Experiment 2: Oscillation and Damping in the LRC Circuit

Introduction

In this laboratory you will construct an LRC series circuit and apply a constant voltage over it. You will view the voltage drop over the various elements of the circuit with the oscilloscope. You must change the order of the circuit elements in order to avoid shorting your circuit, but you will only construct one type of circuit throughout this experiment (LRC series). Also, this laboratory does not introduce much new physics to you since many of these topics have been covered in the previous two experiments. On the other hand, this experiment contains several new definitions and a more complicated differential equation, which result in a longer mathematical analysis.

1 Physics

1.1 Review of Kirchhoff's Law

Kirchhoff's Law states that in any closed loop of a circuit the algebraic sum of the voltages of the elements in that loop will be zero. "Algebraic" simply means signed. Elements in the circuit may either increase (add) voltage or drop (subtract) voltage.

1.2 Voltage Drops Over Various Circuit Elements

Resistors, capacitors and inductors have well known voltage drops at direct current (DC) flows through those elements. Ohm's Law describes that the voltage drop across a resistor is proportional to the current and the resistance:

$$V_R = IR \tag{1}$$

The voltage drop across a capacitor is proportional to the charge held on either side of the capacitor. The charge is not always useful in equations mainly in terms of current, but luckily the charge on a capacitor is the integrated current over time:

$$V_C = \frac{Q}{C} = \frac{1}{C}\int I dt \tag{2}$$

An inductor is a tightly wound series of coils through which the current flows. A fairly uniform magnetic field is created on the interior of these coils. If the current changes so does the magnetic field and an induced current is produced. The previous statement is a result of the well-known physical law known as Faraday's Law. The voltage drop is proportional to the change in the magnetic field and therefore the change in the current:

$$V_L = L\frac{dI}{dt} \tag{3}$$

Also, the coils in inductors often have non-negligible resistance.

1.3 Energy Storage in Capacitors and Inductors

Where resistors simply give off energy by radiating heat, capacitors and inductors store energy. The energy stored in each is listed below:

$$E_C = \frac{1}{2}CV^2 \qquad \textbf{(4)}$$

$$E_L = \frac{1}{2}LI^2 \qquad \textbf{(5)}$$

2 Mathematical Circuit Analysis

2.1 The LRC Series Circuit

In Figure 1 below the circuit you will later construct is shown. Using Kirchhoff's Law we have:

$$V_S + V_L + V_C + V_R = 0 \qquad \textbf{(6)}$$

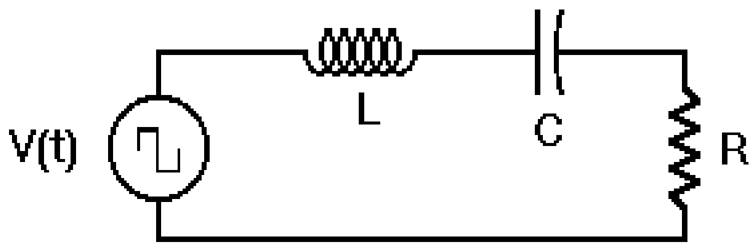

Figure 1 *LRC circuit for this experiment*

Using Equations 1, 2 and 3 in Equation 6 results in:

$$V_S - L\frac{dI}{dt} - \frac{1}{C}\int Idt - IR = 0 \qquad \textbf{(7)}$$

Now let us assume that V_S is constant in time. In this experiment you will be using a square wave with a large period to produce constant voltage. Next let us differentiate equation 7 with respect to time in order to eliminate the integral and the constant term V_S. Equation 7 becomes:

$$-L\frac{d^2I}{dt^2} - \frac{1}{C}I - R\frac{dI}{dt} = 0 \quad \rightarrow \quad \frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0 \qquad \textbf{(8)}$$

Assuming a solution of the form $I = I_0 e^{\alpha t}$ we substitute into equation 8 and obtain

$$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0 \qquad \textbf{(9)}$$

Solving using the quadratic formula we have:

$$\alpha = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} \qquad \textbf{(10)}$$

When the resistor is absent and there is no resistance from any other source the solution is:

$$\alpha = \pm\sqrt{-\frac{1}{LC}} = \pm i\omega_0 \text{ where } \omega_0 = \sqrt{\frac{1}{LC}} \qquad \textbf{(11)}$$

> *Question 2.1*
>
> What is the propagated error of ω_0 (equation 11)? That is, what is $\Delta\omega_0$?

Our goal in defining ω_0, besides making broader connection to other physical circuits, is to simplify α. Other helpful term which will help us simplify α is listed below with the simplified version of α:

$$\tau \equiv \frac{L}{R} \qquad \textbf{(12)}$$

$$\alpha = -\frac{1}{2\tau} \pm \sqrt{\frac{1}{4\tau^2} - \omega_0^{\,2}} \qquad \textbf{(13)}$$

This approach has given us a general solution to the differential equation (8), but depending on the relation between $1/2\omega_0$ & τ, the solution can be very different. Let us define a critical value of the time constant τ[1].

$$\tau_C \equiv \frac{1}{2\omega_0} \qquad \textbf{(14)}$$

If $\tau = \tau_C$, there is no oscillation at all because the discriminant in Equation 13 is 0, and the system is called <u>critically</u> <u>damped</u>. For $\tau < \tau_C$ the discriminant is positive and the system is called <u>overdamped</u>. In either case, we say that the oscillator is <u>aperiodic</u> which means that there is no period.

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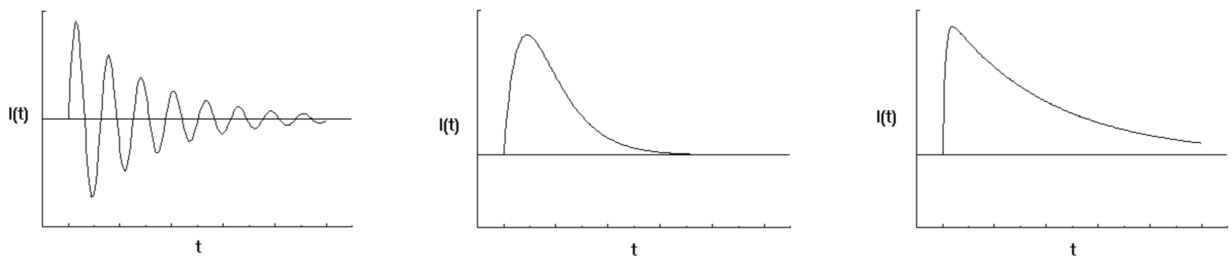
[1] Please take note that τ is not defined the same way in this experiment as it was in experiment 1. τ is NOT the decay time in this experiment.

The case that is of main interest to us is when $\tau > \tau_C$, called underdamped. The discriminant is negative and this yields an imaginary part to α. Figure 2 illustrates the behavior of each of the three cases beginning at $t = 0$ from rest with an initial displacement of I_0 for various values of the quantity τ.



Underdamped ($\tau > \tau_C$) Critical ($\tau = \tau_C$) Overdamped ($\tau < \tau_C$)

Figure 3 *Three régimes for damping behavior.*

2.2 The Underdamped Oscillator

For the underdamped case, the imaginary part of the solution corresponds to the angular frequency of the oscillatory part of the current (and voltage).

$$\alpha = -\frac{1}{2\tau} \pm i\omega \quad \textbf{(15)}$$

where:

$$\omega = \sqrt{\omega_0{}^2 - \frac{1}{4\tau^2}} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad \textbf{(16)}$$

and thus, the solution to the differential equation (8) is:

$$I(t) = I_0 e^{-t/2\tau} e^{\pm i\omega t} \quad \textbf{(17)}$$

This solution is the product of the two types of functions that you were exposed to in Experiment 1: a decaying exponential and an oscillator (from the imaginary argument). Since we are not going to worry about "boundary conditions" of what happens when the square wave makes its transition, we are allowed to pick some convenient piece of $e^{i\omega t}$, for example sin(ωt), as the "physical" solution:

$$I(t) = I_0 e^{-t/2\tau} \sin \omega t \quad \textbf{(18)}$$

If one wished to verify this equation, she or he could do so by inserting this into equation 8. The result of result of such an exercise would be the left side equally exactly zero after inserting equations 12 and 15.

Question 2.2

Could we use $\cos(\omega t)$ instead of $\sin(\omega t)$ in equation 18? Why or why not?

Observe that R has two effects on the circuit. First, it causes the amplitude of the oscillation (i.e., the maximum excursion during a cycle) to decrease steadily from one cycle to the next. The factor $e^{-t/2\tau}$ is responsible for this; it is commonly called the envelope of the oscillation, for a reason evident in Figure 4. Secondly, the frequency of the oscillation is altered, since we see that ω^2 and ${\omega_0}^2$ now differ by $1/4\tau^2$.

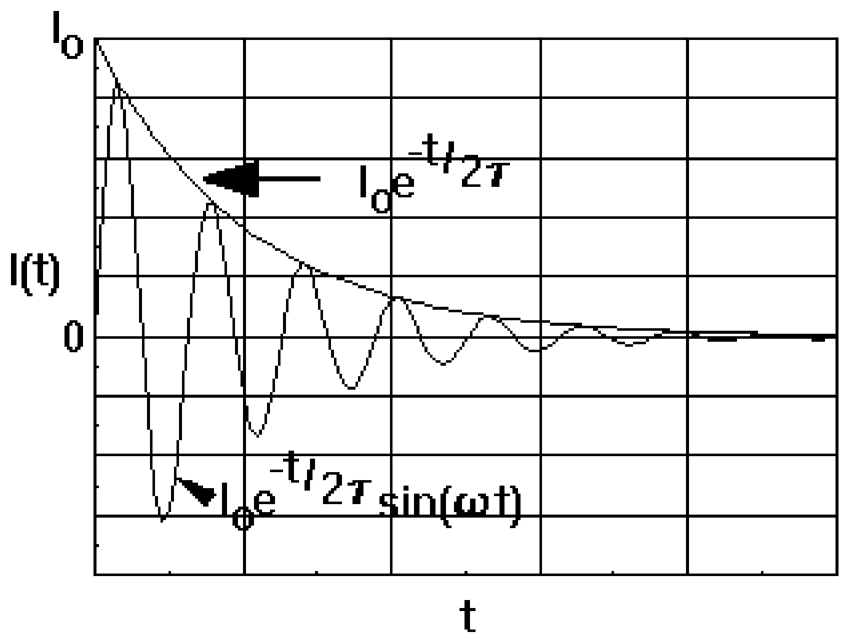


Figure 4 *Exponentially underdamped current I(t) with envelope defined by time constant* $\tau = L/R$.

In the previous experiment, the decay time of the RC circuit was stated as $\tau = RC$. In this experiment, we have used the Greek letter τ to represent L/R, which is not the decay time of the envelope of the underdamped LRC series circuit. The decay time of the LRC series circuit is the time when the exponential factor has become 1/e, i.e., when the exponent is minus one. Thus, the decay time, t_D, is:

$$t_D = 2\tau \quad \textbf{(19)}$$

We will now show that for cases where the decay is not too rapid, or more specifically, when several oscillations occur in a decay time, the difference between the damped and undamped frequencies

(ω and ω_0) is very slight. First suppose there is some number (n) of oscillations in one decay time (n is not necessarily an integer). Then, since there is one oscillation every $T = 2\pi/\omega$ seconds,

$$n = \frac{t_D}{T} = \frac{\omega t_D}{2\pi} = \frac{\omega\tau}{\pi} \tag{20}$$

and so

$$\omega\tau = \pi n \tag{21}$$

The ratio of ω_0 to ω can be obtained from equation 16 by first squaring both sides and then dividing the equation by ω^2:

$$\frac{\omega_0^2}{\omega^2} = 1 + \frac{1}{4\omega^2\tau^2} \tag{22}$$

We note that $\omega_0/\omega = f_0/f$ and use equation 21 to obtain

$$\left(\frac{\omega_0}{\omega}\right)^2 = \left(\frac{f_0}{f}\right)^2 = 1 + \frac{1}{4\pi^2 n^2} \tag{23}$$

2.3 The Quality Factor

The most widely used measure of oscillator decay is the "quality factor" Q, which can be defined by

$$Q = \omega_0\tau \tag{24}$$

It is only well defined (or, rather, there is only a well-agreed definition of Q) when it is somewhat greater than one. In this case it is also true that $\omega \approx \omega_0$. Starting with equation 22 we obtain

$$\left(\frac{\omega_0}{\omega}\right)^2 = 1 + \frac{1}{4Q^2}\left(\frac{\omega_0}{\omega}\right)^2 \tag{25}$$

which can be solved for ω_0/ω to give

$$\frac{\omega_0}{\omega} = \frac{f_0}{f} = \left(1 - \frac{1}{4Q^2}\right)^{-1/2} \tag{26}$$

For example, if $Q = 2$, $\frac{\omega_0}{\omega} = 1.033$ and ω_0 is just 3.3% larger than ω. Thus, for Q not too small, we may approximate ω by ω_0. Using this approximation, we can combine Equations 14 and 11 to give the convenient expression

$$Q = \pi n \tag{27}$$

where n is the number of cycles per decay time.

The quality factor, $\boldsymbol{Q}$, is often a source of confusion so it may be helpful to analyze $\boldsymbol{Q}$ using an alternate definition as well. This definition describes $\boldsymbol{Q}$'s relationship to energy lost during each cycle.

Let us analysis an undamped oscillator in terms of energy conservation. Take $R = 0$ for a moment. Now we can show energy is conserved and is moving back and forth between L and C. For a sinusoidally varying current

$$I = I_0 \sin \omega_0 t \qquad \textbf{(28)}$$

and using equation 5 the energy stored in the inductor becomes

$$E_L = \frac{1}{2} L I_0^2 \sin^2 \omega t \qquad \textbf{(29)}$$

Then for capacitance

$$V_C = \int \frac{1}{C} I dt = -\frac{I_0}{\omega_0 C} \cos \omega_0 t \qquad \textbf{(30)}$$

and the energy stored in the capacitor is

$$E_C = \frac{1}{2} \frac{I_0^2}{\omega_0^2 C} \cos^2 \omega_0 t \qquad \textbf{(31)}$$

The total energy in the circuit is conserved since the sum is constant as shown below.

$$\begin{aligned} E_C + E_L &= \frac{1}{2} I_0^2 \left(L \sin^2(\omega_0 t) + \frac{\cos^2(\omega_0 t)}{C / LC} \right) \\ &= \frac{L I_0^2}{2} \end{aligned} \qquad \textbf{(32)}$$

Here we have used the fact that since $R = 0,\ \omega^2 = \omega_0{}^2 = \dfrac{1}{LC}$.

You may see from the preceding steps that the energy in the inductor is a maximum when the energy in the capacitor is zero, and vice versa, because one is a sine and the other a cosine function. A consequence of this is that you only need to keep track of peak energy in the inductor to follow the cycle-to-cycle energy situation in the circuit.

Now, consider the situation where $R \neq 0$. In this case, the stored energy is eventually lost because the current is flowing back and forth from inductor to capacitor and the power loss ($I^2 R$) in the resistor equals the rate of energy decrease. The amount of energy which the resistor removes each cycle of oscillation is

$$E_{loss/cycle} = \int_0^T I^2 R dt = I_0{}^2 R \int_0^T \sin^2(\omega t) dt = I_0^2 \frac{RT}{2} \quad \textbf{(33)}$$

We are now in a position to define Q in an energy picture and compare to the previous definition. We may say, as an alternate definition,

$$Q = 2\pi \frac{\text{Energy of Oscillation}}{\text{Energy lost in one cycle}} \quad \textbf{(34)}$$

which, using Equations 32 and 33, translates to

$$Q = 2\pi \frac{\frac{1}{2} L I_0{}^2}{\frac{1}{2} R I_0{}^2 T} = 2\pi f \frac{L}{R} = \omega\tau \quad \textbf{(35)}$$

which agree with Equation 24 when $\omega \approx \omega_0$. Thus, the quality factor is the inverse of the fractional energy loss each radian of a cycle.

Let us summarize the descriptive formulas

- Time dependence $\quad I(t) = I_0 e^{-t/2\tau} \sin\omega t \quad \text{for } \tau = L/R$ **(36)**
- Ringing frequency $\quad \omega = \sqrt{\omega_0{}^2 - \frac{1}{4\tau^2}} \quad \text{for } \omega_0 = \frac{1}{\sqrt{LC}}$ **(37)**
- Decay time $\quad t_D = 2\tau = 2\frac{L}{R}$ **(38)**
- Quality factor $\quad Q = \omega_0 \tau = \omega_0 \frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$ **(39)**
- Critical damping $\quad R_{critical} = 2\omega_0 L = 2\sqrt{\frac{L}{C}}$ **(40)**

Question 2.3

What is the propagated error of the quality factor, ***Q***, as a function of measured values and uncertainties for R, L and C?? That is, what is ΔQ?

3 The Experiment

3.1 Target Values for R, L and C

With the mathematical analysis in the preceding sections in mind you will probably guess that it can be tricky to choose R, L and C so that you view a "nice" underdamped oscillator. Below is a table with the listed options for R, L and C.

Circuit Element	Options
Resistor (R)	10, 51, 100, 200, 510, and 1000 Ω There is also a variable resistor.*
Inductor (L)	0.55, 2.5, 5 and 10mH (milli Henry)
Capacitor (C)	0.001, 0.01, 0.1 and 1.0μF (micro Farad)

Recall that both the inductor and the signal generator also have resistance, and thus the total resistance is

$$R_{total} = R + R_L + R_S \qquad \textbf{(41)}$$

Assume for preliminary calculations that $R_S = 50\Omega$ and $R_L = 25\Omega$.

We would like to see several oscillations in a single decay time. This corresponds to n being between one and four. More oscillations would make it difficult to make measurements. Since $Q = \pi n$, then the appropriate range for the value of Q is $4 < Q < 15$.

Also, the frequency is limited by the peak value that the signal generator can produce. Its peak linear frequency is ~1Mhz (thus, $f < 1$ Mhz and $\omega < 10^7$ rad/s).

Question 2.4 – (MANDATORY)

What are your values for R, C, and L such that the requirements above are fulfilled?

Hint- Determine C and L by guessing and checking in the ω_0 formula first. Recall $\omega \approx \omega_0$. Then guess R until you find Q in the correct range.

3.2 Determining Q and ω Experimentally

First calibrate your oscilloscope. See Experiment 0 for details.

Measure the capacitance and resistance of the pertinent elements in your circuit. Assume the inductance of the inductor is correct with a 10% margin of error.

Construct the *LRC* circuit as diagrammed in Figure 5. To measure the natural oscillation frequency for your circuit in the lab, you must first get the oscillations going and obtain a suitable display on your scope screen. This can be difficult but just experiment with the large scale knobs until you have the display you desire.

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* When using the variable resistor, measure your final resistance with the multimeter.

It is recommended that you use the signal generator as an "external trigger" to provide a stable scope trace. To do so, use a BNC cable to connect the TTY outlet from the signal generator (the middle one) to the external trigger input on the lower right of the oscilloscope. There are several switches above the external trigger input. The first two must be lowered to external. When they are set to auto, the oscilloscope uses its internal ground.

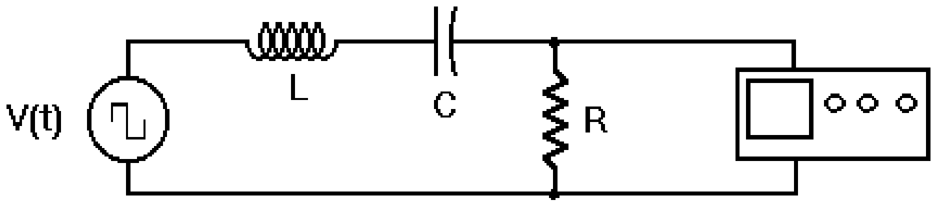

Figure 5 *Addition of the oscilloscope to the LRC circuit*

By measuring the time between two nodes (= 0.5*T) on the oscilloscope you can determine the period, *T*, of the oscillatory part of the voltage function. This is the inverse of the frequency, *f*.

How can you determine empirically the circuit ***Q*** from measurements of the scope trace? One simple method is to make use of Equation 27. Think through in advance each measurement that you will want to perform in the lab.

Finding the decay time, t_D, may be useful to you in finding ***Q***. The decay time can be found accurately by measuring the amplitude of the "peaks" (which define the envelope of the decay) at various times during the oscillations (refer back to Figure 2). Measure the amplitude (with uncertainty) and time of the "peaks" you observe, trying to obtain at least ten points. Using ORIGIN plot the amplitude (with error bars) as a function of time, and fit the data to an exponential decay (i.e. $V_R(t) = V_0 e^{-t/2\tau}$). From the coefficients of the fit, determine a value (with uncertainty) of the decay time (Equation 38). From the decay time and Equation 20, determine the value of the ***Q*** from Equation 27.

3.3 Critical Resistance at Critical Damping

Critical damping occurs when $\tau = \tau_C = 1/2\omega_0$ and in general $\tau = L/R$. You can determine the critical damping resistance empirically in the lab by inserting the variable resistor into your circuit and slowly increasing its resistance while watching the scope trace to see when the oscillations just cease. *L* and ω_0 will remained fixed during this process. By summing the variable resistance to the other circuit resistances (i.e. resistance from the signal generator and inductor), you can compare the actual measured result with the result of your calculation.

Question 2.5

Derive the formula for critical resistance (Equation 40). Read section 3.3.

3.4 Unknown Inductor

Finally, in the laboratory there are several different "unknown" inductors identified only by a number. Select one of these and determine its inductance as well as you can use only the equipment listed above. Devise your own method and describe it in your report. Be sure to give the unknown inductor number, its value as determined by your measurements, and the error limits you feel are appropriate.

Think about the techniques you have used thus far in the lab and examine the summary of equations searching for *L*.

Analysis

Compute the following:

- Theoretical resonance frequency from L, R and C;
- Theoretical quality factor from L, R and C;
- Theoretical τ from L and R;
- Experimental quality factor from experimental data (from n);
- Comparison of resonance frequency to theoretical resonance frequency;
- Comparison of quality factor to theoretical quality factor;
- Comparison of fitted τ to theoretical τ.

Discuss your procedure for finding the inductance of the unknown inductor. State your experimental inductance.

Conclusions

Highlight the themes of the lab and the physics the experiment verifies. You should discuss the errors you encounter in the lab and how you could improve the lab if you had to repeat it. If your results are unexpected or your t-values are high, you should identify possible explanations.

For this experiment, specifically discuss your choice of method for determining the unknown inductor's inductance. Was it a good choice? Was there a better way?

Hints on reports

- Write neatly—if your TA cannot read it, you could lose points.
- Be organized—if your TA cannot find it, you could lose points.
- Report your data, including plots—if your data is not in your report, your TA does know you did it.
- Record uncertainty.
- Propagate uncertainty.
- Write your final answers with proper significant figures.