Lecture 12

Introduction to Fluid Mechanics II.

Momentum flux

$$\frac{\partial}{\partial t}(\rho v_i) = \rho \frac{\partial v_i}{\partial t} + \frac{\partial \rho}{\partial t} v_i$$

rate of change in momentum components $i = 1, 2, 3$

continuity eq. $$\frac{\partial \rho}{\partial t} = - \frac{\partial (\rho v_k)}{\partial x_k}$$

fixed unit volume!

Euler eq. $$\frac{\partial v_i}{\partial t} = - v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial p}{\partial x_i}$$

$$\frac{\partial}{\partial t}(\rho v_i) = - \rho v_k \frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - v_i \frac{\partial (\rho v_k)}{\partial x_k}$$

$$= - \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_k}(\rho \cdot v_i v_k)$$

write $$\frac{\partial p}{\partial x_i} = \delta_{ik} \cdot \frac{\partial p}{\partial x_k}$$

summation over repeated indices

$$\frac{\partial}{\partial t}(\rho \cdot v_i) = - \frac{\partial \Pi_{ik}}{\partial x_k}$$

rate of change in momentum

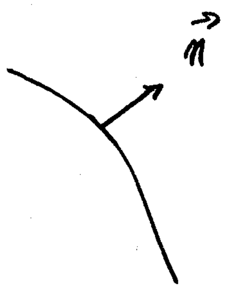
$$\Pi_{ik} = p \cdot \delta_{ik} + \rho v_i v_k$$

What is the meaning of Π_{ik} ?

$$\frac{\partial}{\partial t} \int \rho v_i \, dV = - \int \frac{\partial \Pi_{ik}}{\partial x_k} \, dV$$

rate of change of momentum (ith component) in fixed volume $= - \oint \Pi_{ik} \, df_k$

momentum flowing out through bounding surface of volume



$$df_k = n_k \cdot \underbrace{df}_{\text{area of surface element}}$$

$\Pi_{ik} \cdot n_k$ flux of ith momentum component through unit surface area

$$\Pi_{ik} n_k = p \cdot n_i + \rho v_i v_k \cdot n_k$$

$$p \cdot \vec{n} + \rho \vec{v} \, (\vec{v} \cdot \vec{n})$$ in vector form

momentum flux in direction of $\vec{n}$

Π_{ik} momentum flux density tensor

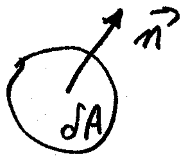
ith component of the amount of momentum flowing in unit time through unit area perpendicular to the x_k-axis

Stress tensor

If an element of mass of fluid is acted on by short-range forces arising from interaction with fluid molecules outside this element, these short range forces can act only on a thin layer

$\vec{\Sigma}(\vec{n}, \vec{x}, t)\, \delta A$ force exerted on surface element

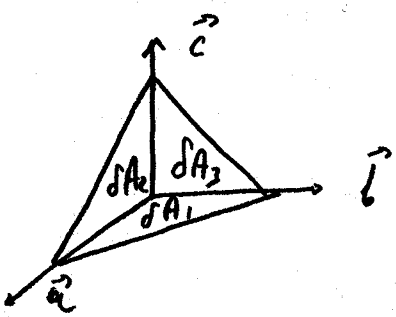
surface area



$\vec{\Sigma}$ is stress exerted by the fluid on the side of the surface element to which $\vec{n}$ points, on the fluid on the side which $\vec{n}$ points away from

$-\vec{\Sigma}(\vec{n}, \vec{x}, t)\, \delta A$ force exerted across the surface element on the fluid on the side to which $\vec{n}$ points

$= \vec{\Sigma}(-\vec{n}, \vec{x}, t)\, \delta A$



tetrahedron volume element

$$\vec{\Sigma}(\vec{n})\, \delta A + \vec{\Sigma}(-\vec{a})\, \delta A_1 + \vec{\Sigma}(-\vec{b})\, \delta A_2 + \vec{\Sigma}(-\vec{c})\, \delta A_3$$

total force on volume element $\quad$ $\vec{x}$ dependence negligible

$$\delta A_1 = \vec{a} \cdot \vec{n}\, \delta A$$

$$\delta A_2 = \vec{b} \cdot \vec{n}\, \delta A$$

$$\delta A_3 = \vec{c} \cdot \vec{n}\, \delta A$$

$$\left[\Sigma_i (n) - \left\{ a_j \Sigma_i(\vec{a}) + b_j \Sigma_i(\vec{b}) + c_j \Sigma_i(\vec{c}) \right\} n_j \right] \delta A$$

i^{th} component of total force

δV much smaller than δA in $\delta V \to 0$ limit

mass × acceleration in tettahedron is $\sim \delta V$

= resultant of body forces $\sim \delta V$ + resultant of surface forces $\sim \delta A$

$\Downarrow$

$$\Sigma_i (\vec{n}) = \left\{ a_j \Sigma_i(\vec{a}) + b_j \Sigma_i(\vec{b}) + c_j \Sigma_i(\vec{c}) \right\} n_j$$

$\vec{n}$, $\vec{\Sigma}$ do not depend on choice of reference axes $\Rightarrow$ $\{\ \}_{ij}$ must be represented by reference independent tensor

$$\Sigma_i (\vec{n}) = \sigma_{ij}\, n_j$$

σ_{ij} stress tensor

It is the i^{th} component of the force per unit area exerted across a plane surface element normal to the j-direction, at position $\vec{x}$ in the fluid and at time t

$\sigma_{ij} = \sigma_{ji}$ six independent components

symmetric

3 diagonal normal stresses

non-diagonal tangential stresses

$\sigma_{ij} = -p\,\delta_{ij}$ for fluid at rest

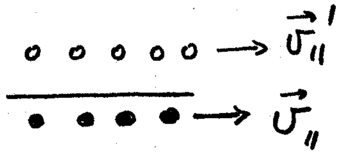
static fluid pressure

For moving fluid the combined effect of momentum flux by passage of molecules across the surface element (momentum transport) and forces exerted between molecules on the two sides is represented by the local stress in the fluid.

If velocity is constant in neighborhood of volume element, stress is normal to surface for all orientations of surface element.

If continuum velocity is not uniform, the tangential components of stress may not be zero.

Consider when fluid velocity has component in surface element and a magnitude which varies only normal to the surface element:



shearing motion

tangential component of stress

sign of stress tries to eliminate velocity difference

Transport of momentum across surface constitutes internal friction, viscous fluid motion

Navier - Stokes Eq.

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \Pi_{ik}}{\partial x_k} \qquad \text{Euler eq.}$$

added term!

$$\Pi_{ik} = p\,\delta_{ik} + \rho v_i v_k - \sigma'_{ik} = -\sigma_{ik} + \rho v_i v_k$$

$$\sigma_{ik} = -p\,\delta_{ik} + \sigma'_{ik}$$

viscous stress tensor

σ'_{ik} vanishes for $\vec{v} = \text{const}$

first order in velocity gradient

$$\sigma'_{ik} = \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_l}{\partial x_l} \right) + \zeta \delta_{ik} \frac{\partial v_l}{\partial x_l}$$

coefficient of viscosity (η) second viscosity (ζ)

$\eta > 0, \quad \zeta > 0$

σ'_{ik} vanishes if $\vec{v} = \vec{\Omega} \times \vec{r}$ for body in uniform rotation

$$\rho \left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} \right) = - \frac{\partial p}{\partial x_i} +$$

$$+ \frac{\partial}{\partial x_k} \left\{ \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_l}{\partial x_l} \right) \right\} + \frac{\partial}{\partial x_i} \left(\zeta \frac{\partial v_l}{\partial x_l} \right)$$

if η, ζ constant:

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \text{grad}) \vec{v} \right] = - \text{grad}\, p + \eta \Delta \vec{v} + \left(\zeta + \frac{1}{3} \eta \right) \text{grad div}\, \vec{v}$$

Navier - Stokes Eq.

Simplifies for incompressible fluid

$\rho = \text{const}$

$$\underbrace{\frac{\partial \rho}{\partial t}}_{0} + \rho \operatorname{div} \vec{v} + \vec{v} \cdot \underbrace{\operatorname{grad} \rho}_{0} = 0$$

$$\operatorname{div} \vec{v} = 0 \qquad \text{incompressible}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \operatorname{grad}) \vec{v} = -\frac{1}{\rho} \operatorname{grad} p + \frac{\eta}{\rho} \Delta \vec{v}$$

$$\sigma_{ik} = -p \delta_{ik} + \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right)$$

$\nu = \frac{\eta}{\rho}$ kinetic viscosity $\qquad$ η dynamic viscosity

	η (g/cm sec)	ν (cm^2/sec)	20°C
Water	0.010	0.010	
Air	0.00018	0.150	
Alcohol	0.018	0.022	
Glycerine	8.5	6.8	
Mercury	0.0156	0.0012	