2\, t} \ , \tag{10.160}$$

with $A = A_0$ and $\phi = 2A_0^2\, t$. Let's linearize about these solutions, writing

$$A(x,t) = A_0 + \delta A(x,t) \tag{10.161}$$

$$\phi(x,t) = 2A_0^2\, t + \delta\phi(x,t) \ . \tag{10.162}$$

Our coupled PDEs then yield

$$4A_0^2\,\delta A \pm \delta A_{xx} - A_0\,\delta\phi_t = 0 \tag{10.163}$$

$$\delta A_t \pm A_0\,\delta\phi_{xx} = 0 \ . \tag{10.164}$$

Fourier transforming, we obtain

$$\begin{pmatrix} 4A_0^2 \mp k^2 & iA_0\,\omega \\ -i\omega & \mp A_0 k^2 \end{pmatrix} \begin{pmatrix} \delta\hat{A}(k,\omega) \\ \delta\hat{\phi}(k,\omega) \end{pmatrix} = 0 \ . \tag{10.165}$$

Setting the determinant to zero, we obtain

$$\omega^2 = \mp 4A_0^2\, k^2 + k^4 \ . \tag{10.166}$$

For NLS($-$), we see that $\omega^2 \geq 0$ for all k, meaning that the initial solution $\psi_0(x,t)$ is stable. For NLS($+$), however, we see that wavevectors $k \in \big[-2A_0\,,\, 2A_0\big]$ are *unstable*. This is known as the *Benjamin-Feir instability*.

10.3.3 Soliton solutions for NLS(+)

Let's consider moving soliton solutions for NLS(+). We try a two-parameter solution of the form

$$A(x,t) = A(x - ut) \tag{10.167}$$
$$\phi(x,t) = \phi(x - vt) \ . \tag{10.168}$$

This results in the coupled ODEs

$$A_{xx} - A\,\phi_x^2 + vA\,\phi_x + 2A^3 = 0 \tag{10.169}$$
$$A\,\phi_{xx} + 2A_x\phi_x - uA_x = 0 \ . \tag{10.170}$$

Multiplying the second equation by $2A$ yields

$$\Big((2\phi_x - u)A^2\Big)_x = 0 \qquad \Longrightarrow \qquad \phi_x = \tfrac{1}{2}u + \frac{P}{2A^2} \ , \tag{10.171}$$

where P is a constant of integration. Inserting this in the first equation results in

$$A_{xx} + 2A^3 + \tfrac{1}{4}(2uv - u^2)A + \tfrac{1}{2}(v - u)PA^{-1} - \tfrac{1}{4}PA^{-3} = 0 \ . \tag{10.172}$$

We may write this as

$$A_{xx} + W'(A) = 0 \ , \tag{10.173}$$

where

$$W(A) = \tfrac{1}{2}A^4 + \tfrac{1}{8}(2uv - u^2)A^2 + \tfrac{1}{2}(v - u)P\ln A + \tfrac{1}{8}PA^{-2} \tag{10.174}$$

plays the role of a potential. We can integrate eqn. 10.173 to yield

$$\tfrac{1}{2}A_x^2 + W(A) = E \ , \tag{10.175}$$

where E is second constant of integration. This may be analyzed as a one-dimensional mechanics problem.

The simplest case to consider is $P = 0$, in which case

$$W(A) = \tfrac{1}{2}\big(A^2 + \tfrac{1}{2}uv - \tfrac{1}{4}u^2\big)A^2 \ . \tag{10.176}$$

If $u^2 < 2uv$, then $W(A)$ is everywhere nonnegative and convex, with a single global minimum at $A = 0$, where $W(0) = 0$. The analog mechanics problem tells us that A will oscillate between $A = 0$ and $A = A^*$, where $W(A^*) = E > 0$. There are no solutions with $E < 0$. If $u^2 > 2uv$, then $W(A)$ has a double well shape[6]. If $E > 0$ then the oscillations are still between $A = 0$ and $A = A^*$, but if $E < 0$ then there are two positive solutions to $W(A) = E$. In this latter case, we may write

$$F(A) \equiv 2\big[E - W(A)\big] = \big(A^2 - A_0^2\big)\big(A_1^2 - A^2\big) \ , \tag{10.177}$$

[6] Although we have considered $A > 0$ to be an amplitude, there is nothing wrong with allowing $A < 0$. When $A(t)$ crosses $A = 0$, the phase $\phi(t)$ jumps by $\pm\,\pi$.

where $A_0 < A_1$ and

$$E = -\tfrac{1}{2} A_0^2 \, A_1^2 \tag{10.178}$$

$$\tfrac{1}{4}u^2 - \tfrac{1}{2}uv = A_0^2 + A_1^2 \ . \tag{10.179}$$

The amplitude oscillations are now between $A = A_0^*$ and $A = A_1^*$. The solution is given in terms of Jacobi elliptic functions:

$$\psi(x,t) = A_1 \, \exp\Big[\tfrac{i}{2}u\,(x - vt)\Big] \, \mathrm{dn}\big(A_1(x - ut - \xi_0)\,,\,k\big) \ , \tag{10.180}$$

where

$$k^2 = 1 - \frac{A_0^2}{A_1^2} \ . \tag{10.181}$$

The simplest case is $E = 0$, for which $A_0 = 0$. We then obtain

$$\psi(x,t) = A^* \, \exp\Big[\tfrac{i}{2}u\,(x - vt)\Big] \, \mathrm{sech}\big(A^*(x - ut - \xi_0)\big) \ , \tag{10.182}$$

where $4A^{*2} = u^2 - 2uv$. When $v = 0$ we obtain the stationary breather solution, for which the entire function $\psi(x,t)$ oscillates uniformly.

10.3.4 Dark solitons for NLS(−)

The small oscillations of NLS(−) are stable, as we found in our phonon calculation. It is therefore perhaps surprising to note that this system also supports solitons. We write

$$\psi(x,t) = \sqrt{\rho_0}\, e^{2i\rho_0 t}\, Z(x,t) \ , \tag{10.183}$$

which leads to

$$iZ_t = Z_{xx} + 2\rho_0\big(1 - |Z|^2\big)Z \ . \tag{10.184}$$

We then write $Z = X + iY$, yielding

$$X_t = Y_{xx} + 2\rho_0\big(1 - X^2 - Y^2\big)Y \tag{10.185}$$

$$-Y_t = X_{xx} + 2\rho_0\big(1 - X^2 - Y^2\big)X \ . \tag{10.186}$$

We try $Y = Y_0$, a constant, and set $X(x,t) = X(x - Vt)$. Then

$$-V X_\xi = 2\rho_0 Y_0\big(1 - Y_0^2 - X^2\big) \tag{10.187}$$

$$0 = X_{\xi\xi} + 2\rho_0\big(1 - Y_0^2 - X^2\big)X \tag{10.188}$$

Thus,

$$X_\xi = -\frac{2\rho_0 Y_0}{V}\,\big(1 - Y_0^2 - X^2\big) \tag{10.189}$$

from which it follows that

$$\begin{aligned} X_{\xi\xi} &= \frac{4\rho_0 Y_0}{V}\, X X_\xi \\ &= -\frac{8\rho_0^2 Y_0^2}{V}\big(1 - Y_0^2 - X^2\big) X = \frac{4\rho_0 Y_0^2}{V^2}\, X_{\xi\xi} \ . \end{aligned} \tag{10.190}$$

Thus, in order to have a solution, we must have

$$V = \pm 2\sqrt{\rho_0}\, Y_0 \ . \tag{10.191}$$

We now write $\xi = x - Vt$, in which case

$$\pm\sqrt{\rho_0}\, d\xi = \frac{dX}{1 - Y_0^2 - X^2} \ . \tag{10.192}$$

From this point, the derivation is elementary. One writes $X = \sqrt{1 - Y_0^2}\, \widetilde{X}$, and integrates to obtain

$$\widetilde{X}(\xi) = \mp \tanh\Big(\sqrt{1 - Y_0^2}\ \sqrt{\rho_0}\,(\xi - \xi_0)\Big) \ . \tag{10.193}$$

We simplify the notation by writing $Y_0 = \sin\beta$. Then

$$\psi(x,t) = \sqrt{\rho_0}\, e^{i\alpha}\, e^{2i\rho_0 t}\bigg[\cos\beta\ \tanh\Big(\sqrt{\rho_0}\ \cos\big(\beta(x - Vt - \xi_0)\big)\Big) - i\sin\beta\bigg] \ , \tag{10.194}$$

where α is a constant. The density $\rho = |\psi|^2$ is then given by

$$\rho(x,t) = \rho_0\bigg[1 - \cos^2\!\beta\ \mathrm{sech}^2\Big(\sqrt{\rho_0}\ \cos\big(\beta(x - Vt - \xi_0)\big)\Big)\bigg] \ . \tag{10.195}$$

This is called a *dark soliton* because the density $\rho(x,t)$ is minimized at the center of the soliton, where $\rho = \rho_0 \sin^2\!\beta$, which is smaller than the asymptotic $|x| \to \infty$ value of $\rho(\pm\infty, t) = \rho_0$.