6-6 $$\psi(x) = A\cos kx + B\sin kx$$
$$\frac{\partial\psi}{\partial x} = -kA\sin kx + kB\cos kx$$
$$\frac{\partial^2\psi}{\partial x^2} = -k^2 A\cos kx - k^2 B\sin kx$$
$$\left(\frac{-2m}{\hbar^2}\right)(E-U)\psi = \left(\frac{-2mE}{\hbar^2}\right)(A\cos kx + B\sin kx)$$

The Schrödinger equation is satisfied if $\frac{\partial^2\psi}{\partial x^2} = \left(\frac{-2m}{\hbar^2}\right)(E-U)\psi$ or

$$-k^2(A\cos kx + B\sin kx) = \left(\frac{-2mE}{\hbar^2}\right)(A\cos kx + B\sin kx).$$

Therefore $E = \frac{\hbar^2 k^2}{2m}$.

6-9 $E_n = \frac{n^2 h^2}{8mL^2}$, so $\Delta E = E_2 - E_1 = \frac{3h^2}{8mL^2}$

$$\Delta E = (3)\frac{(1240 \text{ eV nm}/c)^2}{8(938.28\times10^6 \text{ eV}/c^2)(10^{-5} \text{ nm})^2} = 6.14 \text{ MeV}$$
$$\lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV nm}}{6.14\times10^6 \text{ eV}} = 2.02\times10^{-4} \text{ nm}$$

This is the gamma ray region of the electromagnetic spectrum.

6-10 $E_n = \frac{n^2 h^2}{8mL^2}$

$$\frac{h^2}{8mL^2} = \frac{(6.63\times10^{-34} \text{ Js})^2}{8(9.11\times10^{-31} \text{ kg})(10^{-10} \text{ m})^2} = 6.03\times10^{-18} \text{ J} = 37.7 \text{ eV}$$

(a) $E_1 = 37.7$ eV

$E_2 = 37.7\times2^2 = 151$ eV

$E_3 = 37.7\times3^2 = 339$ eV

$E_4 = 37.7\times4^2 = 603$ eV

(b) $hf = \frac{hc}{\lambda} = E_{n_i} - E_{n_f}$

$$\lambda = \frac{hc}{E_{n_i} - E_{n_f}} = \frac{1\,240 \text{ eV}\cdot\text{nm}}{E_{n_i} - E_{n_f}}$$

For $n_i = 4$, $n_f = 1$, $E_{n_i} - E_{n_f} = 603 \text{ eV} - 37.7 \text{ eV} = 565 \text{ eV}$, $\lambda = 2.19$ nm

$n_i = 4$, $n_f = 2$, $\lambda = 2.75$ nm

$n_i = 4$, $n_f = 3$, $\lambda = 4.70$ nm

$n_i = 3$, $n_f = 1$, $\lambda = 4.12$ nm

$n_i = 3$, $n_f = 2$, $\lambda = 6.59$ nm

$n_i = 2$, $n_f = 1$, $\lambda = 10.9$ nm

6-12 $\Delta E=\frac{hc}{\lambda}=\left(\frac{h^2}{8mL^2}\right)\left[2^2-1^2\right]$ and $L=\left[\frac{(3/8)h\lambda}{mc}\right]^{1/2}=7.93\times10^{-10}\text{ m}=7.93\text{ Å}.$

6-13 (a) Proton in a box of width $L=0.200\text{ nm}=2\times10^{-10}\text{ m}$

$$E_1=\frac{h^2}{8m_pL^2}=\frac{\left(6.626\times10^{-34}\text{ J}\cdot\text{s}\right)^2}{8\left(1.67\times10^{-27}\text{ kg}\right)\left(2\times10^{-10}\text{ m}\right)^2}=8.22\times10^{-22}\text{ J}$$
$$=\frac{8.22\times10^{-22}\text{ J}}{1.60\times10^{-19}\text{ J/eV}}=5.13\times10^{-3}\text{ eV}$$

(b) Electron in the same box:

$$E_1=\frac{h^2}{8m_eL^2}=\frac{\left(6.626\times10^{-34}\text{ J}\cdot\text{s}\right)^2}{8\left(9.11\times10^{-31}\text{ kg}\right)\left(2\times10^{-10}\text{ m}\right)^2}=1.506\times10^{-18}\text{ J}=9.40\text{ eV}.$$

(c) The electron has a much higher energy because it is much less massive.

6-14 (a) Still, $\frac{n\lambda}{2}=L$ so $p=\frac{h}{\lambda}=\frac{nh}{2L}$

$$K=\left[c^2p^2+\left(mc^2\right)^2\right]^{1/2}-\left(mc^2\right)=E-mc^2$$
$$E_n=\left[\left(\frac{nhc}{2L}\right)^2+\left(mc^2\right)^2\right]^{1/2},$$
$$K_n=\left[\left(\frac{nhc}{2L}\right)^2+\left(mc^2\right)^2\right]^{1/2}-mc^2$$

(b) Taking $L=10^{-12}\text{ m}$, $m=9.11\times10^{-31}\text{ kg}$, and $n=1$ we find $K_1=4.69\times10^{-14}\text{ J}$. The nonrelativistic result is

$$E_1=\frac{h^2}{8mL^2}=\frac{\left(6.63\times10^{-34}\text{ J}\cdot\text{s}\right)^2}{8\left(9.11\times10^{-31}\text{ kg}\right)\left(10^{-24}\text{ m}^2\right)}=6.03\times10^{-14}\text{ J}$$

Comparing this with K_1, we see that this value is too big by 29%.

6-16 (a) $\psi(x)=A\sin\left(\frac{\pi x}{L}\right)$, $L=3$ Å. Normalization requires

$$1=\int_0^L|\psi|^2\,dx=\int_0^L A^2\sin^2\left(\frac{\pi x}{L}\right)dx=\frac{LA^2}{2}$$

so $A=\left(\frac{2}{L}\right)^{1/2}$

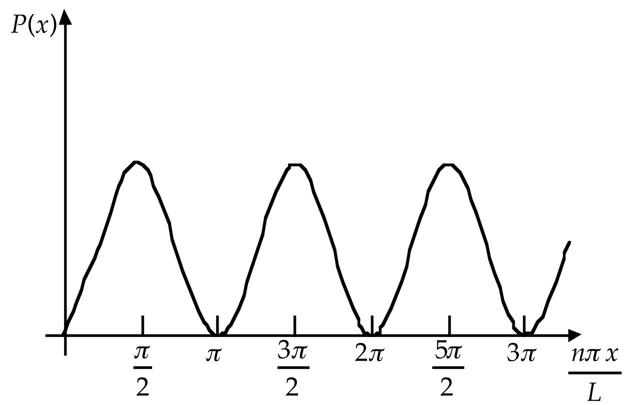
$$P = \int_0^{L/3} |\psi|^2 dx = \left(\frac{2}{L}\right)\int_0^{L/3} \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{2}{\pi}\int_0^{\pi/3} \sin^2 \phi d\phi = \frac{2}{\pi}\left[\frac{\pi}{6} - \frac{(3)^{1/2}}{8}\right] = 0.195\,5 .$$

(b) $\psi = A\sin\left(\frac{100\pi x}{L}\right)$, $A = \left(\frac{2}{L}\right)^{1/2}$

$$P = \frac{2}{L}\int_0^{L/3} \sin^2\left(\frac{100\pi x}{L}\right) dx = \frac{2}{L}\left(\frac{L}{100\pi}\right)\int_0^{100\pi/3} \sin^2 \phi d\phi = \frac{1}{50\pi}\left[\frac{100\pi}{6} - \frac{1}{4}\sin\left(\frac{200\pi}{3}\right)\right]$$

$$= \frac{1}{3} - \left[\frac{1}{200\pi}\right]\sin\left(\frac{2\pi}{3}\right) = \frac{1}{3} - \frac{\sqrt{3}}{400\pi} = 0.331\,9$$

6-18 Since the wavefunction for a particle in a one-dimension box of width L is given by $\psi_n = A\sin\left(\frac{n\pi x}{L}\right)$ it follows that the probability density is $P(x) = |\psi_n|^2 = A^2 \sin^2\left(\frac{n\pi x}{L}\right)$, which is sketched below:



From this sketch we see that $P(x)$ is a *maximum* when $\frac{n\pi x}{L} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots = \pi\left(m + \frac{1}{2}\right)$ or when

$$x = \frac{L}{n}\left(m + \frac{1}{2}\right) \qquad m = 0, 1, 2, 3, \ldots, n.$$

Likewise, $P(x)$ is a *minimum* when $\frac{n\pi x}{L} = 0, \pi, 2\pi, 3\pi, \ldots = m\pi$ or when

$$x = \frac{Lm}{n} \qquad m = 0, 1, 2, 3, \ldots, n$$

6-24 After rearrangement, the Schrödinger equation is $\frac{d^2\psi}{dx^2} = \left(\frac{2m}{\hbar^2}\right)\{U(x) - E\}\psi(x)$ with $U(x) = \frac{1}{2}m\omega^2 x^2$ for the quantum oscillator. Differentiating $\psi(x) = Cxe^{-\alpha x^2}$ gives

$$\frac{d\psi}{dx} = -2\alpha x\psi(x) + C^{-\alpha x^2}$$

and

$$\frac{d^2\psi}{dx^2} = -\frac{2\alpha x d\psi}{dx} - 2\alpha\psi(x) - (2\alpha x)Ce^{-\alpha x^2} = (2\alpha x)^2\psi(x) - 6\alpha\psi(x).$$

Therefore, for $\psi(x)$ to be a solution requires

$(2\alpha x)^2 - 6\alpha = \frac{2m}{\hbar^2}\{U(x) - E\} = \left(\frac{m\omega}{\hbar}\right)^2 x^2 - \frac{2mE}{\hbar^2}$. Equating coefficients of like terms gives $2\alpha = \frac{m\omega}{\hbar}$ and $6\alpha = \frac{2mE}{\hbar^2}$. Thus, $\alpha = \frac{m\omega}{2\hbar}$ and $E = \frac{3\alpha\hbar^2}{m} = \frac{3}{2}\hbar\omega$. The normalization integral is $1 = \int_{-\infty}^{\infty}|\psi(x)|^2 dx = 2C^2\int x^2 e^{-2\alpha x^2} dx$ where the second step follows from the symmetry of the integrand about $x = 0$. Identifying a with 2α in the integral of Problem 6-32 gives $1 = 2C^2\left(\frac{1}{8\alpha}\right)\left(\frac{\pi}{2\alpha}\right)^{1/2}$ or $C = \left(\frac{32\alpha^3}{\pi}\right)^{1/4}$.

6-25 At its limits of vibration $x = \pm A$ the classical oscillator has all its energy in potential form: $E = \frac{1}{2}m\omega^2 A^2$ or $A = \left(\frac{2E}{m\omega^2}\right)^{1/2}$. If the energy is quantized as $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$, then the corresponding amplitudes are $A_n = \left[\frac{(2n+1)\hbar}{m\omega}\right]^{1/2}$.

6-29 (a) Normalization requires

$1 = \int_{-\infty}^{\infty}|\psi|^2 dx = C^2\int_0^{\infty} e^{-2x}\left(1 - e^{-x}\right)^2 dx = C^2\int_0^{\infty}\left(e^{-2x} - 2e^{-3x} + e^{-4x}\right)dx$. The integrals are elementary and give $1 = C^2\left\{\frac{1}{2} - 2\left(\frac{1}{3}\right) + \frac{1}{4}\right\} = \frac{C^2}{12}$. The proper units for C are those of $(\text{length})^{-1/2}$ thus, normalization requires $C = (12)^{1/2}\ \text{nm}^{-1/2}$.

(b) The most likely place for the electron is where the probability $|\psi|^2$ is largest. This is also where ψ itself is largest, and is found by setting the derivative $\frac{d\psi}{dx}$ equal zero:

$$0 = \frac{d\psi}{dx} = C\left\{-e^{-x} + 2e^{-2x}\right\} = Ce^{-x}\left\{2e^{-x} - 1\right\}.$$

The RHS vanishes when $x = \infty$ (a minimum), and when $2e^{-x} = 1$, or $x = \ln 2$ nm. Thus, the most likely position is at $x_p = \ln 2\ \text{nm} = 0.693$ nm.

(c) The average position is calculated from

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\psi|^2 dx = C^2\int_0^{\infty} xe^{-2x}\left(1 - e^{-x}\right)^2 dx = C^2\int_0^{\infty} x\left(e^{-2x} - 2e^{-3x} + e^{-4x}\right)dx.$$

The integrals are readily evaluated with the help of the formula $\int_0^\infty xe^{-ax}\,dx = \frac{1}{a^2}$ to get $\langle x\rangle = C^2\left\{\frac{1}{4} - 2\left(\frac{1}{9}\right) + \frac{1}{16}\right\} = C^2\left\{\frac{13}{144}\right\}$. Substituting $C^2 = 12\text{ nm}^{-1}$ gives

$$\langle x\rangle = \frac{13}{12}\text{ nm} = 1.083\text{ nm}.$$

We see that $\langle x\rangle$ is somewhat greater than the most probable position, since the probability density is skewed in such a way that values of x larger than x_p are weighted more heavily in the calculation of the average.

6-31 The symmetry of $|\psi(x)|^2$ about $x = 0$ can be exploited effectively in the calculation of average values. To find $\langle x\rangle$

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2\,dx$$

We notice that the integrand is antisymmetric about $x = 0$ due to the extra factor of x (an odd function). Thus, the contribution from the two half-axes $x > 0$ and $x < 0$ cancel exactly, leaving $\langle x\rangle = 0$. For the calculation of $\langle x^2\rangle$, however, the integrand is symmetric and the half-axes contribute equally to the value of the integral, giving

$$\langle x\rangle = \int_0^\infty x^2|\psi|^2\,dx = 2C^2\int_0^\infty x^2 e^{-2x/x_0}\,dx.$$

Two integrations by parts show the value of the integral to be $2\left(\frac{x_0}{2}\right)^3$. Upon substituting for C^2, we get $\langle x^2\rangle = 2\left(\frac{1}{x_0}\right)(2)\left(\frac{x_0}{2}\right)^3 = \frac{x_0^2}{2}$ and $\Delta x = \left(\langle x^2\rangle - \langle x\rangle^2\right)^{1/2} = \left(\frac{x_0^2}{2}\right)^{1/2} = \frac{x_0}{\sqrt{2}}$. In calculating the probability for the interval $-\Delta x$ to $+\Delta x$ we appeal to symmetry once again to write

$$P = \int_{-\Delta x}^{+\Delta x}|\psi|^2\,dx = 2C^2\int_0^{\Delta x} e^{-2x/x_0}\,dx = -2C^2\left(\frac{x_0}{2}\right)e^{-2x/x_0}\Big|_0^{\Delta x} = 1 - e^{-\sqrt{2}} = 0.757$$

or about 75.7% independent of x_0.

6-32 The probability density for this case is $|\psi_0(x)|^2 = C_0^2 e^{-ax^2}$ with $C_0 = \left(\frac{a}{\pi}\right)^{1/4}$ and $a = \frac{m\omega}{\hbar}$.

For the calculation of the average position $\langle x\rangle = \int_{-\infty}^{\infty} x|\psi_0(x)|^2\,dx$ we note that the integrand is an odd function, so that the integral over the negative half-axis $x < 0$ exactly cancels that over the positive half-axis ($x > 0$), leaving $\langle x\rangle = 0$. For the calculation of $\langle x^2\rangle$, however, the integrand $x^2|\psi_0|^2$ is symmetric, and the two half-axes contribute equally, giving

$$\langle x^2 \rangle = 2C_0^2 \int_0^\infty x^2 e^{-ax^2}\,dx = 2C_0^2\left(\frac{1}{4a}\right)\left(\frac{\pi}{a}\right)^{1/2}.$$

Substituting for C_0 and a gives $\langle x^2 \rangle = \frac{1}{2a} = \frac{\hbar}{2m\omega}$ and $\Delta x = \left(\langle x^2 \rangle - \langle x \rangle^2\right)^{1/2} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$.

6-33 (a) Since there is no preference for motion in the leftward sense vs. the rightward sense, a particle would spend equal time moving left as moving right, suggesting $\langle p_x \rangle = 0$.

(b) To find $\langle p_x^2 \rangle$ we express the average energy as the sum of its kinetic and potential energy contributions: $\langle E \rangle = \left\langle \frac{p_x^2}{2m} \right\rangle + \langle U \rangle = \frac{\langle p_x^2 \rangle}{2m} + \langle U \rangle$. But energy is sharp in the oscillator ground state, so that $\langle E \rangle = E_0 = \frac{1}{2}\hbar\omega$. Furthermore, remembering that $U(x) = \frac{1}{2}m\omega^2 x^2$ for the quantum oscillator, and using $\langle x^2 \rangle = \frac{\hbar}{2m\omega}$ from Problem 6-32, gives $\langle U \rangle = \frac{1}{2}m\omega^2\langle x^2 \rangle = \frac{1}{4}\hbar\omega$. Then

$$\langle p_x^2 \rangle = 2m\left(E_0 - \langle U \rangle\right) = 2m\left(\frac{\hbar\omega}{4}\right) = \frac{m\hbar\omega}{2}.$$

(c) $$\Delta p_x = \left(\langle p_x^2 \rangle - \langle p_x \rangle^2\right)^{1/2} = \left(\frac{m\hbar\omega}{2}\right)^{1/2}$$

6-34 From Problems 6-32 and 6-33, we have $\Delta x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$ and $\Delta p_x = \left(\frac{m\hbar\omega}{2}\right)^{1/2}$. Thus,

$\Delta x \Delta p_x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\left(\frac{m\hbar\omega}{2}\right)^{1/2} = \frac{\hbar}{2}$ for the oscillator ground state. This is the minimum uncertainty product permitted by the uncertainty principle, and is realized only for the ground state of the quantum oscillator.