Physics 2D — Quiz 4 Solutions — Fall 2009

Problem 1

The average k of this wavepacket is $k_0 = \frac{k_1+k_2}{2} = 300\ \text{Å}^{-1}$.

The spread in k is $\Delta k = k_2 - k_1 = 2\ \text{Å}^{-1} \Rightarrow k_1 = k_0 - \frac{\Delta k}{2}$, $k_2 = k_0 + \frac{\Delta k}{2}$

(a) From the uncertainty principle

$$\Delta x \Delta k \sim 1 \Rightarrow \boxed{\Delta x \sim \frac{1}{\Delta k} = 0.5\ \text{Å}}$$

More precisely, for this $a(k)$, $\Delta x \Delta k = 4\pi \Rightarrow \Delta x = 2\pi\ \text{Å}$

(b) The average momentum of this electron is $p = \hbar k_0$

$$pc = \hbar c k_0 = 1973\ \text{eV Å} \times 300\ \text{Å}^{-1} = 591{,}900\ \text{eV}.$$

If we use $p = mv$ we get $\frac{v}{c} = \frac{pc}{mc^2} = \frac{591{,}900}{0.511\times 10^6} = 1.158 > c$

not right, this electron is relativistic. Use relativistic formulas:

$$E = \sqrt{p^2c^2 + m^2c^4} = \sqrt{591{,}900^2 + mc^2} = 781{,}963\ \text{eV}$$

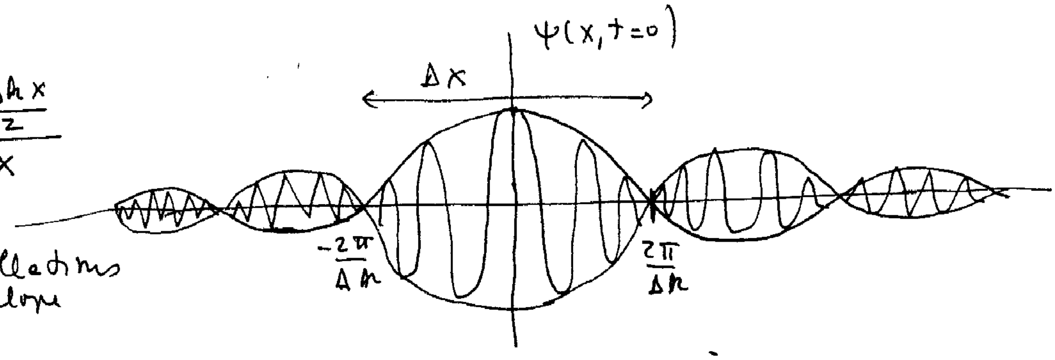
$$E = \gamma mc^2 \Rightarrow \gamma = 1.530 = \frac{1}{\sqrt{1-v^2/c^2}} \Rightarrow \boxed{\frac{v}{c} = \sqrt{1-\frac{1}{\gamma^2}} = 0.757}$$

Alternative solution: $p = \gamma m v \Rightarrow pc = \frac{v/c}{\sqrt{1-v^2/c^2}} mc^2$, solve for

$$\frac{v}{c} = \frac{pc/mc^2}{\sqrt{1+(pc/mc^2)^2}} = 0.757$$

(c)
$$\psi(x,t=0) = \int_{k_1}^{k_2} dk\, e^{ikx} = \frac{e^{ik_2x} - e^{ik_1x}}{ix} = e^{ik_0x}\frac{e^{i\frac{\Delta k}{2}x} - e^{-i\frac{\Delta k}{2}x}}{ix} =$$

$$= 2e^{ik_0x}\frac{\sin\frac{\Delta k x}{2}}{\Delta k\, x}$$



$k_0 \gg \Delta k \Rightarrow$ a lot of oscillations inside envelope

Problem 2 : $U(x) = C\frac{x^2}{a^2}$

The size is determined by minimizing total energy (kinetic + potential).

If the average distance between proton and neutron is x_0, the average potential energy is $\langle U\rangle \sim C\frac{x_0^2}{a^2}$. This is minimized if $x_0 = 0$, however that is impossible because it gives infinite kinetic energy.

If the average distance is x_0, the uncertainty in position is $\Delta x \sim x_0$, from the uncertainty principle $\Delta p \sim \frac{\hbar}{\Delta x} = \frac{\hbar}{x_0}$, the average kinetic energy is

$$\langle K\rangle \sim \frac{\langle p^2\rangle}{2m} = \frac{\langle \Delta p^2\rangle}{2m} = \frac{\hbar^2}{2m x_0^2} \Rightarrow \text{total energy is}$$

$$E = U + K = \frac{\hbar^2}{2m x_0^2} + C\frac{x_0^2}{a^2}. \quad \text{Minimize } \left(\frac{dE}{dx_0} = 0\right) \text{ and}$$

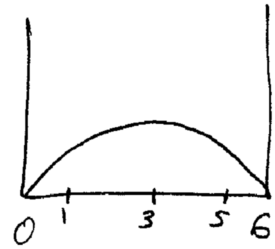
Set $x_0 = a$: $-\frac{\hbar^2}{m a^3} + 2C\cdot\frac{a}{a^2} = 0 \Rightarrow a^2 = \frac{\hbar^2}{2C}$

$$C = \frac{\hbar^2}{2ma^2} = \frac{(\hbar c)^2}{2mc^2a^2} = \frac{1973^2 eV^2 Å^2}{2\times 939\times 10^6 eV \times 10^{-10} Å^2} \Rightarrow \boxed{C = 20.7\ MeV}$$

(b) If instead $a = 0.529$ Å $\Rightarrow \boxed{C = 0.0074\ eV}$

Problem 3

$$E_1 = \frac{\hbar^2 \pi^2}{2 m_e L^2} = \frac{1973^2 \pi^2}{2 \times 0.511 \times 10^6 \times 6^2} \text{ eV} \Rightarrow$$



$$\boxed{E_1 = 1.044 \text{ eV}}$$

(b) $\psi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)$

$$|\psi_1(x)|^2 = \frac{2}{L} \sin^2 \frac{\pi x}{L}$$

Center of box: $x = \frac{L}{2}$, $\sin \frac{\pi}{2} = 1 \Rightarrow |\psi_1(L/2)|^2 = \frac{2}{L}$

At $x = 1A$: $x = \frac{L}{6}$, $\sin \frac{\pi}{6} = \frac{1}{2} \Rightarrow |\psi_1(\frac{L}{6})|^2 = \frac{2}{L} \times \frac{1}{4}$

If electron is at distance 1A from well it is at $x = 1A$ or $x = 5A$, probability is same; so

$$\frac{\text{prob (1A from well)}}{\text{prob (center)}} = \frac{2 \times \frac{1}{4}}{1} = 0.5 = \frac{1}{2}$$

$\Rightarrow$ **electron is twice as likely to be at center than at distance 1A from a well.**

(c) Largest wavelength $\Rightarrow$ smallest energy difference $\Rightarrow$ transition $n=1$ to $n=2$.

$$\frac{hc}{\lambda} = E_2 - E_1 = 2^2 E_1 - E_1 = 3 E_1 = 3.133 \text{ eV}$$

$$\lambda = \frac{hc}{E_2 - E_1} = \frac{12{,}400}{3.133} \text{ Å} \Rightarrow \boxed{\lambda = 3958 \text{ Å}}$$