6-35 Applying the momentum operator $[p_x] = \left(\frac{\hbar}{i}\right)\frac{d}{dx}$ to each of the candidate functions yields

(a) $[p_x]\{A\sin(kx)\} = \left(\frac{\hbar}{i}\right)k\{A\cos(kx)\}$

(b) $[p_x]\{A\sin(kx) - A\cos(kx)\} = \left(\frac{\hbar}{i}\right)k\{A\cos(kx) + A\sin(kx)\}$

(c) $[p_x]\{A\cos(kx) + iA\sin(kx)\} = \left(\frac{\hbar}{i}\right)k\{-A\sin(kx) + iA\cos(kx)\}$

(d) $[p_x]\{e^{ik(x-a)}\} = \left(\frac{\hbar}{i}\right)ik\{e^{ik(x-a)}\}$

In case (c), the result is a multiple of the original function, since

$$-A\sin(kx) + iA\cos(kx) = i\{A\cos(kx) + iA\sin(kx)\}.$$

The multiple is $\left(\frac{\hbar}{i}\right)(ik) = \hbar k$ and is the eigenvalue. Likewise for (d), the operation $[p_x]$ returns the original function with the multiplier $\hbar k$. Thus, (c) and (d) are eigenfunctions of $[p_x]$ with eigenvalue $\hbar k$, whereas (a) and (b) are not eigenfunctions of this operator.

6-37 (a) Normalization requires

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\Psi|^2\, dx = C^2 \int_{-\infty}^{\infty} \{\psi_1^* + \psi_2^*\}\{\psi_1 + \psi_2\}dx \\ &= C^2\left\{\int |\psi_1|^2\, dx + \int |\psi_2|^2\, dx + \int \psi_2^*\psi_1 dx + \int \psi_1^*\psi_2\, dx\right\} \end{aligned}.$$

The first two integrals on the right are unity, while the last two are, in fact, the same integral since $\psi_1$ and $\psi_2$ are both real. Using the waveforms for the infinite square well, we find

$$\int \psi_2\psi_1 dx = \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)dx = \frac{1}{L}\int_0^L \left\{\cos\left(\frac{\pi x}{L}\right) - \cos\left(\frac{3\pi x}{L}\right)\right\}dx$$

where, in writing the last line, we have used the trigonometric exponential identities of sine and cosine. Both of the integrals remaining are readily evaluated, and are zero. Thus, $1 = C^2\{1 + 0 + 0 + 0\} = 2C^2$, or $C = \frac{1}{\sqrt{2}}$. Since $\psi_{1,2}$ are stationary states, they develop in time according to their respective energies $E_{1,2}$ as $e^{-iEt/\hbar}$. Then $\Psi(x, t) = C\{\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar}\}$.

(c) $\Psi(x,\, t)$ is a stationary state only if it is an eigenfunction of the energy operator $[E] = i\hbar \frac{\partial}{\partial t}$. Applying $[E]$ to $\Psi$ gives

$$[E]\Psi = C\left\{ i\hbar\left(\frac{-iE_1}{\hbar}\right)\psi_1 e^{-iE_1 t/\hbar} + i\hbar\left(\frac{-iE_2}{\hbar}\right)\psi_2 e^{-iE_2 t/\hbar} \right\} = C\left\{E_1\psi_1 e^{-iE_1 t/\hbar} + E_2\psi_2 e^{-iE_2 t/\hbar}\right\}.$$

Since $E_1 \neq E_2$, the operations $[E]$ does *not* return a multiple of the wavefunction, and so $\Psi$ is not a stationary state. Nonetheless, we may calculate the average energy for this state as

$$\begin{aligned}\langle E\rangle &= \int \Psi^*[E]\Psi dx = C^2 \int \left\{\psi_1^* e^{+iE_1 t/\hbar} + \psi_2^* e^{+iE_2 t/\hbar}\right\}\left\{E_1\psi_1 e^{-iE_1 t/\hbar} + E_2\psi_2 e^{-iE_2 t/\hbar}\right\}dx \\ &= C^2\left\{E_1 \int |\psi_1|^2 dx + E_2 \int |\psi_2|^2 dx\right\}\end{aligned}$$

with the cross terms vanishing as in part (a). Since $\psi_{1,2}$ are normalized and $C^2 = \frac{1}{2}$ we get finally $\langle E\rangle = \frac{E_1 + E_2}{2}$.

7-1 (a) The reflection coefficient is the ratio of the reflected intensity to the incident wave intensity, or $R = \frac{|(1/2)(1-i)|^2}{|(1/2)(1+i)|^2}$. But $|1-i|^2 = (1-i)(1-i)^* = (1-i)(1+i) = |1+i|^2 = 2$, so that $R = 1$ in this case.

(b) To the left of the step the particle is free. The solutions to Schrödinger's equation are $e^{\pm ikx}$ with wavenumber $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$. To the right of the step $U(x) = U$ and the equation is $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(U - E)\psi(x)$. With $\psi(x) = e^{-kx}$, we find $\frac{d^2\psi}{dx^2} = k^2\psi(x)$, so that $k = \left[\frac{2m(U-E)}{\hbar^2}\right]^{1/2}$. Substituting $k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$ shows that $\left[\frac{E}{(U-E)}\right]^{1/2} = 1$ or $\frac{E}{U} = \frac{1}{2}$.

(c) For 10 MeV protons, $E = 10$ MeV and $m = \frac{938.28 \text{ MeV}}{c^2}$. Using $\hbar = 197.3 \text{ MeV fm}/c\,(1 \text{ fm} = 10^{-15} \text{ m})$, we find

$$\delta = \frac{1}{k} = \frac{\hbar}{(2mE)^{1/2}} = \frac{197.3 \text{ MeV fm}/c}{\left[(2)(938.28 \text{ MeV}/c^2)(10 \text{ MeV})\right]^{1/2}} = 1.44 \text{ fm}.$$

7-2 (a) To the left of the step the particle is free with kinetic energy $E$ and corresponding wavenumber $k_1 = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$:

$$\psi(x) = Ae^{ik_1 x} + Be^{-ik_1 x} \qquad x \le 0$$

To the right of the step the kinetic energy is reduced to $E-U$ and the wavenumber is now $k_2 = \left[\frac{2m(E-U)}{\hbar^2}\right]^{1/2}$

$$\psi(x) = Ce^{ik_2x} + De^{-ik_2x} \qquad x \geq 0$$

with $D = 0$ for waves incident on the step from the left. At $x = 0$ both $\psi$ and $\frac{d\psi}{dx}$ must be continuous: $\psi(0) = A + B = C$

$$\left.\frac{d\psi}{dx}\right|_0 = ik_1(A-B) = ik_2C.$$

(b) Eliminating $C$ gives $A + B = \frac{k_1}{k_2}(A-B)$ or $A\left(\frac{k_1}{k_2} - 1\right) = B\left(\frac{k_1}{k_2} + 1\right)$. Thus,

$$R = \left|\frac{B}{A}\right|^2 = \frac{(k_1/k_2 - 1)^2}{(k_1/k_2 + 1)^2} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

$$T = 1 - R = \frac{4k_1k_2}{(k_1 + k_2)^2}$$

(c) As $E \to U$, $k_2 \to 0$, and $R \to 1$, $T \to 0$ (no transmission), in agreement with the result for any energy $E < U$. For $E \to \infty$, $k_1 \to k_2$ and $R \to 0$, $T \to 1$ (perfect transmission) suggesting correctly that very energetic particles do not *see* the step and so are unaffected by it.

7-3 With $E = 25\text{ MeV}$ and $U = 20\text{ MeV}$, the ratio of wavenumber is $\frac{k_1}{k_2} = \left(\frac{E}{E-U}\right)^{1/2} = \left(\frac{25}{25-20}\right)^{1/2} = \sqrt{5} = 2.236$. Then from Problem 7-2 $R = \frac{(\sqrt{5}-1)^2}{(\sqrt{5}+1)^2} = 0.146$ and $T = 1 - R = 0.854$. Thus, 14.6% of the incoming particles would be reflected and 85.4% would be transmitted. For electrons with the same energy, the transparency and reflectivity of the step are unchanged.

7-4 The reflection coefficient for this case is given in Problem 7-2 as

$$R = \left|\frac{B}{A}\right|^2 = \frac{(k_1/k_2 - 1)^2}{(k_1/k_2 + 1)^2} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}.$$

The wavenumbers are those for electrons with kinetic energies $E = 54.0\text{ eV}$ and $E - U = 54.0\text{ eV} + 10.0\text{ eV} = 64.0\text{ eV}$:

$$\frac{k_1}{k_2} = \left(\frac{E}{E-U}\right)^{1/2} = \left(\frac{54\text{ eV}}{64\text{ eV}}\right)^{1/2} = 0.918\,6.$$

Then, $R = \frac{(0.918\,6-1)^2}{(0.918\,6+1)^2} = 1.80\times 10^{-3}$ is the fraction of the incident beam that is reflected at the boundary.

7-5 (a) The transmission probability according to Equation 7.9 is $\frac{1}{T(E)} = 1 + \left[\frac{U^2}{4E(U-E)}\right]\sinh^2 \alpha L$ with $\alpha = \frac{[2m(U-E)]^{1/2}}{\hbar}$. For $E \ll U$, we find $(\alpha L)^2 \approx \frac{2mUL^2}{\hbar^2} \gg 1$ by hypothesis. Thus, we may write $\sinh \alpha L \approx \frac{1}{2}e^{\alpha L}$. Also $U - E \approx U$, giving $\frac{1}{T(E)} \approx 1 + \left(\frac{U}{16E}\right)e^{2\alpha L} \approx \left(\frac{U}{16E}\right)e^{2\alpha L}$ and a probability for transmission $P = T(E) = \left(\frac{16E}{U}\right)e^{-2\alpha L}$.

(b) Numerical Estimates: $(\hbar = 1.055\times 10^{-34}\ \text{Js})$

1) For $m = 9.11\times 10^{-31}$ kg, $U - E = 1.60\times 10^{-21}$ J, $L = 10^{-10}$ m; $\alpha = \frac{[2m(U-E)]^{1/2}}{\hbar} = 5.12\times 10^{8}\ \text{m}^{-1}$ and $e^{-2\alpha L} = 0.90$

2) For $m = 9.11\times 10^{-31}$ kg, $U - E = 1.60\times 10^{-19}$ J, $L = 10^{-10}$ m; $\alpha = 5.12\times 10^{9}\ \text{m}^{-1}$ and $e^{-2\alpha L} = 0.36$

3) For $m = 6.7\times 10^{-27}$ kg, $U - E = 1.60\times 10^{-13}$ J, $L = 10^{-15}$ m; $\alpha = 4.4\times 10^{14}\ \text{m}^{-1}$ and $e^{-2\alpha L} = 0.41$

4) For $m = 8$ kg, $U - E = 1$ J, $L = 0.02$ m; $\alpha = 3.8\times 10^{34}\ \text{m}^{-1}$ and $e^{-2\alpha L} = e^{-1.5\times 10^{33}} \approx 0$

7-16 Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about $3\,755.8\ \text{MeV}/c^2$, the first approximation to the decay length $\delta$ is

$$\delta \approx \frac{\hbar}{(2mU)^{1/2}} = \frac{197.3\ \text{MeV fm}/c}{\left[2\left(3\,755.8\ \text{MeV}/c^2\right)(30\ \text{MeV})\right]^{1/2}} = 0.415\,6\ \text{fm}.$$

This gives an effective width for the (infinite) well of $R + \delta = 9.415\,6$ fm, and a ground state energy $E_1 = \frac{\pi^2(197.3\ \text{MeV fm}/c)^2}{2\left(3\,755.8\ \text{MeV}/c^2\right)(9.415\,6\ \text{fm})^2} = 0.577$ MeV. From this $E$ we calculate $U - E = 29.42$ MeV and a new decay length

$$\delta = \frac{197.3\ \text{MeV fm}/c}{\left[2\left(3\,755.8\ \text{MeV}/c^2\right)(29.42\ \text{MeV})\right]^{1/2}} = 0.419\,7\ \text{fm}.$$

This, in turn, increases the effective well width to 9.419 7 fm and lowers the ground state energy to $E_1 = 0.576\ \text{MeV}$. Since our estimate for $E$ has changed by only 0.001 MeV, we may be content with this value. With a kinetic energy of $E_1$, the alpha particle in the ground state has speed $v_1 = \left(\frac{2E_1}{m}\right)^{1/2} = \left[\frac{2(0.576\ \text{MeV})}{(3\,755.8\ \text{MeV}/c^2)}\right]^{1/2} = 0.017\ 5c$. In order to be ejected with a kinetic energy of 4.05 MeV, the alpha particle must have been preformed in an excited state of the nuclear well, not the ground state.

7-17 The collision frequency $f$ is the reciprocal of the transit time for the alpha particle crossing the nucleus, or $f = \frac{v}{2R}$, where $v$ is the speed of the alpha. Now $v$ is found from the kinetic energy which, inside the nucleus, is not the total energy $E$ but the difference $E - U$ between the total energy and the potential energy representing the bottom of the nuclear well. At the nuclear radius $R = 9\ \text{fm}$, the Coulomb energy is

$$\frac{k(Ze)(2e)}{R} = 2Z\left(\frac{ke^2}{a_0}\right)\left(\frac{a_0}{R}\right) = 2(88)(27.2\ \text{eV})\left(\frac{5.29\times 10^4\ \text{fm}}{9\ \text{fm}}\right) = 28.14\ \text{MeV}\ .$$

From this we conclude that $U = -1.86\ \text{MeV}$ to give a nuclear barrier of $30\ \text{MeV}$ overall. Thus an alpha with $E = 4.05\ \text{MeV}$ has kinetic energy $4.05 + 1.86 = 5.91\ \text{MeV}$ inside the nucleus. Since the alpha particle has the combined mass of 2 protons and 2 neutrons, or about $3\,755.8\ \text{MeV}/c^2$ this kinetic energy represents a speed

$$v = \left(\frac{2E_k}{m}\right)^{1/2} = \left[\frac{2(5.91)}{3\,755.8\ \text{MeV}/c^2}\right]^{1/2} = 0.056c\ .$$

Thus, we find for the collision frequency $f = \frac{v}{2R} = \frac{0.056c}{2(9\ \text{fm})} = 9.35\times 10^{20}\ \text{Hz}$.

7-18 Any one conduction electron of the metal is virtually free to move about with a speed $v$ fixed by its kinetic energy $E_k = \frac{1}{2}mv^2$, but the average energy per electron available for motion in any specific direction (say, normal to the surface) is reduced from this by the factor 1/3 to account for the random directions of travel:

$$\langle E_k \rangle = \frac{1}{2}m\left\{\langle v_x^2 \rangle + \langle v_y^2 \rangle + \langle v_z^2 \rangle\right\} = \frac{3}{2}m\langle v_x^2 \rangle, \text{ or } \frac{1}{2}m\langle v_x^2 \rangle = \frac{1}{3}\langle E_k \rangle.$$

For a sample with dimension $L$ normal to the surface, the time elapsed between collisions with this surface is $\frac{2L}{|v_x|}$, for any one electron. The reciprocal of this time is the collision frequency. For two electrons, collisions occur twice as often, and so forth, so that the collision frequency for $N$ electrons is $\frac{N|v_x|}{2L}$. Making the identification $|v_x|^2 = \langle v_x^2 \rangle$ allows us to write the collision frequency $f$ in terms of electron energy as $f = \frac{N}{2L}\left(\frac{2E_k}{3m}\right)^{1/2}$. The density of copper is $8.96\ \text{g/cm}^3$, so one cubic centimeter represents an amount of copper equal to 8.96 g, or the equivalent of $\frac{8.96}{63.54} = 0.141$ moles (the atomic weight of copper is 63.54). Since each mole contains a number of atoms equal to Avogadro's number $N_A = 6.02 \times 10^{23}$, the number of copper atoms in our sample is $0.141N_A$ or about $8.49 \times 10^{22}$, which is also the number $N$ of conduction electrons.

The most energetic electrons in copper have kinetic energies of about 7 eV. Using this for $E_k$, $L = 1$ cm, and $N = 8.49 \times 10^{22}$ gives for the collision frequency $f = 3.85 \times 10^{30}$ Hz.

## Addendum to problem 16

The ground state energy is $E_1 = 0.576\,MeV$, so (7.10) gives:

$T \sim \exp(-\frac{2}{\hbar}\sqrt{2m}\int_{9fm}^{19fm}\sqrt{U-E_1}\,dx) = \exp(-\frac{2}{\hbar c}10\,fm\sqrt{2mc^2(U-E_1)}) = \exp(-\frac{2}{197.3\,Mev\,fm}10\,fm\sqrt{2*3755.8\,MeV*(30-0.576)\,Mev}) \approx e^{-47.6}$

For the $n = 6$ state we first need to approximately determine the energy by the same iterative procedure. Note that this procedure need not always start with $\delta$, we may start with the energy as well, and as in this case we expect the energy to be much closer to $U$, we do start with the energy (the final result, of course, should not depend on the starting point if the process converges, however the number of steps to reach the desired accuracy does). So we first treat the well as infinite and calculate $E_6$:

$E_6 = \frac{36\pi^2\hbar^2}{2mL^2} = \frac{36\pi^2(\hbar c)^2}{2mc^2L^2} = \frac{36\pi^2(197.3\,MeV\,fm)^2}{2*3755.8\,MeV*(9\,fm)^2} \approx 22.732\,MeV$

Now calculate $\delta$ with this energy:

$\delta = \frac{\hbar}{\sqrt{2m(U-E)}} = \frac{\hbar c}{\sqrt{2mc^2(U-E)}} = \frac{197.3\,MeV\,fm}{\sqrt{2*3755.8\,MeV*(30-22.732)\,MeV}} \approx 0.844\,fm$

Using this $\delta$ we calculate $E_6$ again (note that we use $L+\delta$ rather than $L+2\delta$ as the well is semi-infinite), and keep doing this:

$E_6 = \frac{36\pi^2\hbar^2}{2m(L+\delta)^2} = \frac{36\pi^2(\hbar c)^2}{2mc^2(L+\delta)^2} = \frac{36\pi^2(197.3\,MeV\,fm)^2}{2*3755.8\,MeV*(9.844\,fm)^2} \approx 19.00\,MeV$

$\delta = \frac{197.3\,MeV\,fm}{\sqrt{2*3755.8\,MeV*(30-19.00)\,MeV}} \approx 0.686\,fm$

$E_6 = \frac{36\pi^2(197.3\,MeV\,fm)^2}{2*3755.8\,MeV*(9.686\,fm)^2} \approx 19.626\,MeV$

$\delta = \frac{197.3\,MeV\,fm}{\sqrt{2*3755.8\,MeV*(30-19.626)\,MeV}} \approx 0.707\,fm$

$E_6 = \frac{36\pi^2(197.3\,MeV\,fm)^2}{2*3755.8\,MeV*(9.707\,fm)^2} \approx 19.541\,MeV$

$\delta = \frac{197.3\,MeV\,fm}{\sqrt{2*3755.8\,MeV*(30-19.541)\,MeV}} \approx 0.704\,fm$

$E_6 = \frac{36\pi^2(197.3\,MeV\,fm)^2}{2*3755.8\,MeV*(9.704\,fm)^2} \approx 19.553\,MeV$

We stop here as the last two values of $E_6$ are very close to each other. Using this energy we calculate the transmission coefficient exactly as for the ground state:

$T \sim \exp(-\frac{2}{\hbar}\sqrt{2m}\int_{9fm}^{19fm}\sqrt{U-E_6}\,dx) = \exp(-\frac{2}{\hbar c}10\,fm\sqrt{2mc^2(U-E_6)}) = \exp(-\frac{2}{197.3\,Mev\,fm}10\,fm\sqrt{2*3755.8\,MeV*(30-19.553)\,Mev}) \approx e^{-28.4}$