Phy 212 C. HW 1

1. Consider a bosonic system whose single particle basis is denoted as $\psi_\alpha, \psi_\beta, \cdots$. The particle number in each state is denoted $n_\alpha, n_\beta, \cdots$. The many-body eigenstates in the particle-number occupation representation is denoted $|n_\alpha n_\beta \cdots\rangle$. We use $a_\alpha^\dagger$ and $a_\alpha$ as the creation and annhilation operators for the state $\alpha$, which satisfies

$[a_\alpha, a_\beta^\dagger] = i\delta_{\alpha\beta}$.

a) Prove
$$a_\alpha^\dagger |n_\alpha n_\beta \cdots\rangle = \sqrt{n_\alpha+1}\ |n_\alpha+1, n_\beta, \cdots\rangle$$
$$a_\alpha |n_\alpha n_\beta \cdots\rangle = \sqrt{n_\alpha}\ |n_\alpha-1, n_\beta, \cdots\rangle$$

b) let $\hat{n}_\alpha = a_\alpha^\dagger a_\alpha$, Prove $\hat{n}_\alpha |n_\alpha n_\beta \cdots\rangle = n_\alpha |n_\alpha n_\beta \cdots\rangle$

c) Prove $$|n_\alpha \cdot n_\beta \cdots\rangle = \frac{1}{\sqrt{n_\alpha! n_\beta! \cdots}} (a_\alpha^\dagger)^{n_\alpha} (a_\beta^\dagger)^{n_\beta} \cdots |0_\alpha 0_\beta \cdots\rangle$$

## Schwinger Boson:



(2) Let us consider two independent harmonic oscillators to form a system. Let us use $n_1$, $n_2$ to denote the number of phonos, and $a_1^\dagger, a_1, a_2^\dagger, a_2$ for the creation / annhilation operators, $n_1 = a_1^\dagger a_1$, $n_2 = a_2^\dagger a_2$ are phonon number operator. In the phonon number representation, the eigen states basis are denoted as $|n_1 n_2\rangle$. We write in the matrix form $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $a^\dagger = (a_1^\dagger, a_2^\dagger)$

and define $\vec{J} = \frac{1}{2} a_\alpha^\dagger \vec{\sigma}_{\alpha\beta} a_\beta$, where $\vec{\sigma}$ is the usual Pauli matrix.

1) define $J_\pm = J_x \pm i J_y$, show $J_+ = a_1^\dagger a_2$, $J_- = a_2^\dagger a_1$

2) Prove $[J_z, J_\pm] = \pm J_\pm$, and $[J_+, J_-] = 2J_z$, which are the angular momentum algebra.

3) Calculate $J^2 = J_x^2 + J_y^2 + J_z^2$, and express it in terms of $n_1$ and $n_2$.

4) Show that the state $|n_1 n_2\rangle$ is a eigenstate of $J^2$, $J_z$ with $j = \frac{1}{2}(n_1 + n_2)$ and $j_z = \frac{1}{2}(n_1 - n_2)$,

5) Prove the identity in angular momentum theory $J_\pm |jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\ |j, m \pm 1\rangle$ by using the Schwinger bosonic representation defined above.

6) Define operators $K_+ = a_1^\dagger a_2^\dagger$, $K_- = a_1 a_2$, calcute their effects when acting on $|n_1 n_2\rangle$, and then interpreate in the angular momentum representation.

3: Explicitly verify the equation of continuity

$$\frac{\partial}{\partial t}\rho(r,t) + \nabla\cdot\vec{j}(r,t) = 0$$ for a free Fermion gas.

The Hamiltonian $H_0 = \sum_{k\sigma} \varepsilon_k C^{+}_{k\sigma} C_{k\sigma}$. The density operatator is defined as $\rho(r,t) = e^{iHt}\rho(r)e^{-iHt}$, where $\rho(r) = \psi^{+}_{\sigma}(r)\psi(r) = \frac{1}{V}\sum_{kq\sigma} e^{-iqr} C^{+}_{k+q\sigma} C_{k\sigma}$

The current operator $j(r,t) = e^{iHt} j(r) e^{-iHt}$ and

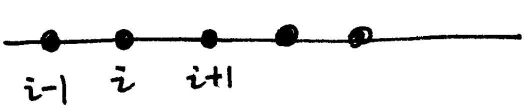
$$j(r,t) = \frac{-i\hbar}{m}\left[\psi^{+}_{\sigma}(r)\nabla\psi_{\sigma}(r) - (\nabla\psi^{+}_{\sigma}(r))\psi(r)\right]$$

$$= \frac{1}{V}\sum_{kq\sigma} e^{-iqr}\frac{(\vec{k}+\frac{1}{2}\vec{q})}{m} C^{+}_{k+q\sigma} C_{k\sigma}.$$

4: Do the Cooper problem again as in the lecture notes and find the gap value. (If you had done this problem 60 years before, you would be awarded the Nobel prize).

5: Tight binding model. Consider in solid, we denote $c^{+}_{i}, c_i$ for the creation/annhillation operators for electrons on site "i".

The hopping Hamitonian is




$$H = -t\sum_{i\sigma} C^{+}_{i\sigma} C_{i+1\sigma}.$$

By introducing the discrete Fourier transform to diagnalize the spectra $H = \sum_{k\sigma}\varepsilon(k) C^{+}_{k\sigma} C_{k\sigma}$, where $C_k = \frac{1}{\sqrt{N}}\sum_i e^{i\vec{k}\cdot\vec{r}_i} C_i$.

number of sites