Physics 2D | Quiz 5 Solutions | Fall 2009

Problem 1

$$\psi(x) = C x e^{-x/2} \qquad x \geq 0$$

$$1 = \int dx\, |\psi(x)|^2 = C^2 \int_0^\infty dx\, x^2 e^{-x} = 2C^2 \implies$$

$$\boxed{C = \frac{1}{\sqrt{2}}} \text{ (a)}$$

(b) $$\langle x \rangle = \int dx\, x |\psi(x)|^2 = \frac{1}{2}\int_0^\infty dx\, x^3 e^{-x} = \frac{3!}{2} = 3$$

$$\implies \boxed{\langle x \rangle = 3} \text{ (b)}$$

(c) $$p = \frac{\hbar}{i}\frac{d}{dx} \ ; \quad \frac{d}{dx}\left(x e^{-x/2}\right) = e^{-x/2} - \frac{x}{2} e^{-x/2} \ ;$$

$$\frac{d^2}{dx^2} x e^{-x/2} = -\frac{1}{2} e^{-x/2} - \frac{1}{2} e^{-x/2} + \frac{1}{4} x e^{-x/2} = -e^{-x/2} + \frac{1}{4} x e^{-x/2}$$

$$\langle p \rangle = \int dx\, \psi^*(x)\, p\, \psi(x) = \frac{\hbar}{i}\cdot\frac{1}{2}\cdot\int_0^\infty dx\, x e^{-x/2}\left(e^{-x/2} - \frac{x}{2} e^{-x/2}\right) =$$

$$= \frac{\hbar}{i}\cdot\frac{1}{2}\cdot\int_0^\infty dx\left(x e^{-x} - \frac{x^2}{2} e^{-x}\right) = \frac{\hbar}{i}\cdot\frac{1}{2}\cdot\left(1! - \frac{2!}{2}\right) = 0$$

$$\implies \boxed{\langle p \rangle = 0}$$

$$\langle p^2 \rangle = -\hbar^2 \int dx\, \psi^*(x) \frac{d^2}{dx^2}\psi(x) = -\frac{\hbar^2}{2}\int dx\, x e^{-x/2}\left(-e^{-x/2} + \frac{1}{4} x e^{-x/2}\right) =$$

$$= -\frac{\hbar^2}{2}\int_0^\infty dx\left(-x e^{-x} + \frac{1}{4} x^2 e^{-x}\right) = -\frac{\hbar^2}{2}\left(-1 + \frac{2}{4}\right) = -\frac{\hbar^2}{2}\left(-1 + \frac{1}{2}\right) = \frac{\hbar^2}{4}$$

$$\implies \boxed{\langle p^2 \rangle = \frac{\hbar^2}{4}} \ , \quad \boxed{\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\hbar}{2}}$$

Problem 2

$$\Psi(x) = C x e^{-3x}$$

$$Q = \frac{1}{x} - \frac{d}{dx}$$

$$Q\Psi = \left(\frac{1}{x} - \frac{d}{dx}\right) C x e^{-3x} = C e^{-3x} - C \frac{d}{dx}\left(x e^{-3x}\right) =$$

$$= \cancel{C e^{-3x}} - \cancel{C e^{-3x}} + 3 C x e^{-3x} = 3 C x e^{-3x} = 3 \Psi(x)$$

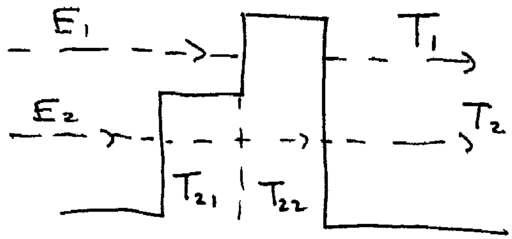
$$\Rightarrow \boxed{Q\Psi = q\Psi \quad \text{with} \quad q = 3 \text{ the eigenvalue}} \quad \Psi \text{ is the eigenfunction}$$

(b) Since $Q^2 \Psi = q^2 \Psi \Rightarrow$

$$\boxed{\langle Q^2 \rangle = \langle Q \rangle^2 = q^2}$$

$$\boxed{\Delta Q = \sqrt{\langle Q^2 \rangle - \langle Q \rangle^2} = 0}$$

Problem 3



$$T = e^{-2\sqrt{\frac{2m}{\hbar^2}}\sqrt{U-E}\,\cdot\,\Delta x}$$ for a constant potential.

For electrons with $E_1 = 4eV$, $T = T_1 = e^{-2\sqrt{\frac{2m}{\hbar^2}}\sqrt{(5eV-4eV)}\,\Delta x}$

with $\Delta x = 1\,\text{Å}$.

For electrons with energy $E_2 = 3eV$, we see from the picture that

$$T_2 = T_{21} \cdot T_{22}, \text{ with } T_{21} = T_1 \text{ since } \sqrt{U-E} =$$

$$= \sqrt{3eV - 2eV} = \sqrt{5eV-4eV}, \text{ and } \Delta x = 1\,\text{Å also.}$$

$$T_{22} = e^{-2\sqrt{\frac{2m}{\hbar^2}}\sqrt{5eV-2eV}\,\cdot\,\Delta x} = e^{-2\sqrt{\frac{1}{3.81}}\sqrt{3}\,\cdot\,1} = e^{-1.775}$$

So: $$\frac{T_2}{T_1} = \frac{T_1 \cdot T_{22}}{T_1} = T_{22} = e^{-1.775} = 0.170$$

So if 1000 electrons get transmitted with energy $E_1 = 4eV$,

$1000 \times 0.170 = \boxed{170 \text{ electrons get transmitted w/ energy } E_2 = 2eV}$