Lect 12. Non-abelian Berry phase / holonomy

In this lecture, we generalize the Berry phase to systems with energy level degeneracy. We will see the Berry connection becomes a matrix, not just a phase, and non-abelian structure appears. Suppose $|\eta_\alpha(R)\rangle$ $(\alpha=1,\cdots N)$ is an N-fold degenerate set of ortho-normal instaneous eigenstates.

Let us write the eigenstate

$$|\psi_a(t)\rangle = \sum_b |\eta_b(R(t))\rangle\, u_{ba}(t)\, e^{-i\int_0^t dt'\, E(t')/\hbar}$$

at $t=0$, $|\psi_a(0)\rangle = |\eta_a(R(0))\rangle$ and $u_{ba}(0)=\delta_{ba}$.

$$i\hbar\frac{\partial}{\partial t}|\psi_a(t)\rangle = \sum_b i\hbar\frac{\partial}{\partial t}|\eta_b(R(t))\rangle\, u_{ba}(t)\, e^{-i\int_0^t dt'\, E(t')/\hbar}$$

$$+\sum_b i\hbar\, |\eta_b(R(t))\rangle \frac{\partial}{\partial t} u_{ba}(t)\, e^{-i\int_0^t dt'\, E(t')/\hbar}$$

$$+\sum_b |\eta_b(R(t))\rangle\, u_{ba}(t)\, E(t')\, e^{-i\int\cdots} = H(t)|\psi_a(t)\rangle = \sum_b |\eta_b(R(t))\rangle\, u_{ba}\, E(t')\, e^{-i\int_0^t dt'\, E/\hbar}$$

$$\Rightarrow \sum_b \frac{\partial}{\partial t}|\eta_b(R(t))\rangle\, u_{ba}(t) = -\sum_b |\eta_b(R(t))\rangle \frac{\partial}{\partial t} u_{ba}(t)$$

$$\sum_b \langle\eta_c(R(t))|\eta_b(R(t))\rangle \frac{\partial}{\partial t} u_{ba}(t) = \sum_b \langle\eta_c(R(t))|\frac{\partial}{\partial t}|\eta_b(R(t))\rangle\, u_{ba}(t)$$

$$\Rightarrow \frac{\partial}{\partial t}\dot{u}_{ca} = -\sum_b \langle\eta_c|\frac{d}{dt}|\eta_b\rangle\, u_{ba}$$

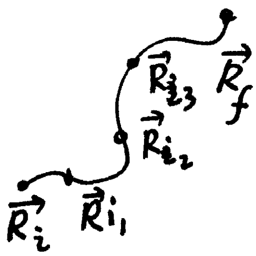
define non-Abeliean gauge field

$$A_{ab,\mu} = -i \langle \eta_a(\vec{R}) | \nabla_{R_\mu} | \eta_b(\vec{R}) \rangle$$

$$\Rightarrow \frac{\partial}{\partial R_\mu} \mathcal{U} = -i A_\mu \cdot \mathcal{U}$$ where $\mathcal{U}$, A_μ are $N \times N$ matrix

$$\Rightarrow \mathcal{U}(\vec{R}_f) = T_R \exp\left[-i \int_{\vec{R}_i}^{\vec{R}_f} dR_\mu A_\mu(\vec{R}) \right] \mathcal{U}(\vec{R}_i)$$

path ordered operator



$$T_R \exp\left[-i \int_{\vec{R}_i}^{\vec{R}_f} d\vec{R}\, A_\mu(\vec{R})\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{\vec{R}_i}^{\vec{R}_f} dR_{n,\mu_n} \int_{\vec{R}_i}^{\vec{R}_f} dR_{\mu_{n-1}} \cdots \int_{\vec{R}_i}^{\vec{R}_f} dR_{1,\mu_1}\, T_R\left[A_{\mu_n}(\vec{R}_n) A_{\mu_{n-1}}(\vec{R}_{n-1}) \cdots A_{\mu_1}(\vec{R}_1) \right]$$

where $T_R\left[A_{\mu_n}(\vec{R}_n) \cdots A_{\mu_1}(\vec{R}_1)\right] = A_{\mu_{i_n}}(\vec{R}_{i_n}) A_{\mu_{i_{n-1}}}(\vec{R}_{i_{n-1}}) \cdots A_{\mu_{i_1}}(\vec{R}_{i_1})$

and along the path from $\vec{R}_i$ to $\vec{R}_f$, $\underbrace{\vec{R}_{i_n} > \vec{R}_{i_{n-1}} > \cdots > \vec{R}_{i_1}}_{\text{the sequence is as}}$,

which is a permutation of $\vec{R}_n, \cdots \vec{R}_1$ in the right order.

For a close loop, and suppose we state from $\mathcal{U}(0) = 1 \Rightarrow$

The ~~flux~~ non-abelian phase gained is $\boxed{\mathcal{U} = T_R \exp\left[-i \oint d\vec{R}\, A_\mu(\vec{R})\right]}$

Wilson loop

Gauge transformation: For degenerate states $|\eta_a(R)\rangle$

$$\Rightarrow \quad |\eta_a(R)\rangle \rightarrow |\tilde{\eta}'_a(R)\rangle = |\eta_b(R)\rangle \; \omega_{ba}(R)$$

$$\Rightarrow \langle \tilde{\eta}_a(R)| = \langle \eta_b | \omega^*_{ba} = \langle \eta_b | \omega^+_{ab}$$

$$\Rightarrow \quad \tilde{A}_{ab,\mu} = -i \langle \tilde{\eta}_a(R) | \nabla_{R\mu} | \tilde{\eta}_b(R)\rangle$$

$$= -i \langle \eta_{a'} | \omega^+_{aa'} | \nabla_{R\mu} \{ |\eta_{b'}(R)\rangle \, \omega_{b'b}(R) \}$$

$$= \omega^+_{aa'} (-i) \langle \eta_{a'} | \nabla_{R\mu} | \eta_{b'} \rangle \omega_{b'b}$$

$$+ \; \omega^+_{aa'} (-i) \langle \eta_{a'} | \eta_{b'} \rangle \nabla_{R\mu} \omega_{b'b}$$

$$= W^+_{aa'} A_{a'b'} \omega_{b'b} + (-i) \omega^+_{aa'} \nabla_{R\mu} \omega_{a'b}$$

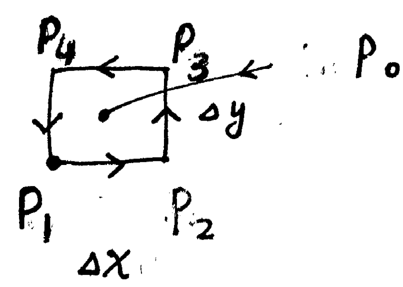
$$\Rightarrow \quad \boxed{\tilde{A}_\mu = W^+ A_\mu W - i \, W^+ \nabla_{R\mu} W}$$

non-abelian gauge transformation

W is an unitary matrix

Non-abelian gauge field strength - Curvature

$$T_R \exp[-i \oint d\vec{R} A] = \exp[-i F_{xy} \Delta x \Delta y]$$



$$1 - i \oint dR_\mu A_\mu + \frac{(-i)^2}{2!} \oint dR_{2,\mu_2} \oint dR_{1,\mu_1} T[A_{\mu_2}(\vec{R}_2) A_{\mu_1}(\vec{R}_1)]$$

$$= 1 - i F_{xy} \Delta x \Delta y$$

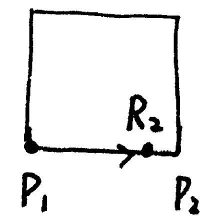
The $\oint dR_\mu A_\mu(\vec{R}) = \Delta x A_x(P_0 - \frac{\Delta y}{2}\hat{e}_y) + \overset{\Delta y}{A_y}(P_0 + \frac{\Delta x}{2}\hat{e}_x) - \overset{\Delta x}{A_x}(P_0 + \frac{\Delta y}{2}\hat{e}_y) - \overset{\Delta y}{A_y}(P_0 - \frac{\Delta x}{2}\hat{e}_x)$

$$\Delta X = \left[A_x(P_0) - \partial_{R_y} A_x \frac{\Delta y}{2}\right] + \left[A_y(P_0) + \partial_{R_x} A_y \frac{\Delta X}{2}\right] \Delta y$$

$$\Delta X\left[- A_x(P_0) - \partial_{R_y} A_x \frac{\Delta y}{2}\right] + \left(A_y(P_0) + \partial_{R_x} A_y \frac{\Delta X}{2}\right] \Delta y$$

$$= \left[\partial_{R_x} A_y - \partial_{R_y} A_x\right] \Delta x \Delta y$$

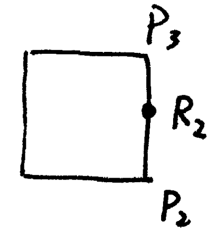
The second term $= (-i)^2 \cdot$



$$\oint dR_{2,\mu_2} \int_{P_1}^{R_2} dR_1\, A_{\mu_2}(R_2)\, A_{\mu_1}(R_1)$$

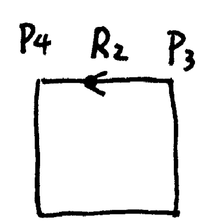
① if R_2 is from $P_1 \to P_2$ $\Rightarrow \int_{P_1}^{P_2} dR_{2,x} \int_{P_1}^{R_2} dR_1 A_x A_x = \frac{(\Delta X)^2}{2} A_x^2$

② if R_2 is from $P_2 \to P_3$



$$\int_{P_2}^{P_3} dR_{2,y} \left[\int_{P_1}^{P_2} dR_{1,x}\, A_y A_x + \int_{P_2}^{R_2} dR_{1,y}\, A_y A_y \right]$$

$$= \Delta y \Delta x\, A_y A_x + \frac{(\Delta y)^2}{2} A_y^2$$

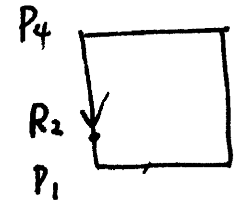


③ if R_2 is from $P_3 \to P_4$

$$\int_{P_3}^{P_4} dR_{2,x} \left[\int_{P_1}^{P_2} dR_{1,x}\, A_x A_x + \int_{P_2}^{P_3} dR_{1,y}\, A_x A_y + \int_{P_3}^{P_4} dR_{1,x}\, A_x A_x \right]$$

$$\ldots (\Delta X)(\Delta y)\, A_x A_y + \left(\frac{\Delta X}{2}\right) A_x^2$$

4 if R_2 is from $R_4 \to R_1$



$$\int_{P_4}^{P_1} dR_{2y} \left[\int_{P_1}^{P_2} dR_{1,x} A_y A_x + \int_{P_2}^{P_3} dR_{1y} A_y A_y \right.$$

$$\left. + \int_{P_3}^{P_4} dR_{1x} A_y A_x + \int_{P_4}^{P_1} dR_{1y} A_y A_y \right]$$

$$= -\Delta y \Delta x \; A_y A_x + -(\Delta y)^2 A_y^2 + (\Delta x \Delta y) A_y A_x + \frac{(\Delta y)^2}{2} A_y^2$$

$\Rightarrow$ Add together $\Rightarrow$ $-\Delta x \Delta y \, [A_x A_y - A_y A_x]$

$$\Rightarrow -i[\partial_{R_x} A_y - \partial_{R_y} A_x]\Delta x \Delta y + \Delta x \Delta y (A_x A_y - A_y A_x] = -i F_{xy} \Delta x \Delta y$$

$$\Rightarrow \boxed{F_{\mu\nu} = \partial_{R_\mu} A_\nu - \partial_{R_\nu} A_\mu + i[A_\mu, A_\nu]}$$

under gauge transformation $\tilde{A}_\mu = W^\dagger A_\mu W - i W^\dagger \nabla_\mu W$

$$\Rightarrow \tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu + i[\tilde{A}_\mu, \tilde{A}_\nu]$$

$$= (\partial_\mu W^\dagger) A_\nu W + W^\dagger \partial_\mu A_\nu W + W^\dagger A_\nu \partial_\mu W - i \, \partial_\mu (W^\dagger \partial_\nu W)$$

$$-(\partial_\nu W^\dagger) A_\mu W - W^\dagger \partial_\nu A_\mu W - W^\dagger A_\mu \partial_\nu W + i \, \partial_\nu (W^\dagger \partial_\mu W)$$

$$+ i[W^\dagger A_\mu W, W^\dagger A_\nu W] + [W^\dagger A_\mu W, W^\dagger \partial_\nu W] + [W^\dagger \partial_\mu W, W^\dagger A_\nu W]$$

$$-i[W^\dagger \partial_\mu W, W^\dagger \partial_\nu W]$$

check

$$\partial_\mu W^\dagger A_\nu W + W^\dagger A_\nu \partial_\mu W + [W^\dagger \partial_\mu W,\ W^\dagger A_\nu W]$$

$$- \partial_\mu W^\dagger A_\nu W - W^\dagger A_\nu \partial_\mu W$$

$$= 0$$

$$- \partial_\nu W^\dagger A_\mu W \mp W^\dagger \partial_\nu A_\mu W + [W^\dagger A_\mu W,\ W^\dagger \partial_\nu W]$$

$$W^\dagger A_\mu \partial_\nu W + \partial_\nu W^\dagger A_\mu W$$

$$= 0$$

$$- \partial_\mu (W^\dagger \partial_\nu W) + \partial_\nu (W^\dagger \partial_\mu W) - [W^\dagger \partial_\mu W,\ W^\dagger \partial_\nu W]$$

$$= - \partial_\mu W^\dagger \partial_\nu W + \partial_\nu W^\dagger \partial_\mu W + \partial_\mu W^\dagger \partial_\nu W - \partial_\nu W^\dagger \partial_\mu W = 0$$

$$\Rightarrow \boxed{\tilde{F}_{\mu\nu} = W^\dagger (\partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu]) W = W^\dagger F_{\mu\nu} W .}$$

we used $W^\dagger \partial_\mu W = - \partial_\mu W^\dagger W$ above.

$W \partial_\mu W^\dagger = - \partial_\mu W W^\dagger$

Example: qudratic Zeeman for spin-3/2 system

$$H = (S \cdot B)^2.$$

each energy level is doubly degenerate due to time-reversal symmetry

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$$H = B^2 e^{-i\varphi S_z} e^{-i\theta S_y} S_z^2 e^{i\theta S_y} e^{i\varphi S_z}$$

we denote $|a\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $|b\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $|c\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $|d\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ eigenstates of S_z

$$|\eta_a\rangle = e^{-i\varphi S_z} e^{-i\theta S_y} |a\rangle$$

where θ, φ are the direction of B-field.

$$\frac{\partial}{\partial\theta} |\eta_b\rangle = e^{-i\varphi S_z} e^{-i\theta S_y} (-i) S_y |b\rangle$$

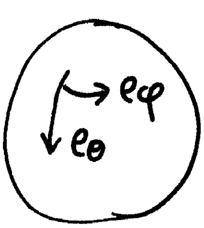
$$A_{ab,\theta} = -i\langle \eta_a | \frac{\partial}{\partial\theta} | \eta_b\rangle = -\langle a| e^{i\theta S_y} e^{i\varphi S_z} e^{-i\varphi S_z} e^{-i\theta S_y} S_y |b\rangle$$

$$= -\langle a | S_y | b\rangle$$

$$\frac{1}{\sin\theta\,\partial\varphi} |\eta_b\rangle = -i \frac{S_z}{\sin\theta} e^{-i\varphi S_z} e^{-i\theta S_y} |b\rangle$$

$$e^{i\theta S_y} e^{i\varphi S_z} S_z e^{-i\varphi S_z} e^{-i\theta S_y} = e^{i\theta S_y} S_z e^{-i\theta S_y} = -\sin\theta S_x + \cos\theta S_z$$

$$A_{ab,\varphi_\theta} = -\langle a | \frac{1}{\sin\theta} [\cos\theta S_z - \sin\theta S_x] | b\rangle$$



along $\hat{e}_\theta \Rightarrow A_{ab,\theta} = -\langle a | S_y | b\rangle$

$$= -\begin{pmatrix} 0 & -\frac{\sqrt{3}i}{2} & 0 & 0 \\ \frac{\sqrt{3}i}{2} & 0 & -i & 0 \\ 0 & i & 0 & -\frac{\sqrt{3}i}{2} \\ 0 & 0 & \frac{\sqrt{3}i}{2} & 0 \end{pmatrix}$$

non-abelian

along $\hat{e}_\varphi$: $A_{ab,\varphi} = \frac{-1}{\sin\theta} \langle a | \cos\theta S_z - \sin\theta S_x | b \rangle$

$$= \frac{-1}{\sin\theta} \begin{pmatrix} \frac{3}{2}\cos\theta & -\frac{\sqrt{3}}{2}\sin\theta & 0 & 0 \\ \frac{\sqrt{3}}{2}\sin\theta & \frac{1}{2}\cos\theta, & -\sin\theta & 0 \\ 0 & -\sin\theta & -\frac{1}{2}\sin\theta & -\frac{\sqrt{3}}{2}\sin\theta \\ 0 & 0 & -\frac{\sqrt{3}}{2}\sin\theta & 0 \end{pmatrix}$$

Take the $\pm 1/2$ part

$$\vec{A} = (-)\left[\frac{1}{\sin\theta} \left[-\sin\theta\, \sigma_1 + \frac{\cos\theta}{2} \sigma_3 \right] \hat{e}_\varphi + \sigma_2 \hat{e}_\theta \right] = -\vec{A}^i \left(\frac{\sigma^i}{2}\right)$$

$$\vec{A}^1 = -2\hat{e}_\varphi \qquad \vec{A}^2 = 2\hat{e}_\theta , \quad \vec{A}^3 = \operatorname{ctg}\theta\ \hat{e}_\varphi$$

The non-abelian field strength

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \epsilon^{ijk} A^j_\mu A^k_\nu$$

define $F^i{}_\lambda = \frac{1}{2}\epsilon_{\mu\nu\lambda} F^i_{\mu\nu}$

$$= (\nabla \times \vec{A}^i)_\lambda + \frac{1}{2}\epsilon_{\lambda\mu\nu}\epsilon_{ijk} A^j_\mu A^k_\nu$$

$$\vec{F}^3 = F^3{}_r \hat{e}_r + F^3{}_\theta \hat{e}_\theta + F^3{}_\varphi \hat{e}_\varphi$$

$$= \nabla\times\vec{A}^3 + \hat{e}_r \frac{1}{2}\epsilon^{3jk}[A^j_\theta A^k_\varphi - A^j_\varphi A^k_\theta]$$

$$+ \hat{e}_\theta \frac{1}{2}\epsilon^{3jk}[A^j_\varphi A^k_r - A^j_r A^k_\theta]$$

$$+ \hat{e}_\varphi \frac{1}{2}\epsilon^{3jk}[A^j_r A^k_\theta - A^j_\theta A^k_r]$$

$$= \nabla\times\vec{A}^3 + \frac{\hat{e}_r}{2}[A^1_\theta A^2_\varphi - A^2_\theta A^1_\varphi - A^1_\varphi A^2_\theta + A^2_\varphi A^1_\theta]$$

$$\nabla \times \vec{A}^3 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, \mathrm{ctg}\,\theta)\,\hat{e}_r = -\hat{e}_r$$

$$\vec{F}^3 = -\hat{e}_r + \frac{\hat{e}_r}{2}\left[0 - 2(-2) - (-2)2 + 0\right] = 3\hat{e}_r$$

Similarly

$$\vec{F}^1 = (\nabla \times \vec{A}^1) + \frac{\hat{e}_r}{2}\left[A^2_\theta A^3_\varphi - A^3_\theta A^2_\varphi - A^2_\varphi A^3_\theta + A^3_\varphi A^2_\theta\right]$$

$$= \nabla \times (-2\hat{e}_\varphi) + \frac{\hat{e}_r}{2}\left[2\,\mathrm{ctg}\,\theta + 2\,\mathrm{ctg}\,\theta\right]$$

$$= -2\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\,\hat{e}_r + 2\,\mathrm{ctg}\,\theta\,\hat{e}_r = 0$$

Similarly $\vec{F}^2 = (\nabla \times \vec{A}^2) + \frac{\hat{e}_r}{2}\left(A^3_\theta A^1_\varphi - A^1_\theta A^3_\varphi - A^3_\varphi A^1_\theta + A^1_\varphi A^3_\theta\right)$

$$= (\nabla \times 2\hat{e}_\theta) + \frac{\hat{e}_r}{2}\left(0 - 0 - \mathrm{ctg}\,\theta \cdot 0 + (-2)\cdot 0\right)$$

$$= 0$$