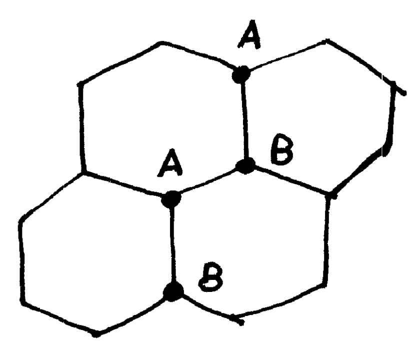
Final Exam Quantum mechanics (III) 212C

1. Berry curvature, Chern number, Quantum anomalous Hall effect.

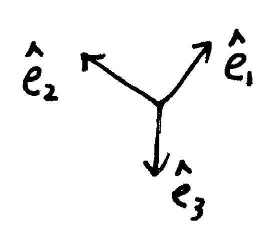
Quantum Hall effect is an extremely important topic in condensed matter physics on which two Nobel Prizes have been issued. Although fractional quantum Hall effect is closely related to Landau levels, Haldane showed that Landau level is not necessary for integer quantum Hall effect. Such an effect without Landau level but from non-trivial topological band structures (Landau level is only one convenient example), is call quantum anomalous Hall effect. Below we will use the toy-model defined in the Honey-comb lattic to illustrate this point.

Again we follow the honeycomb lattice defined in the midterm.

But now we assume A and B sublattices are with different on site potential $\varepsilon_A = -\varepsilon_B = \Delta$. The Hamiltonian in the tight binding reads



$$H_0 = -t \sum_{\substack{i \in A,\\ j=1,2,3}} \{ C_i^\dagger C_{i+\hat{e}_j a} + h.c. \}$$



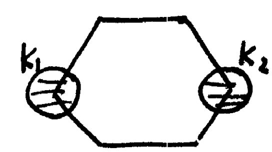
$$+ \Delta \sum_{i \in A} C_i^\dagger C_i - \Delta \sum_{i \in B} C_j^\dagger C_j$$

1. Follow the same definition in the Middle term, prove that

$$H_0 = \sum_{k \in BZ} \left(C^\dagger_{A,k} \;\; C^\dagger_{B,k} \right) H(k) \begin{pmatrix} C_{A,k} \\ C_{B,k} \end{pmatrix}, \quad \text{where}$$

$$H(k) = a(k)\tau_1 + b(k)\tau_2 + c(k)\tau_3 .$$

Prove that when you expand $a(k)$, $b(k)$, $c(k)$ around K_1 and K_2 you get two massive Dirac cones. The mass term comes from $C(k)$. The two mass terms are with the same sign and $\propto \Delta$.



2. assume Δ is small, calculate the Berry $\overbrace{\text{curvatures}}^{\Omega_-(k_x,k_y)}$ around K_1 and K_2 for the low energy band, where their distributions concentrate. Show that the flux $\iint_{\text{around } K_1} dk_x dk_y \, \Omega_-(k_x, k_y)$ and $\iint_{\text{around } K_2} dk_x dk_y \, \Omega_-(k_x, k_y)$

are opposite in sign, and their magnitude is $\to \pi$.

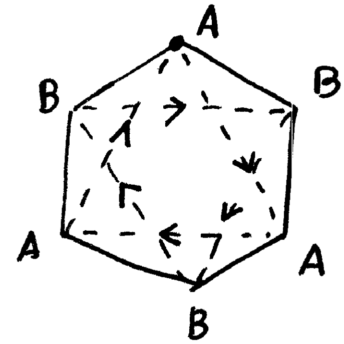
The total flux in the BZ has to be quantized

$$\iint_{BZ} dk_x dk_y \, \Omega_-(k_x, k_y) = 2\pi C,$$

what is the value of the Chern number that you can infer?

Suppose you increase Δ, will "C" change or not? Why?

Now let us set $\Delta = 0$, but add a next-nearest-neighbour (NNN) hopping which is complex-valued, thus breaks time-reversal symmetry.



Following the dashed bonds (NNN), along the direction of the arrows $t' = |t'| e^{i\delta}$, δ is a const and $\delta \neq 0$ and π. Along the opposite direction of the arrows, the hopping amplitude $t'^{*} = |t'| e^{-i\delta}$.

The new hopping Hamiltonian changes into

$$H_0' = -t \sum_{\substack{i \in A \\ j \in 1,2,3}} \{ c_i^{\dagger} c_{i+\hat{e}_j a} + h.c. \}$$

$$- |t'| \sum_{\substack{i \in A \\ j \in NNN}} e^{i\theta_{ij}} c_i^{\dagger} c_j + h.c \quad - |t'| \sum_{\substack{i \in B \\ j \in NNN}} \{ e^{i\theta_{ij}} c_i^{\dagger} c_j + h.c. \}$$

where $\theta_{ij} = \delta$ if $j \to i$ follows the arrows.

$-\delta$ if $j \to i$ is against the arrow directions

2 redo the calculation in 1, show

$$H(k) = a(k)\tau_1 + b(k)\tau_2 + c(k)\tau_3 + d(k)\, I$$

where I is the 2×2 identity matrix. Expand $a(k), b(k), c(k)$ around $K_{1,2}$, Again you will get two massive Dirac cones at K_1, K_2. The mass terms arise from $C(k)$, which proportional to $|t' \sin\delta|$, but are with opposite signs.

4 assume $|t' \sin\delta| \ll t$, redo the calculation in 2.

what is the conclution for the Chern number for the lower energy band

$$\iint_{BZ} dk_x dk_y \, \Omega_-(k_x, k_y) = 2\pi C_- \, ?$$

5 Compare the two different situations in 2 and 4.

From the results of HW 3.

this is important!

$$\sigma_{xy} = \frac{e^2}{\hbar} \iint \frac{dk_x dk_y}{(2\pi)^2} \, f(\varepsilon_+)\Omega^+ + f(\varepsilon_-)\Omega^-.$$

Consider the situation of a band insulater in which $f(\varepsilon_+) = 0$ and $f(\varepsilon_-) = 1$. (the higher band is empty, and the lower band is ful-filled),

Prove $\sigma_{xy} = \dfrac{e^2}{h} \cdot C_-$.

In which case 2 or 4, you get non-zero, quantized σ_{xy}?
Prob Prob

2. Anomaly (Chiral)

For Dirac particles in the even space-time dimensions, if they are massless, then γ_5 is a good quantum number. We denote ψ_R and ψ_L as eigenstates of γ_5 with eigenvalue ± 1. Here we will show with the appearance of external E-M field. The particle number of $\psi_R^\dagger \psi_R$ and $\psi_L^\dagger \psi_L$ are no-longer conserved at the quantum level. This phenomenon is called chiral anomaly. "Chirality" here means the eigenstates of γ_5, which is different from parity.

① 1+1 dimension. $H = c\vec{\alpha}\cdot\vec{p} + \beta m$, where

we choose

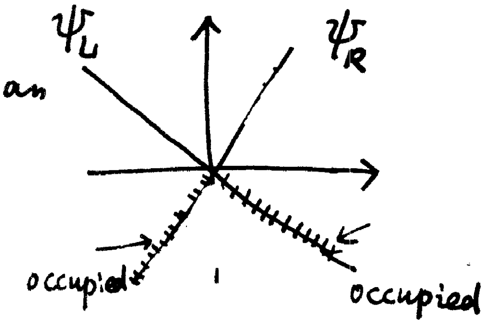
$\beta = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\alpha = i\beta\sigma_2 = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$

$\gamma_5 = (i)\sigma_1\sigma_2 = -\sigma_3$

thus if $m=0$, $H = \begin{pmatrix} -cp & 0 \\ 0 & +cp \end{pmatrix}$

and γ_5 is conserved. The component of eigenvalues of $\gamma_5 = \pm 1$, are called right/left movers.



In the second quantized representation, Hamiltonian reads

$$H = \int_{-L}^{+L} dx\, \psi^\dagger \left[c(-i\hbar\nabla_x)\alpha \right] \psi$$

$$= \int_{-L}^{+L} dx \underbrace{-i}_{c} \left[\psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L \right],$$ where $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ is the field operator for Dirac particles

Now let us add the gauge field A_x, but set $A_t = 0$.

$$H = \int_{-L}^{+L} -ic \, [\psi_R^{\dagger} (\partial_x - \frac{ie}{\hbar c} A_x) \psi_R - \psi_L^{\dagger} (\partial_x - \frac{ie}{\hbar c} A_x) \psi_L .$$

For a const A_x, it's easy to diagonalize the spectrum, the eigenstates are $\begin{pmatrix} e^{ik_n x} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ e^{ik_n x} \end{pmatrix}$ with $k_n = \frac{2\pi n}{L}$ $n = -\infty, \cdots \infty$

where "L" is the length of system and we use the peroidical boundary condition. Show that the spectra read

$$\psi_R : \quad E_n = (k_n - eA_x)$$

$$n = -\infty, \cdots 0, \cdots \infty$$

$$\psi_L : \quad E_n = -(k_n - eA_x)$$

To find the ground state of such a system, we fill all the negative energy levels and interpret holes in the filled states as anti-particles.

Now let us adibatically change the value of A_x. Prove that if A_x changes $\frac{2\pi}{L} \cdot \frac{\hbar c}{e}$, the spectra comes back to its original from. Show that In this process, each level of ψ_R moves down to the next position, and each level of ψ_L moves up to the next position.

Show that $$N_R - N_L = \int_{-L}^{L} dx \int_{t_{initial}}^{t_{final}} dt \quad \frac{e}{2\pi} \frac{1}{\hbar c} \epsilon^{\mu\nu} F_{\mu\nu} \qquad (*)$$

where $N_R = \int dx \;\; \psi_R^\dagger(x)\, \psi_R(x)$, $\quad N_L = \int dx \;\; \psi_L^\dagger(x)\, \psi_L(x)$.

In other words. N_R, N_L are no longer conserved. Prove that the meaning of the equation (*) is just

$$\boxed{\partial_t \underbrace{j}_{\text{electric current}} = eE}$$

② 3+1 dimension Chiral anomaly, we choose

$$\gamma_{1,2,3} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\vec{\alpha} = \gamma^0 \vec{\gamma} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}.$$

$$H_D = -i\hbar\, \vec{\alpha} \cdot \vec{\nabla} = \begin{pmatrix} i\hbar \vec{\nabla}\cdot\vec{\sigma} & 0 \\ 0 & -i\hbar \vec{\nabla}\cdot\vec{\sigma} \end{pmatrix} \leftarrow \text{massless Dirac particles}$$

and the second quantized form

$$H = \int d^3x \quad \psi_R^\dagger (-i\hbar \vec{\sigma}\cdot\nabla)\, \psi_R - \psi_L^\dagger (-i\hbar \vec{\sigma}\cdot\nabla)\, \psi_L$$

a) Write down the second quantized form of Hamiltonian in the external E-M field.

b) Focus on the "R"-branch, solve the eigen-value problem

$$-i\hbar\, \vec{\sigma}\cdot\vec{D}\, \psi_R = E\, \psi_R \quad , \quad \text{where } \vec{D} = \vec{\partial} - \frac{ie}{\hbar c}\vec{A}.$$

For the background field, we choose $A_0 = 0$, $A_x = 0$, $A_y = Bx$, $A_z = Et$

check these give $B_x = B_y = 0$ $B_z = B$; $E_x = E_y = 0$; $E_z = E$.

The system is homogenous along "y" and "z" direction, write down

$$\psi_R = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} e^{i(k_y y + k_z z)} \quad ,$$

prove that the eigen-equation changes to (we also set $E=0$ here)

$$\begin{pmatrix} k_z \, , & -i\partial_x - i(k_y - eBx) \\ -i\partial_x + i(k_y - eBx) \, , & -k_z \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = E \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}.$$

- Solve the eigen-spectrum for ψ_R, and also eigen vector.

Hint: you can define

$$a^\dagger = \frac{l}{\sqrt{2}}\left(-i\partial_x - i(k_y - eBx)\right) \quad \text{where } l^{-2} = \frac{eB}{\hbar c}$$

$$a = \frac{l}{\sqrt{2}}\left(-i\partial_x + i(k_y - eBx)\right)$$

if $(eB > 0)$.

Check that a, a^{+} do satisfy $[a, a^{+}] = 1$ for $eB > 0$.

Express the eigen-equation in terms of a^{+}, a, and express the eigen-vector of $(-i\hbar\vec{\sigma}\cdot\vec{D} - E)\psi_R = 0$ in terms of $|n\rangle$, which satisfies $a^{+}a|n\rangle = n|n\rangle$, and Solve the eigen-spectra.

Prove that the eigen-spectra are

for $n=0$, $E(k_z, k_y) = k_z$

$n = 1, 2, 3, \cdots$ $\quad E(k_z, k_y) = \pm\sqrt{k_z^2 + \frac{2n}{\ell^2}}$

Suppose the system is in a box of size of L, what's the degeneracy of each level?

C) Do the same calculation for the left handed fermion ψ_L.

Compare the result with that in b).

d) Let us turn on A_z, which corresponding to $k_z \to k_z - \frac{eA_z}{c}$.

What's the pattern of spectra flow?

Suppose change A_z by an amount of $\frac{2\pi}{eL}\frac{1}{\hbar c}$, then the system spectra return back to its inital position. Prove the number of right-mover $\left(\underbrace{\int \psi_R^{\dagger}\psi_R d^3x}_{=N_R}\right)$ and left-movers $\left(\int \psi_L^{\dagger}\psi_L d^3x = N_L\right)$.

Changes $\Delta(N_R - N_L) = -\frac{e^2L^3}{2\pi^2}\vec{B}\cdot\vec{E} = -\frac{e^2}{2\pi^2}\int_{-L}^{L} d^3x \int dt\, \vec{E}\cdot\vec{B}$, up to a coefficient of $\hbar, c$. Because $\vec{E}\cdot\vec{B} \propto \varepsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}$, the above result can be written as

$$\Delta(N_R - N_L) \propto \int d^4x \ \varepsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}.$$

e) From the results of Chiral anomaly in 1+1 D and 3+1 D.

$$\Delta(N_R - N_L) \propto \int d^2x \ \varepsilon^{\mu\nu} F_{\mu\nu}$$

$$\Delta(N_R - N_L) \propto \int d^4x \ \varepsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho},$$

Can you guess what's the general result for $(2n-1)+1$ Dimesion?

$$\Delta(N_R - N_L) \propto \int d^{2n}x \ \ ???$$

You do not need to prove.