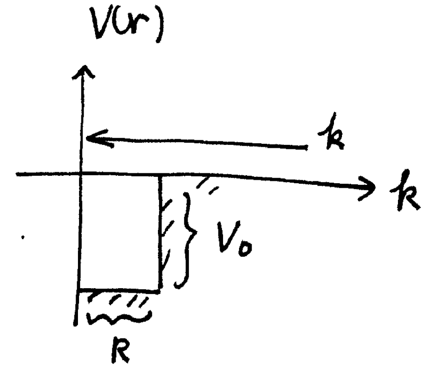
Lect 6.5 More examples and exercises

§ Scattering on the spherical potential well



define $R(kr) = \dfrac{u(kr)}{r}$

$$\begin{cases} \dfrac{d^2u}{dr^2} + (k^2 + k_0^2)\, u = 0 & (r<R) \\[2ex] \dfrac{d^2u}{dr^2} + k^2 u = 0 & (r>R) \end{cases}$$

where $k = \sqrt{\dfrac{2mE}{\hbar^2}}$, $k_0 = \sqrt{\dfrac{2mV_0}{\hbar^2}}$

define $k_1^2 = k^2 + k_0^2$. $\Rightarrow$ $u(r) = \begin{cases} \sin k_1 r & r<R \\ A \sin(kr + \delta_0) & r>R \end{cases}$

From the continuity equation $\left.\dfrac{u'(r)}{u(r)}\right|_{r=R^-} = \left.\dfrac{u'(r)}{u(r)}\right|_{r=R^+}$ $\Rightarrow$

$$k_1 \,\mathrm{ctg}\, k_1 R = k \,\mathrm{ctg}(kR + \delta_0)$$

$$\Rightarrow \quad \mathrm{tg}\,\delta_0 = \frac{k\,\mathrm{tg}(k_1R) - k_1 \mathrm{tg}(kR)}{k_1 + k\,\mathrm{tg}\,k_1R\,\mathrm{tg}\,kR} = \frac{\frac{k}{k_1}\mathrm{tg}(k_1R) - \mathrm{tg}(kR)}{1 + \frac{k}{k_1}\mathrm{tg}(k_1R)\,\mathrm{tg}(kR)}$$

$$= \mathrm{tg}\left[\,\mathrm{tg}^{-1}\left[\frac{k}{k_1}\mathrm{tg}(k_1R)\right] - kR\right]$$

$$\delta_0 = \mathrm{tg}^{-1}\left(\frac{kR}{k_1R}\mathrm{tg}(k_1R)\right) - kR \ , \quad \text{where } k_1 = \sqrt{k^2 + k_0^2}$$

1. let us consider the limit $k \to 0^{+}$, and $k_1 R$ is away from $(n+1/2)\pi$

$$tg^{-1} x = x - \frac{x^3}{3} \Rightarrow tg^{-1}\left[\frac{kR}{k_1 R} tg(k_1 R)\right] = \frac{kR}{k_1 R} tg(k_1 R)$$

$$\delta_0 \sim kR \left(\frac{tg(k_1 R)}{k_1 R} - 1\right] \quad \text{and} \quad \sigma \sim 4\pi R^2 \left(\frac{tg\, k_1 R}{k_1 R} - 1\right)^2$$

2. high energy limit $k \to +\infty$.

$$\frac{k}{k_1} = \frac{1}{\sqrt{1+(k_0/k)^2}} \approx 1 - \frac{k_0^2}{2k^2}, \quad tg(k_1 R) = tg(kR) + \frac{1}{\cos^2 kR} \frac{k_0^2 R}{2k}$$

$$\Rightarrow \frac{k}{k_1} tg(k_1 R) = tg(kR) + \frac{k_0^2 R}{2\cos^2 kR\, k} \quad (\text{keep to } \tfrac{1}{k}\text{'s order})$$

$$tg^{-1}\left[\frac{k}{k_1} tg(k_1 R)\right] = kR + \frac{1}{1+[tg(kR)]^2}\left[\frac{1}{2\cos^2 kR} \cdot \frac{k_0^2 R}{k}\right]$$

$$\to kR + \frac{k_0^2 R}{2k} \quad \text{keep to } \frac{1}{k}\text{'s order}$$

$$\Rightarrow \delta_0 \approx \frac{k_0}{2k}(k_0 R) \quad \text{decays as } \frac{1}{k}.$$

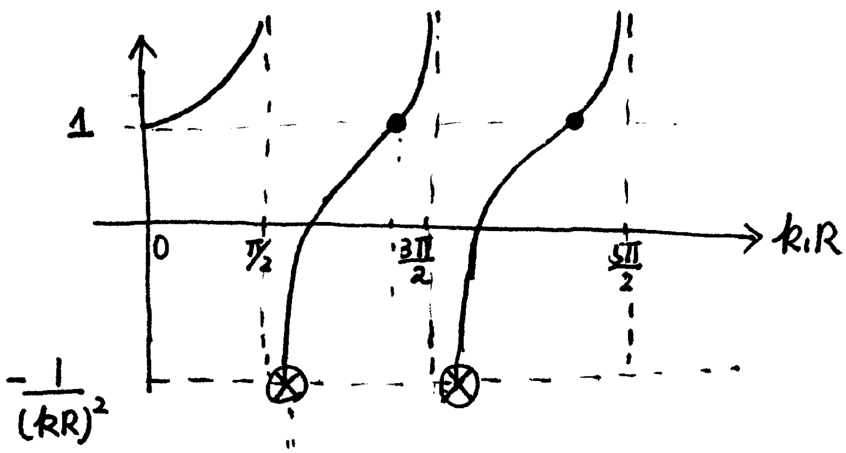
This also make sense: since potential is relatively weak at high energy scattering. However, in this case, we cannot only keep the s-wave channel.

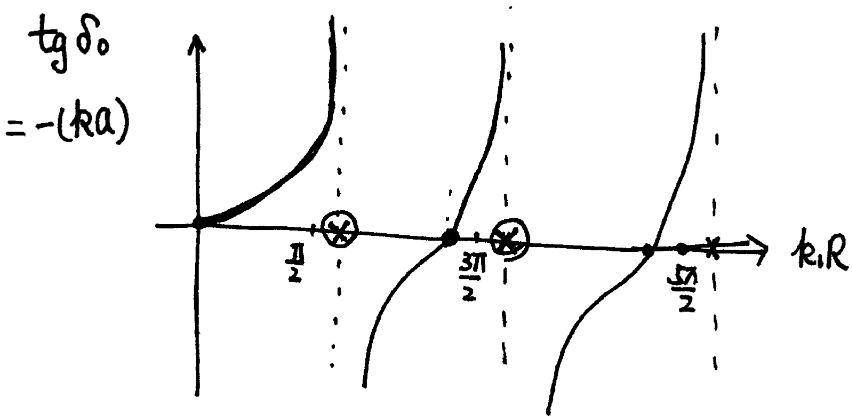
③ low energy limit $k \to 0$

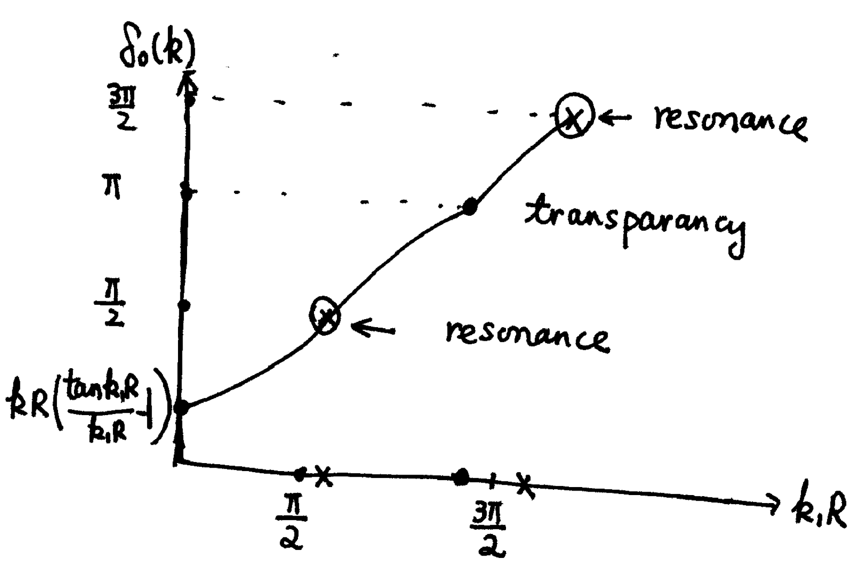
$$\tan\delta_0 = \frac{kR\left[\frac{\tan k_1R}{k_1R} - 1\right]}{1 + (kR)^2\,\frac{\tan k_1R}{k_1R}}.$$

as we increase the potential depth

$$k_1 = \sqrt{k_0^2 + k^2}$$







① The reasonce points ⊗

$$\delta = (n + \tfrac{1}{2})\pi$$

$$\frac{\tan k_1R}{k_1R} = -\frac{1}{(kR)^2}$$

$$\cot k_1R = -\frac{(kR)^2}{k_1R}$$

$$k_1R = (n+\tfrac{1}{2})\pi + \frac{k^2R}{k_1}$$

$$\boxed{k_0R = (n+\tfrac{1}{2})\pi + \frac{k^2}{2k_0}R}$$

The condition of the bond state with $E_b \to 0$ is

$$\boxed{k_0R = (n+\tfrac{1}{2})\pi.}$$

This means that the divergence of "a" appears slightly later than the appearance of the bound states.

③ For a fixed potential depth k_0, the resonance occurs at the incoming k_{Res} satisfying $k_0 R - (n+\frac{1}{2})\pi = \frac{k_{Res}^2}{2k_0} R$.

For k around k_{Res}, we have do expansion

$$k_1 = k_0 + \frac{k^2}{2k_0}$$

$$k_1 R = k_0 R + \frac{k^2}{2k_0} R = (n+\tfrac{1}{2})\pi + \frac{k^2 + k_{Res}^2}{2k_0} R$$

$$kR \operatorname{ctg} \delta_0(k) \approx \frac{k_1 R \operatorname{ctg} k_1 R + (kR)^2}{1 - k_1 R \operatorname{ctg} k_1 R}$$

$$\approx \frac{(k_0 R + \frac{k^2 R}{2k_0})(-)\frac{k^2+k_{Res}^2}{2k_0} R + (kR)^2}{1 - (k_0 R + \frac{k^2 R}{2k_0})(-)\frac{k^2+k_{Res}^2}{2k_0} R}$$

$$= \frac{-\frac{k^2+k_{Res}^2}{2} R^2 + (kR)^2}{1 + k_0 R \frac{k^2+k_{Res}^2}{2k_0} R} \approx \left(\frac{k^2}{2} - \frac{k_{Res}^2}{2}\right) R^2$$

$$= (E_k - E_0)\frac{mR^2}{\hbar^2}$$

$$\Rightarrow \quad f = \frac{\sqrt{4\pi}}{k \operatorname{ctg}\delta - ik} = \frac{\sqrt{4\pi}}{(E_k - E_0)\frac{mR}{\hbar^2} - i\sqrt{\frac{2mE}{\hbar^2}}}$$

$$\sigma = |f|^2 = 4\pi \frac{\left(\frac{\hbar^2}{mR}\right)^2}{(E_k - E_0)^2 + \left[\frac{\hbar^2 k/R}{m}\right]^2} = \frac{4\pi}{k^2} \frac{(\Gamma/2)^2}{(E_k - E_0)^2 + (\Gamma/2)^2}$$

where $\frac{\Gamma}{2} = \frac{\hbar^2 k/R}{m} \simeq \frac{\hbar^2 k_{Res}^2}{m} \cdot \frac{1}{k_{Res} R} = \frac{E_{Res}}{k_{Res} R}$

on the other hand

$$k \operatorname{ctg} \delta_0(k) = \frac{k^2}{2} R - \frac{k^2_{Res}}{2} R = \frac{k^2}{2} R - k_0 \left[k_0 R - (n+\tfrac{1}{2})\pi \right]$$

scattering length

$$= \frac{k^2}{2} R - \frac{1}{a_0}$$

interaction range

correct to k^2 order

the scattering length $a_0 = \dfrac{1}{k_0} \dfrac{1}{k_0 R - (n+\frac{1}{2})\pi}$.