Lect 6: Zero energy scattering, bound states, resonances

§ Zero energy scattering: let us consider a short range scattering potential, the S-wave WF can be written as $\psi = \frac{1}{\sqrt{4\pi}} R_0(r) = \frac{u(r)}{r}$, where $u(r)$ satisfies

$$\frac{d^2}{dr^2} u + \left[k^2 - \frac{2m}{\hbar^2} V(r)\right] u = 0, \quad \text{where } k = \sqrt{\frac{2mE}{\hbar^2}}.$$

at $r > R$ ← interaction range, $\frac{d^2}{dr^2} u = 0$, $(r > R, k \to 0)$, $\Rightarrow$

$$\boxed{u(r) = \underset{\text{const.}}{}\left(1 - \frac{r}{a_0}\right),}$$

a_0 is called scattering length.

more precisely at $r > R$, $u(r) \approx A \sin(kr + \delta_0) = A \sin\delta_0 [\cos kr + \text{ctg}\,\delta_0 \sin kr]$

$$\boxed{u(r) \xrightarrow{k \to 0} \text{const}\,[1 + \text{ctg}\,\delta_0\, k r]}$$

Compare them $\Rightarrow$ $\boxed{k\,\text{ctg}\,\delta_0 = -\frac{1}{a_0}}$.

The scattering amplitude $f_0 = \frac{\sqrt{4\pi}}{k} e^{i\delta_0} \sin\delta_0 = \frac{\sqrt{4\pi}}{k\,\text{ctg}\,\delta_0 - ik} = -\sqrt{4\pi} \frac{a_0}{1 + ika_0}$

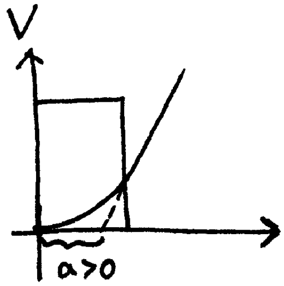
$$\sigma = |f_0|^2 = \frac{4\pi a_0^2}{1 + (ka_0)^2}$$

If a_0 is finite, $\Rightarrow$ $\delta_0 \approx \text{tg}\,\delta_0 \approx -ka_0$ ← hard sphere with radius a_0.

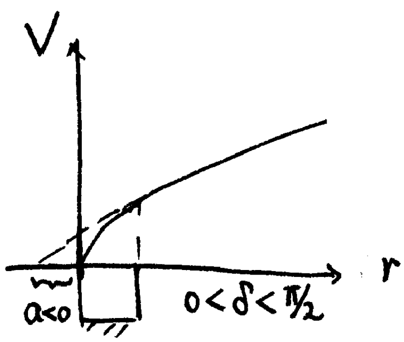
If $a_0 \to \pm\infty$, $\delta_0 = \pm\frac{\pi}{2}$, $f = \frac{\sqrt{4\pi}\, i}{k}$ $\Rightarrow$ $\sigma_0 = \frac{4\pi}{k^2} = \frac{2\pi\hbar^2}{mE} \propto \frac{1}{E}$

resonace scattering

repulsive potential



attractive potential



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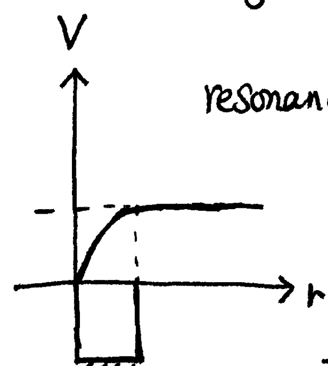
$\sigma = \frac{4\pi}{k^2}$

resonance scattering



$a = \mp\infty$, $\delta_0 = \frac{\pi}{2} \mp 0^+$

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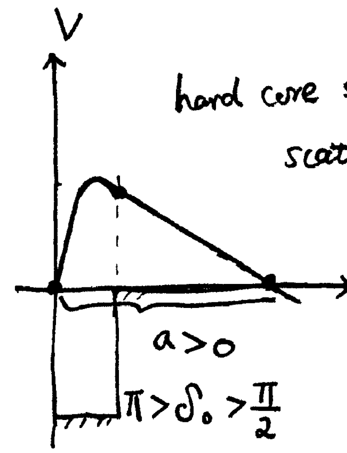
$\sigma = 4\pi a^2$

hard core sphere scattering



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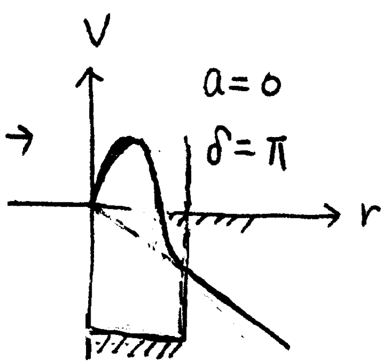
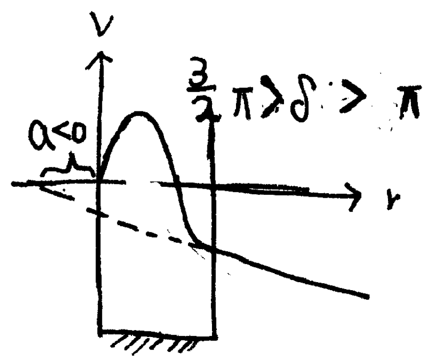
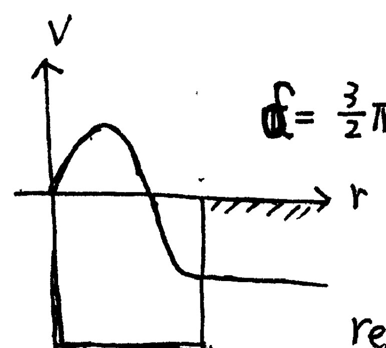
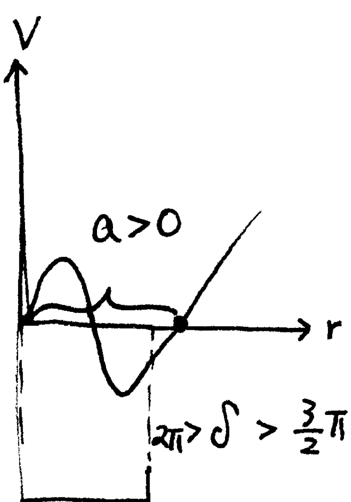


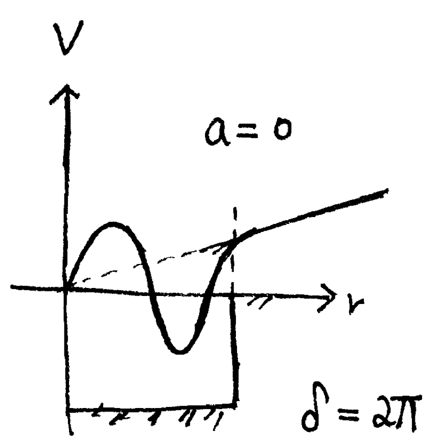




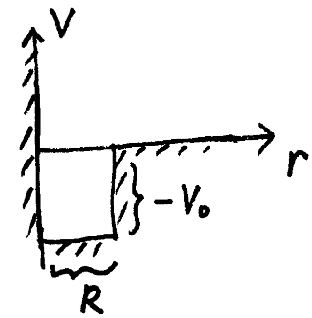
$a = \mp\infty$

resonance scattering





The appearance of resonance scattering has close relation to the bound state in the potential well. let us consider a spherical potential well, we decide the condition for the appearence of <u>a bound state just at the top of well.</u>



$$\begin{cases} \dfrac{d^2}{dr^2}u + k_0^2 u = 0 & (r<R) \quad k_0 \approx \sqrt{\dfrac{2mV}{\hbar^2}} \\ \dfrac{d^2}{dr^2}u - \beta^2 u = 0 & (u>R) \quad \beta \to 0^- \end{cases}$$

$\uparrow$ bound state

at $r<R$: $\quad u = \sin k_0 r \quad \Rightarrow \quad \left.\dfrac{u'}{u}\right|_{r=R} = k_0 \operatorname{ctg} k_0 R$

$r>R$: $\quad u = A e^{-\beta r} \quad\quad \left.\dfrac{u'}{u}\right|_{r=R} = -\beta e^{-\beta R}$

$$\Rightarrow \quad k_0 \operatorname{ctg} k_0 R = -\beta e^{-\beta R} \longrightarrow 0 \quad \text{as } \beta \to 0^-$$

$$\Rightarrow \quad k_0 R \approx (n+\tfrac{1}{2})\pi .$$

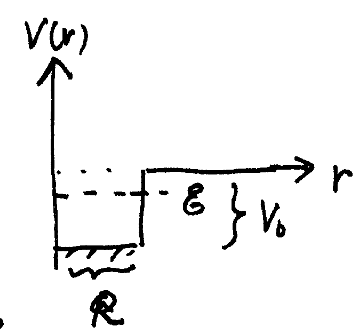
Let me denote $\Delta\delta = k_0 R - (n+\frac{1}{2})\pi \quad \Rightarrow$

$$k_0 R \operatorname{ctg} k_0 R = \left[(n+\tfrac{1}{2})\pi + \Delta\delta\right] \operatorname{ctg}\left[(n+\tfrac{1}{2})\pi + \Delta\delta\right] = -\left[(n+\tfrac{1}{2})\pi\right] \Delta\delta$$

$$\Rightarrow \boxed{\beta R = (n+\tfrac{1}{2}\pi)\,\Delta\delta} \leftarrow \text{localization length.}$$

Let us consider a concrete example: hot neutrino scattering on the proton, we know there is a bound state ($l=0$) with $\varepsilon = -2.23$ Mev. Hot neutron $E_0 \simeq \frac{1}{40}$ eV,

$\uparrow$ binding energy of D-nucleus.



the interaction range $R \approx 2\times10^{-15}$ m. $V_0 = 25-30$ Mev.

Let us discuss the scattering length and its relation to bound state.

First in the center of mass from. $\mu \approx m/2$ $(m_p = m_n)$, and the hot neutron $E \approx \frac{1}{2} E_0 \approx \frac{1}{80}$ eV. $\Rightarrow k = \sqrt{\frac{2\mu E}{\hbar}} = 1.7 \times 10^{10}\ m^{-1}$

$kR \approx 3.5 \times 10^{-5} << 1$, which match the limit of zero energy scattering.

Let us consider both the bound state and the scattering state:

$$\psi_b = \frac{u_b(r)}{r}, \qquad \psi_s = \frac{u_s(r)}{r};$$

$$\Rightarrow \begin{cases} \frac{d^2}{dr^2} u_b + (\mathcal{E}_b - V(r)) \frac{2\mu}{\hbar^2} u_b = 0 & \mathcal{E}_b < 0 \\ \frac{d^2}{dr^2} u_s + (\mathcal{E}_s - V(r)) \frac{2\mu}{\hbar} u_s = 0 & \mathcal{E}_s > 0 \end{cases}$$

Inside the well, $V_0 >> |\mathcal{E}_b|, \mathcal{E}_s$, thus we can neglect the $|\mathcal{E}_b|$ and $\mathcal{E}_s$ and have $u_b \simeq u_s \approx \sin(k_0 r)$, $k_0 = \sqrt{\frac{2\mu}{\hbar^2} V_0}$.
$(r<R)$

Out side the well, $\begin{cases} u_b = c e^{-\beta r} & \beta = \sqrt{\frac{2\mu |\mathcal{E}_b|}{\hbar^2}} \\ u_s \approx A \sin(kr + \delta_0) & k = \sqrt{\frac{2\mu E}{\hbar^2}} \end{cases}$ for $r > R$

match boundry condition $\frac{u'}{u}\Big|_{r=R}$ continuous.

$$\frac{u_b'}{u_b}\Big|_{r=R^+} = \frac{u_s'}{u_s}\Big|_{r=R^+} = k_0 \operatorname{ctg} k_0 R$$

$$\Rightarrow -\beta R e^{-\beta R} = kR \operatorname{ctg}(kR + \delta_0) = k_0 R \operatorname{ctg} k_0 R \qquad \text{and } \beta R,\ kR \to 0$$

$$\Rightarrow -\beta R = kR \operatorname{ctg} \delta_0 \Rightarrow \frac{-1}{a_0} = k \operatorname{ctg} \delta_0 = -\beta \Rightarrow \boxed{a_0 = \frac{1}{\beta}}$$

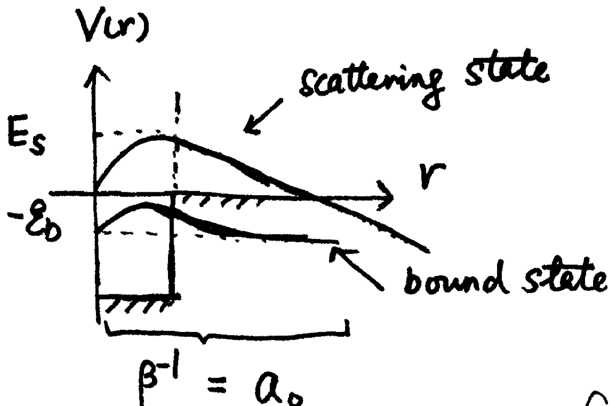
$$f_0 = \sqrt{4\pi} \frac{1}{k \operatorname{ctg} \delta_0 - ik} = \frac{-\sqrt{4\pi}}{\beta + ik}$$

$$\sigma = \quad f_0^2 = \frac{4\pi\hbar^2}{2\mu(|\varepsilon_b| + E)},$$

the scattering length and cross section have no dependence on R, V_0, these microscopic details. This is the virtue of resonace scattering.

★ When there is just a true bound state below the top of the well, the scattering length is positive and a_0 equals the localization length of the shallow bound state β^{-1}.

Suppose we make the well shallower, i.e. to make the bound state shallower, we will have even larger positive scattering length, $a_0 \to +\infty$.



Resonance scattering with positive scattering length: $a_0 = \beta^{-1} > 0$

Scattering state v.s. bound state.

Condition for the first resonance

$$\boxed{k_0 R = \frac{\pi}{2} + 0^+}.$$

★ If the well is made even shallower, $k_0 R < \frac{\pi}{2}$, there will be no bound state.

let us denote $\Delta\delta = k_0 R - \frac{\pi}{2} \to 0^- \Rightarrow$

$$kR\, \mathrm{ctg}(kR + \delta_0) = \frac{\pi}{2}\, \mathrm{ctg}\left(\frac{\pi}{2} + \Delta\delta\right) = -\frac{\pi}{2}\Delta\delta$$

$$\Rightarrow kR\, \mathrm{ctg}\, \delta_0 = -\frac{\pi}{2}\Delta\delta \quad \Rightarrow \quad \frac{-1}{a_0} = k\, \mathrm{ctg}\, \delta_0 = -\frac{\pi}{2}\frac{\Delta\delta}{R}$$

$$\boxed{a_0 = \frac{2}{\pi}\frac{R}{\Delta\delta} < 0,}$$

we can arrive at very large negative scattering length $\gg R$.

☆ If we do analytic continuation for the scattering amplitude $\underbrace{f_0}_{(k)} = -\sqrt{4\pi}\,\frac{1}{\beta + ik}$, i.e., take k as a complex variable).

The bound state $\underset{e^{-\beta r} = e^{i(i\beta)r}}{k = i\beta}$ appears at a pole of $f_0(k)$.

This is a general statement, and we will prove it later.

☆ Width of the resonance:

As we have seen, a_0 diverges $\to \pm\infty$, as approching the resonance.

In other words $k\,\mathrm{ctg}\,\delta_0 = -\frac{1}{a_0}$ is continous. Let us take k_0R as parameter to study the behavior close to resonace. Let us denote $\theta = k_0 R$, and look at the parameter regime $\theta \sim \frac{\pi}{2} = \theta_0$

$$\mathrm{ctg}\,\delta_0(\theta) = \mathrm{ctg}\,\delta_0(\theta_0) - \frac{1}{\sin^2\delta_0(\theta_0)}(\theta - \theta_0)\cdot\frac{d\delta}{d\theta}$$

$$= -\left.\frac{d\delta(\theta)}{d\theta}\right|_{\theta=\theta_0}(\theta - \theta_0)$$

define $\Gamma = \frac{2}{\frac{d\delta(\theta)}{d\theta}}$ $\Rightarrow$ $\mathrm{ctg}\,\delta_0(\theta) = -\frac{2}{\Gamma}(\theta - \theta_0)$

↖ resonce width

$$\Rightarrow f_0 = \frac{\sqrt{4\pi}}{k\,\mathrm{ctg}\,\delta_0 - ik} = \frac{\sqrt{4\pi}}{k}\,\frac{1}{-\frac{2}{\Gamma}(\theta-\theta_0) - i} = -\frac{\sqrt{4\pi}}{k}\,\frac{\Gamma/2}{(\theta-\theta_0) + \frac{\Gamma}{2}i}$$

$$\Rightarrow \boxed{\sigma_0 = f_0^2 = \frac{4\pi}{k^2}\,\frac{(\Gamma/2)^2}{(\theta-\theta_0)^2 + \frac{\Gamma}{2}i}}$$ Breit-Wigner formula

★ $\boxed{\text{Perfect transmition:}}$

we also see as $k_0 R \sim n\pi$, then the phase shift $\rightarrow$ $\delta_0 \sim n\pi$ thus the scattering length $a_0 \rightarrow 0$. This was observed for low energy electron scattering on some inert gases. This is called Ramsauer - Townsend effect, which was found before the establishment before wave-mechanics.