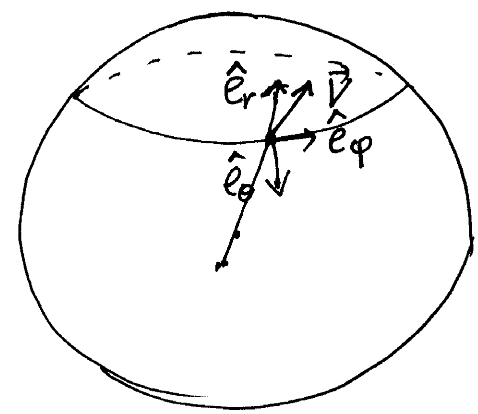
Lect 11. Geometric intepretation of Berry Phase and applications : arXiv: cond-mat 0508236

§ Parallel transport

Let us consider a beetle moves a unit vector $\vec{V}$ along a closed path C on a sphere. We put the constraint that $\vec{V}$ needs staying on the tangent plane, i.e. $\boxed{\vec{V}\cdot\vec{e}_r = 0}$

$\hat{e}_r$ is the normal vector.



When $\vec{V}$ moves, we require that the orientation is perturbed minimally.

On a curved surface, $\vec{V}$ cannot be unchanged. But the change $d\vec{V}$ should have no projection in the tangent plane, i.e $\boxed{d\vec{V}\times\hat{e}_r = 0}$ $\boxed{d\vec{V} \parallel \hat{e}_r}$.

It is obvious that if $\vec{V}$ moves in a plane, the above two conditions is equivalent to $d\vec{V}\equiv 0$, i.e. parallel transport in a plane.

Now let us study the "parallel transport in a sphere".

Let us write, at time t.

$$\vec{V}(t) = \cos\alpha(t)\,\hat{e}_\theta + \sin\alpha(t)\,\hat{e}_\varphi \Rightarrow \vec{V}\cdot\vec{e}_r = 0.$$

, and define $\vec{\omega} = \hat{e}_r \times \vec{V}(t)$

If $\vec{V}$ is doing parallel transport, so does $\vec{W}$

Proof: $d\vec{W} = d(\hat{e}_r \times \vec{V}(t)) = d\hat{e}_r \times \vec{V} + \underbrace{\hat{e}_r \times d\vec{V}}_{=0}$

$$d\vec{W} \times \hat{e}_r = (d\hat{e}_r \times \vec{V}) \times \hat{e}_r = \underbrace{(d\hat{e}_r \cdot \hat{e}_r)}_{=0} \vec{V} + \underbrace{(\vec{V} \cdot \hat{e}_r)}_{=0} d\hat{e}_r = 0$$

$$d\vec{V} \times \hat{e}_r = 0 \Rightarrow [-\sin\alpha \, d\alpha \, \hat{e}_\theta + \cos\alpha \, d\alpha \, \hat{e}_\varphi + \cos\alpha \, d\hat{e}_\theta + \sin\alpha \, d\hat{e}_\varphi] \times \hat{e}_r = 0$$

$$\sin\alpha \, [-\hat{e}_\theta \, d\alpha \times \hat{e}_r + d\hat{e}_\varphi \times \hat{e}_r] + \cos\alpha \, [d\alpha \, \hat{e}_\varphi \times \hat{e}_r + d\hat{e}_\theta \times \hat{e}_r] =$$

$$\sin\alpha \, [d\alpha \, \hat{e}_\varphi + d\hat{e}_\varphi \times \hat{e}_r] + \cos\alpha \, [+d\alpha \, \hat{e}_\theta + d\hat{e}_\theta \times \hat{e}_r] = 0$$

This is valid for arbitary $\alpha \Rightarrow$

$$d\alpha \, \hat{e}_\theta = -d\hat{e}_\theta \times \hat{e}_r \Rightarrow d\alpha = -\hat{e}_\theta \cdot (d\hat{e}_\theta \times \hat{e}_r) = \hat{e}_\theta \cdot (d\hat{e}_\theta \times (\hat{e}_\theta \times \hat{e}_\varphi))$$

$$= \hat{e}_\theta \cdot [\hat{e}_\theta (d\hat{e}_\theta \cdot \hat{e}_\varphi) - \hat{e}_\varphi (d\hat{e}_\theta \cdot \hat{e}_\theta)]$$

$$= -d\hat{e}_\theta \cdot \hat{e}_\varphi$$

$$\Rightarrow \boxed{d\alpha = -\hat{e}_\varphi \cdot d\hat{e}_\theta = \hat{e}_\theta \cdot d\hat{e}_\varphi}$$

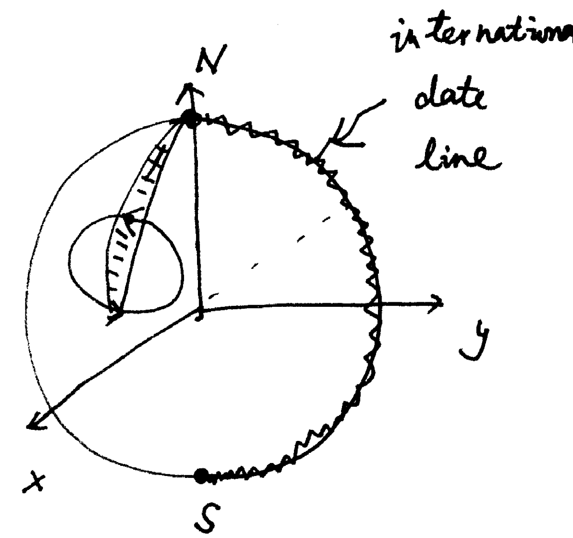
$$\hat{e}_r = \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z$$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{e}_x + \cos\theta\sin\phi\,\hat{e}_y - \sin\theta\,\hat{e}_z$$

$$\hat{e}_\varphi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y$$

$$d\hat{e}_\theta = -\sin\theta\cos\phi\,d\theta\,\hat{e}_x - \sin\theta\sin\phi\,d\theta\,\hat{e}_y - \cos\theta\,d\theta\,\hat{e}_z$$
$$- \cos\theta\sin\phi\,d\phi\,\hat{e}_x + \cos\theta\cos\phi\,d\phi\,\hat{e}_y$$
$$= -d\theta\,\hat{e}_r + \cos\theta\,d\phi\,\hat{e}_\varphi$$

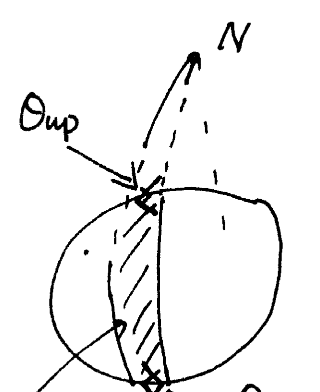
$$\Rightarrow -\hat{e}_\varphi \cdot d\hat{e}_\theta = -\cos\theta\,d\phi$$



Let define the branch cut at $\varphi = \pi$, from north pole to south pole.

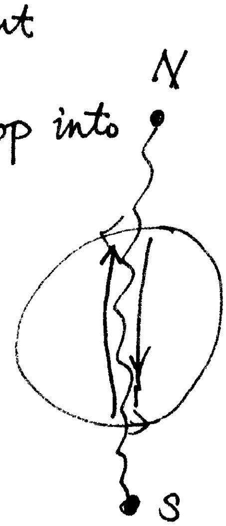
For any loop which do not across the branch cut, we have

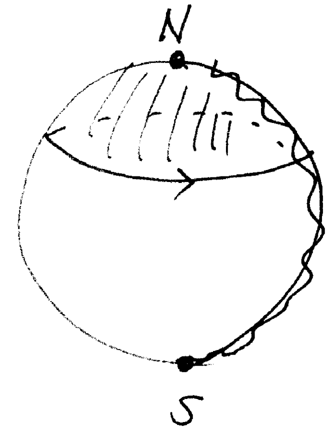
$$\Rightarrow \quad \alpha = \oint_{\triangle} \cos\theta \cdot d\phi = \int dS = \text{Solid angle}$$



$$dS = d\phi\,(\cos\theta_{up} - \cos\theta_{down})$$
$$= d\phi \int_{\theta_{up}}^{\theta_{down}} \sin\theta\,d\theta$$

If the loop arcoss the branch cut twice, we can decompose the loop into two loops, and each of them does not across the branch cut.





If the loop acrosses the branch cut,

i.e.

$$\alpha = \oint -\cos\theta \, d\phi = \int_0^{2\pi} (1-\cos\theta)\, d\phi - 2\pi$$

$$= \underset{\uparrow}{S} - 2\pi$$

solid angle.

So the angle change after a parallel transport along a closed path is the solid angle of the area enclosed by the path.

Analogy to Berry phase.

Let us write $\hat{e} = \dfrac{\hat{e}_\theta + i\hat{e}_\varphi}{\sqrt{2}}$ and $\hat{\psi} = \dfrac{\vec{V} + i\vec{W}}{\sqrt{2}}$.

$$\Rightarrow \quad \hat{\psi}_{(t)} = \hat{e}(t)\, e^{-i\alpha(t)}$$

The condition of parallel transport

$$\boxed{\begin{aligned} d\alpha &= -\hat{e}_\varphi \cdot d\hat{e}_\theta \\ &= +i\hat{e}^* \, d\hat{e} \end{aligned}}$$

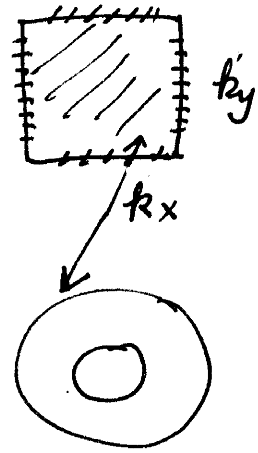
$$d\vec{V} \times \hat{e}_r = 0 \quad \Rightarrow d\vec{V}\cdot\hat{e}_\theta = d\vec{W}\cdot\hat{e}_\theta = 0$$

$$d\vec{W} \times \hat{e}_r = 0 \qquad d\vec{V}\cdot\hat{e}_\varphi = d\vec{V}\cdot\hat{e}_\varphi = 0$$

$$\Rightarrow (\hat{e}_\theta - i\hat{e}_\varphi)(d\vec{V} + i\,d\vec{W}) = 0 \Rightarrow \boxed{\hat{e}^* \cdot d\hat{\psi} = 0}$$

$$\Rightarrow \hat{e}^* \cdot [\hat{e}\,(-i)\,d\alpha + d\hat{e}] = 0$$

* Berry phase defined in Brillouin Zone in solid state



Due to periodicity of the lattice, Bloch theory tells us the eigenstates can be written as

$$\psi_{n,k}(r) = \frac{1}{\sqrt{N}} e^{ikr} \, u_{n,k}(r)$$

index of bands — plane wave factor — periodic wave function.

$u_{n,k}(r)$ satisfies $H(\vec{k}, r, i\hbar\nabla) = e^{-i\vec{k}\cdot\vec{r}} H_{(r, i\hbar\nabla)} e^{i\vec{k}\cdot\vec{r}}$.

We treat $\vec{k}$ as an external parameter, which is defined on a torus.

(BZ actually has no boundary $\longleftrightarrow$ a torus).

$$\vec{A}_n(k) = i\int dr \; u^*_{n,k}(r) \; \nabla_k u_{n,k}(r)$$

$$\Omega^n_{xy}(k) = \nabla_k \times \vec{A}_n(k)$$

By a similarly method defined in last lecture,

$$C \cdot 2\pi = \oiint dk_x \, dk_y \; \Omega_{xy,n}(\vec{k})$$

integer, quantized. If $C \neq 0$, we have a topologically non-trivial band structure.

§ anomalous velocity

The Berry connection is defined in momentum space.

$$x = i\partial_K \longrightarrow i\vec{\partial}_K + \vec{A}(k) = \vec{x}$$

coordinate

so where project to a band, we have $H = \mathcal{E}_n(k) + V(i\partial_k + A)$

Then x_i and x_j doesn't commute any more.

$$[x_i, x_j] = i[\partial_{k_i}\ A_j] + i[A_i, i\partial_{k_j}] = i\epsilon_{ijk}\Omega_K$$

$$\Rightarrow \hbar v_i = -i[x_i, H] = \nabla_k \mathcal{E}_n(k) + (-i)[x_i, V(x)]$$

$$= -i[x_i, \mathcal{E}_n(k) + V(x)]$$

$$[x_i, V(x)] = [x_i, \frac{\partial V}{\partial x_j} x_j] = \frac{\partial V}{\partial x_j} i\epsilon_{ijk}\Omega_K = i\epsilon_{ijk}\frac{\partial V}{\partial x_j}\Omega_K$$

$$\Rightarrow \hbar\vec{v} = \nabla_k \mathcal{E}_n(k) + \frac{\partial V}{\partial \vec{x}} \times \vec{\Omega}_k$$ ← anomalous velocity

⇒ semi-classical Equation of motion

$$\boxed{\begin{aligned} \hbar\dot{x} &= \nabla_k \mathcal{E}_n(k, r) - \dot{k}\times\Omega_k \\ \hbar\dot{k} &= -\nabla_r \mathcal{E}_n(k,r) + e\dot{x}\times\vec{B}(r) \end{aligned}}$$ ← Lorentz force in phase sphace