Lect 7 Analytic properties of Scattering amplitude

§ Radial solutions

$u_l(k, r) = r R_l(k, r)$ should be even functions of k because the radial equation only involves k^2. $u_l(k, r)$ is a standing wave like solution.

For later convenience, we introduce the propagating wave solution

$$f_l(k, r) \xrightarrow{r \to +\infty} i^l e^{-ikr} .$$

Then the solution $f_l(-k, r)$ is another linearly independent solution

$$f_l(-k, r) \xrightarrow{r \to +\infty} i^l e^{ikr} .$$

Due to reality of the radial equation, we have $\boxed{[f_l(-k, r)]^* = (-)^l f_l(k, r)}$.

So far, we take k to be real, if we consider k as complex variable

$$\Rightarrow \quad [f_l(-k^*, r)]^* = (-)^l f_l(k, r) .$$

For $u_l(k, r)$ it satisfies $[u_l(k^*, r)]^* = u_l(k, r)$.

for complex values of k.

§ Jost function

$u_\ell(k,r)$ can be expressed as a linear combination of $f_\ell(\pm k, r)$ as

up to a factor difference

$$u_\ell(k,r) = \frac{i}{2} k^{-\ell-1} [\underbrace{\tilde{f}_\ell(-k)}_{\text{coefficient}} f_\ell(k,r) - (-)^\ell \underbrace{\tilde{f}_\ell(k)}_{\text{coefficient}} f_\ell(-k,r)]$$

Let us compare the asymptotic solution

$$u_\ell(k\, r) = r R_\ell(k.r) \xrightarrow{r\to+\infty} \frac{1}{2k} i^{\ell+1} e^{i\delta_\ell} [e^{-ikr - i\delta_\ell} - (-)^\ell e^{ikr + i\delta_\ell}]$$

it follows that the coeffients $\tilde{f}_\ell(-k)$ and $\tilde{f}_\ell(k)$ are related to the scattering matrix defined as

$$S_\ell(k) = e^{2i\delta_\ell(k)} = 1 + 2ik f_\ell(k) = \frac{\tilde{f}_\ell(k)}{\tilde{f}_\ell(-k)} .$$

$f_\ell(\pm k)$ are called Jost functions as a function of complex varible k.

Take the complex conjugate of $u_\ell(k,r)$

$$u_\ell^*(k,r) = -\frac{i}{2} (k^*)^{-\ell-1} [\tilde{f}_\ell^*(-k) f_\ell^*(k,r) - (-)^\ell \tilde{f}_\ell^*(k) f_\ell^*(-k,r)]$$

$$= -\frac{i}{2} (k^*)^{-\ell-1} [\tilde{f}_\ell^*(-k) (-)^\ell f_\ell(-k^*,r) - \tilde{f}_\ell^*(k) f_\ell(k^*,r)]$$

$u_\ell^*(k,r)$ should be $u_\ell(k^*,r)$

$$= \frac{i}{2} (k^*)^{-\ell-1} [\tilde{f}_\ell(-k^*) f_\ell(k^*,r) - (-)^\ell \tilde{f}_\ell(k^*) f_\ell(-k^*,r)]$$

$$\Rightarrow \boxed{\tilde{f}_\ell(-k^*) = \tilde{f}_\ell^*(k)}$$

for real k $\Rightarrow$ $\tilde{f}_l(-k) = \tilde{f}_l^{*}(k)$, thus $|S_l(k)| = 1$ as required by the unitarity of the S-matrix. The phase of $\tilde{f}_l(k) = |\tilde{f}_l(k)|\, e^{i\delta_l(k)}$ is just the phase shift.

§ Bound states

$k^2 < 0$ but is real, $k = \pm i\chi$ $(\chi > 0)$, we need

$$u_l(i\chi, r) \xrightarrow[r \to +\infty]{} \frac{i}{2}(i\chi)^{-l-1}\, \tilde{f}_l(-i\chi)\, i^l e^{\chi r} - \frac{i}{2}(i\chi)^{-l-1}(-)^l\, \tilde{f}_l(i\chi)\, i^l e^{-\chi r}$$

$$= -\frac{1}{2}(-)^l (\chi)^{-l-1}\, \tilde{f}_l(i\chi)\, e^{-\chi r}$$

$$= \frac{1}{2}(-\chi)^{-l-1}\, \tilde{f}_l(i\chi)\, e^{-\chi r}$$

$\boxed{\text{we need } \tilde{f}_l(-i\chi) = 0\, ;\ \chi > 0}$

Similarly

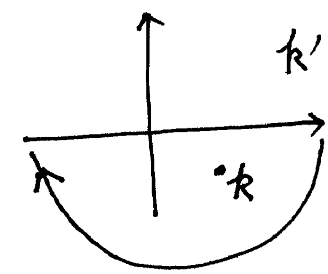
$$u_l(-i\chi, r) \xrightarrow{r \to +\infty} \frac{i}{2}(-i\chi)^{-l-1}\left[\tilde{f}_l(i\chi)\, i^l e^{-\chi r} - (-)^l\, \tilde{f}_l(-i\chi)\, i^l e^{\chi r}\right]$$

$$\ldots^{-l-1}\, \tilde{f} \ldots e^{-\chi r}$$

§ Dispersion relation for the Jost function $\tilde{f}_e(k)$

It can be shown that as $k \to +\infty$, $\tilde{f}_e(k) - 1$ is analytic for $Im\, k \leq 0$ for short range potentials, and $\tilde{f}_e(k) - 1 \to \frac{1}{k}$ as $|k| \to +\infty$. By Cauchy's theorem $\Rightarrow$

$$\tilde{f}_e(k) - 1 = -\frac{1}{2\pi i} \int_c \frac{\tilde{f}_e(k') - 1}{k' - k + i\eta} dk', \qquad Im\, k < 0.$$



$f_e(k')-1$

i.e.

$$Re[\tilde{f}_e(k) - 1] = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{Im[\tilde{f}_e(k') - 1]}{k' - k} dk'$$

$$Im[\tilde{f}_e(k) - 1] = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{Re[\tilde{f}_e(k') - 1]}{k' - k} dk'$$

we need another dispersion relation for $\ln \tilde{f}_e(k)$,

$$\ln \tilde{f}_e(k) = \ln|\tilde{f}_e(k)| + i\delta_e(k)$$

if we assume no bound states, then $\ln f_e(k)$ is also analytic since ~~it~~ $\tilde{f}_e(k)$ has no zeros.

$$\Rightarrow \ln|\tilde{f}_e(k)| = Re[\ln \tilde{f}_e(k)] = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\delta_e(k')}{k' - k} dk$$

$$\ln|\tilde{f}_e(k)| + i\delta_e(k) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\delta_e(k')}{k' - k} dk + i\delta_e(k)$$

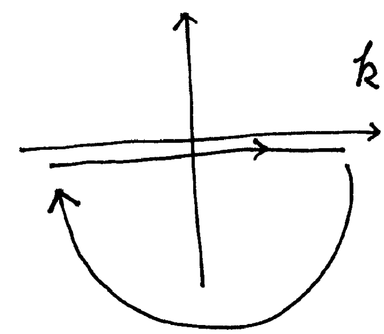
$$= \boxed{-\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\delta_e(k')}{k' - k + i\eta} dk' = \ln \tilde{f}_e(k)}$$

if no bound state

§ Levinson Theorem

We need to build up the connection between the number of bound states of a given l and the corresponding phase shift $\delta_l(0)$ at zero energy. If we assume the Jost function defined as the limit from the positive k.

$|\tilde{f}_l(0)| \neq 0$, for the contour integral



$$-\frac{1}{2\pi i}\int_c \frac{\tilde{f}_l'(k)}{\tilde{f}_l(k)}\, dk = -\frac{1}{2\pi i}\int_c d\ln \tilde{f}_l(k)$$

The integrand has simple poles at zeros of $\tilde{f}_l(k)$, i.e. bound states.

The LHS just gives the numer of bound states n_l.

$$\ln \tilde{f}_l(k) = \ln|\tilde{f}_l(k)| + i\,\delta_l(k),$$

$\ln|\tilde{f}_l(k)|$ is continuous because $|\tilde{f}_l(k)|$ is never zero or diverge on the contour. $\Rightarrow \oint_c d\ln|\tilde{f}_l(k)| = 0$.

However $\delta_l(k)$ has discontinuity at $k=0$. Due to

$$\tilde{f}_l(-k^*) = \tilde{f}_l^*(k) \quad \text{and} \quad \tilde{f}_l(k) = |\tilde{f}_l(k)|\, e^{i\delta_l(k)}$$

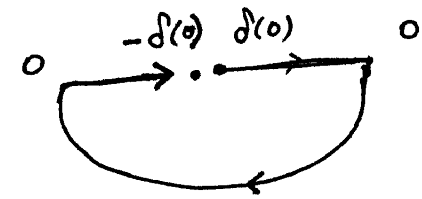
$\Rightarrow$ for k on Real axis $\quad \tilde{f}_l(-k) = \tilde{f}^*(k) = |\tilde{f}_l(k)|\, e^{-i\delta_l(k)}$

i.e. $\boxed{\delta_l(-k) = -\delta_l(k)}$ for $k \neq 0$, also $\delta_l(k) \xrightarrow[k\to\infty]{} 0$ ← high energy scattering

~~if $|\tilde{f}_\ell(0)| \neq 0$, then $\delta_\ell(0)$ is well defined~~

Thus as $k \to 0$, there is a discontinuity of $\delta_\ell(k)$ if $\delta_\ell(k=0) \neq 0$.

$$\Rightarrow \quad -\frac{1}{2\pi i}\oint_c d\ln \tilde{f}_\ell(k) = \frac{i}{-2\pi i}\left(-\delta_\ell(0) \quad -\delta_\ell(0)\right)$$



To maintain the analyticity of $\tilde{f}_\ell(k)$, we need

$\boxed{\delta_\ell(0) \text{ and } -\delta_\ell(0) \text{ differ from each other by } 2n\pi}$

$$\Rightarrow \quad 2\delta_\ell(0) = 2n_\ell\pi \qquad \text{if} \quad f_\ell(0) \neq 0$$

i.e $\delta_\ell(0) = n_\ell\pi$ which agree with the picture we develpoed before.

if $\tilde{f}_\ell(0) = 0$, then the situation is more complicated. Please refer shiff's book. The conclusion is

$$\delta_\ell(0) = \pi(n_\ell + 1/2) \quad \text{if} \quad \tilde{f}_\ell(0) = 0 \text{ and } \ell = 0$$

otherwise $\quad \delta_\ell(0) = n_\ell\pi$

§ Effective interaction range

Let us consider an explicit example of the Jost function

$$\tilde{f}_0(k) = \frac{k+i\chi}{k-i\alpha}$$ which has the right asympotutic behavior

$\tilde{f}_0(k) - 1 \xrightarrow{k\to+\infty} 1/k$. It has zero at $k=-i\chi$ corresponding to bond states with energy $-\frac{\hbar^2\chi^2}{2m}$.

$$\tilde{f}_0(k) = \left(\frac{k^2+\chi^2}{k^2+\alpha^2}\right)^{1/2} e^{i\delta_0(k)}$$, the phase shift $\delta_0(k) = \tan^{-1}\frac{\chi}{k} + \tan^{-1}\frac{\alpha}{k}$

$$\Rightarrow \tan\delta_0(k) = \frac{k(\chi+\alpha)}{k^2-\chi\alpha}$$

$$k \operatorname{ctg}\delta_0(k) = k\,\frac{k^2-\chi\alpha}{k(\chi+\alpha)} = -\frac{\chi\alpha}{\chi+\alpha} + \frac{k^2}{\chi+\alpha}$$

For the low energy scattering if we expand to second order of k^2

$$\boxed{k \operatorname{ctg}\delta_0(k) = -\frac{1}{a} + \frac{1}{2} r_0 k^2,}$$ where r_0 is called interaction range.

$$\Rightarrow \quad r_0 = \frac{2}{\chi+\alpha}$$, r_0 is usually small, so α needs to be large.

$$\boxed{\frac{1}{a} = \frac{\chi\alpha}{\chi+\alpha} = \chi\left(1-\frac{\chi}{\chi+\alpha}\right) = \chi - \frac{r_0}{2}\chi^2}$$

correct to the second order of k^2

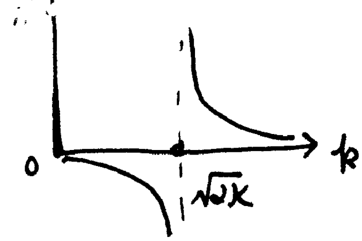
Conside the situation where α is fixed, but χ decreases to zero and becomes negative. At $\chi = 0$, $f_0(0) = 0$, there is a zero energy resonance and the scattering length a divergers. For χ is negative, there is no true bound state, and the scattering length is negative.

For all the three cases, $\delta_0(k)$ increases from zero as k decreases from $+\infty$.

with a bound state,

$\chi > 0$

$$\tan\delta_0(k) \approx \frac{k}{\left(\frac{k^2}{\alpha} - \chi\right)}$$



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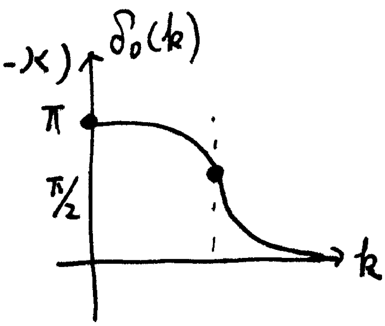
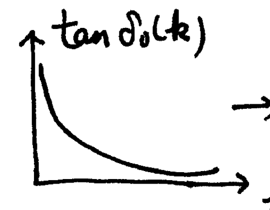


zero energy resonance

$\chi = 0$,

$$\tan\delta_0(k) = \frac{k\alpha}{k^2} = \frac{\alpha}{k}$$



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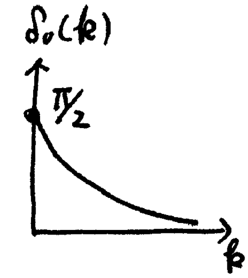
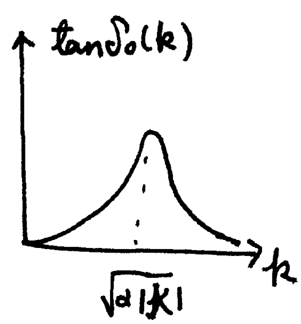


no-bound state

$\chi < 0$

$$\tan\delta_0(k) = \frac{1}{\frac{k}{\alpha} + \frac{|\chi|}{k}}$$



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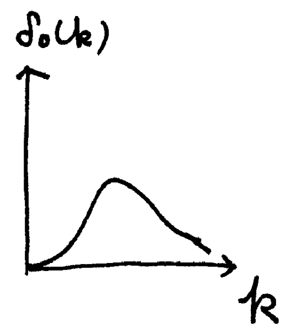
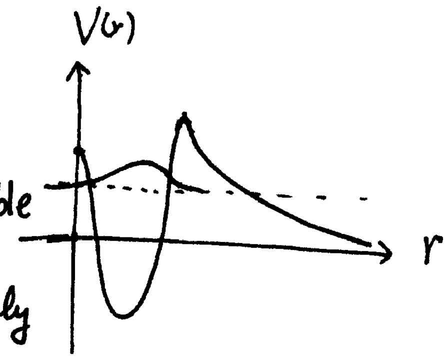


all agree with Levinson theorem.



§ Quasi-bound state

In a system depicted in RHS, it may hold a metastable state which has a finite life time to decay. Rigorously speaking, this is not a stationary state, but called quasi-stationary state.

This corresponds to the following bounding condition that as $r \to +\infty$, we have an out going spherical wave. Since the boundary condition is complex, the eigenvalue isn't necessarily real. Say, we have

$$E = E_0 - \frac{1}{2} i \Gamma \Rightarrow e^{-iEt} = e^{-iE_0 t} e^{-\frac{\Gamma}{2} t} \text{ decay with time.}$$

thus $k = \sqrt{\frac{2mE}{\hbar^2}} = \sqrt{\frac{2m}{\hbar^2}} \sqrt{E_0 - \frac{i}{2}\Gamma} = k_1 - i k_2$

if $\Gamma \ll E_0 \Rightarrow k_1 = \sqrt{\frac{2mE_0}{\hbar^2}}$, $k_2 = \frac{\Gamma}{4E_0} k_1$.

Then the Jost function representation:

$$u_l(k, r) = \frac{i}{2} k^{-l-1} \left[\tilde{f}_l(-k) f_l(k, r) - (-)^l \tilde{f}_l(k) f_l(-k, r) \right]$$

$$\tilde{f}_l(k, r) \xrightarrow{r \to +\infty} i^l e^{-ik_1 r} e^{-k_2 r}$$

$$\tilde{f}_l(-k, r) \xrightarrow{r \to +\infty} i^l e^{ik_1 r} e^{k_2 r}$$

The quasi-stationary condition means the boundary condition that there is no incoming wave, i.e., $\tilde{f}_l(-k) = \tilde{f}_l(k^*) = \underbrace{\tilde{f}_l(k_1 + ik_2)}_{=0}$

Now let me cook an sample Jost function $\tilde{f}_0(k) = \dfrac{k-k_1-ik_2}{k-i\alpha}$

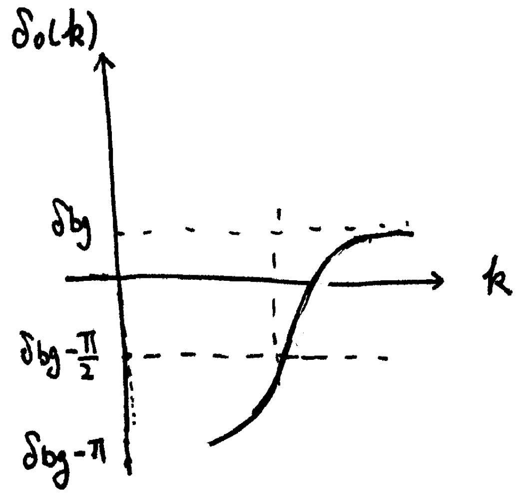
$$\tilde{f}_0(k) = \left[\frac{(k-k_1)+k_2}{k^2+\alpha^2}\right]^{1/2} e^{i\delta_0(k)} \quad \text{where} \quad \delta_0(k) = -\tan^{-1}\frac{k_2}{k-k_1} + \tan^{-1}\frac{\alpha}{k}$$

when the energy approaches $\frac{\hbar^2 k_1^2}{2m}$, the second term behaves as background

$$\delta_0(k) = \delta_{bg} - \tan^{-1}\frac{\Gamma}{2(E-E_0)} \qquad \text{check} \quad \frac{\frac{\Gamma}{2}}{\frac{\hbar^2}{2m}(k+k_p)(k-k_1)}$$

As E passes E_0

$$\approx \frac{\frac{\Gamma k_1}{2\times 2}}{\frac{\hbar^2 k_1^2}{2m}(k-k_1)} = \frac{\frac{\Gamma k_1}{4E_0}}{k-k_1} = \frac{k_2}{k_1-k}$$



The scattering amplitude $f_0(k) = \dfrac{\sqrt{4\pi}}{k\,ctg\,\delta_0 - ik}$. If neglect δ_{bg}

$$\Rightarrow ctg\,\delta_0 = \frac{2(E-E_0)}{-\Gamma} \quad \Rightarrow f_0(k) = \frac{\sqrt{4\pi}}{k}\,\frac{1}{\frac{(E-E_0)}{-\Gamma/2} - i} = \frac{\sqrt{4\pi}}{k}\,\frac{-\Gamma/2}{E-E_0+i\frac{\Gamma}{2}}$$

$$\sigma = \frac{4\pi}{k^2}\,\frac{\left(\frac{\Gamma}{2}\right)^2}{(E-E_0)^2+\left(\frac{\Gamma}{2}\right)^2}$$