Physics 215C – Problem Set #2
due Monday May 4

1. The linear σ model contains the field $\Sigma \equiv \sigma + i\tau^a\pi^a$, which transforms under chiral $SU(2)_L \times SU(2)_R$ symmetry by $\Sigma \to L\Sigma R^\dagger$, where $L = e^{i\alpha_L{}^a\tau^a/2}$ and $R = e^{i\alpha_R{}^a\tau^a/2}$ are general $SU(2)_L$ and $SU(2)_R$ transformations, respectively. The Lagrangian of the model also contains the nucleon doublet

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix},$$

which transforms as $\Psi_L \to L\Psi_L$ and $\Psi_R \to R\Psi_R$ under chiral symmetry. The chiral Lagrangian is

$$\mathcal{L} = i\bar{\Psi}\gamma_\mu\partial^\mu\Psi - g\left(\bar{\Psi}_L\Sigma\Psi_R + \bar{\Psi}_R\Sigma^\dagger\Psi_L\right) + \frac{1}{2}\mathrm{Tr}\ \partial_\mu\Sigma\partial^\mu\Sigma^\dagger - \mathrm{Tr}\ V\left(\Sigma^\dagger\Sigma\right)\ .$$

(a) Show that the 4 scalar fields σ and π^a, $a = 1, 2, 3$, transform as

$$\delta\sigma = (\alpha_A)^a\pi^a,$$

$$\delta\pi^a = -(\alpha_A)^a\sigma - \epsilon^{abc}(\alpha_V)_b\pi_c,$$

where

$$\alpha_V{}^a \equiv \frac{1}{2}\left(\alpha_L{}^a + \alpha_R{}^a\right),$$

$$\alpha_A{}^a \equiv \frac{1}{2}\left(\alpha_R{}^a - \alpha_L{}^a\right),$$

are the parameters describing $SU(2)_V \times SU(2)_A$ transformations.

(b) Show that the chiral transformation is equivalent to

$$\Psi \to e^{i(\alpha_V{}^a + \gamma_5\alpha_A{}^a)\tau^a/2}\ \Psi.$$

(c) Deduce the $SU(2)_V \times SU(2)_A$ algebra from the $SU(2)_L \times SU(2)_R$ algebra.

(d) Find the Nöther currents associated with $SU(2)_L$, $SU(2)_R$, $SU(2)V$ and $SU(2)_A$ transformations.

(e) Show that after spontaneous symmetry breakdown, when $\langle\sigma\rangle = f_\pi$, the axial current J_A^μ acting on a one-pion state gives a non-vanishing matrix element with the vacuum, i.e.

$$\langle 0|J_A^{\mu\,a}|\pi^b\rangle = if_\pi p^\mu\delta^{ab}$$

where p^μ is the pion momentum. Thus, there is a non-zero amplitude for a pion to “disappear” into the vacuum.