Experiment 2: Oscillation and Damping in the LRC Circuit

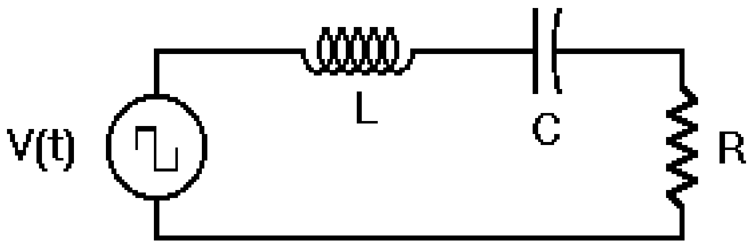
Goal: examine a circuit consisting of one inductor, one resistor, and one capacitor

1. Apply a constant voltage over the LRC circuit and view the voltage drop over the various elements of the circuit with the oscilloscope

2. Examine underdamped, critically damped, and overdamped oscillations.

3. Determine quality factor, frequency, critical resistance, and inductance of unknown inductor.

LRC circuit



Ohm's Law

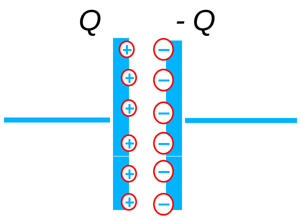
Voltage Drops Over Various Circuit Elements

The voltage drop across a resistor is proportional to the current and the resistance

$$V_R = IR$$

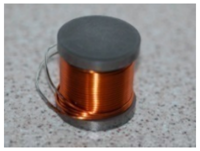
The voltage drop across a capacitor is proportional to the charge held on either side of the capacitor

$$V_C = \frac{Q}{C} = \frac{1}{C}\int I dt$$



The voltage drop across an inductor is proportional to the change in the current

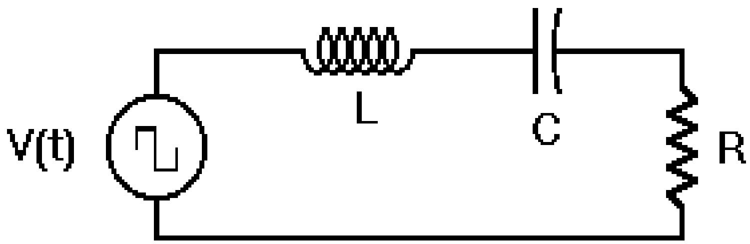
$$V_L = L\frac{dI}{dt}$$



An inductor is a series of coils.
The current flowing through inductor creates magnetic field in the interior of these coils.
A changing magnetic field creates an electric field.

Ampère's circuital law
Faraday's law of induction

LRC circuit



Kirchhoff's Law: $V_S + V_L + V_C + V_R = 0$

$$V_S - L\frac{dI}{dt} - \frac{1}{C}\int Idt - IR = 0$$

in this experiment you will be using a square wave with a large period to produce constant V_S

differentiate equation with respect to time

$$-L\frac{d^2I}{dt^2} - \frac{1}{C}I - R\frac{dI}{dt} = 0 \quad \rightarrow \quad \frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = 0$$

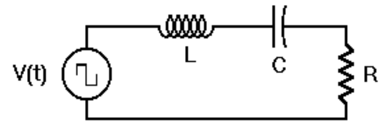
seek solution of the form $I = I_0 e^{\alpha t}$ $\longrightarrow$ $\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0$

$$\downarrow$$

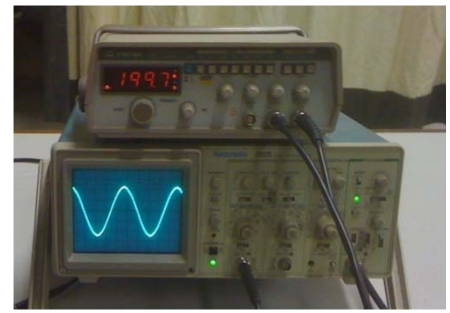
$$\alpha = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

$$I = I_0 e^{\alpha t}$$

$$\alpha = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$



undamped oscillations

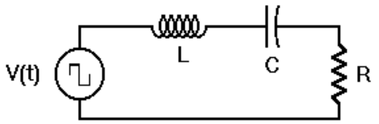


if $R = 0$

$$\alpha = \pm\sqrt{-\frac{1}{LC}} = \pm i\omega_0 \text{ where } \omega_0 = \sqrt{\frac{1}{LC}}$$

$$I = I_0 e^{\alpha t}$$

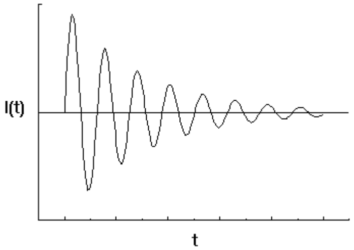
$$\alpha = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$



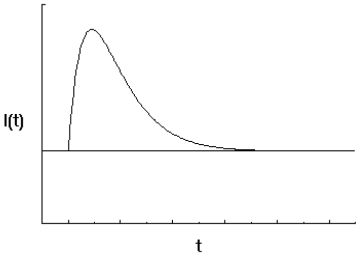
$$\tau \equiv \frac{L}{R}$$

$$\alpha = -\frac{1}{2\tau} \pm \sqrt{\frac{1}{4\tau^2} - \omega_0{}^2}$$

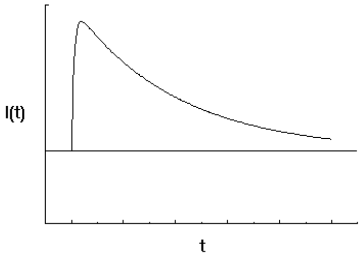
critical value of τ $\tau_C \equiv \frac{1}{2\omega_0}$



underdamped ($\tau > \tau_C$)
the discriminant is negative



critical ($\tau = \tau_C$)
the discriminant is 0

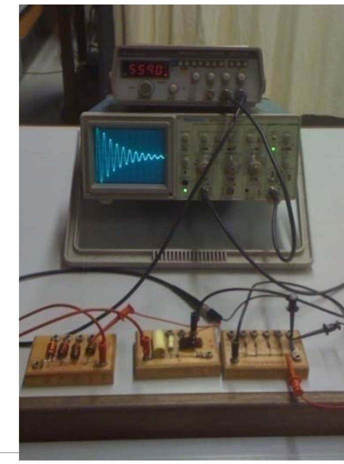


overdamped ($\tau < \tau_C$)
the discriminant is positive

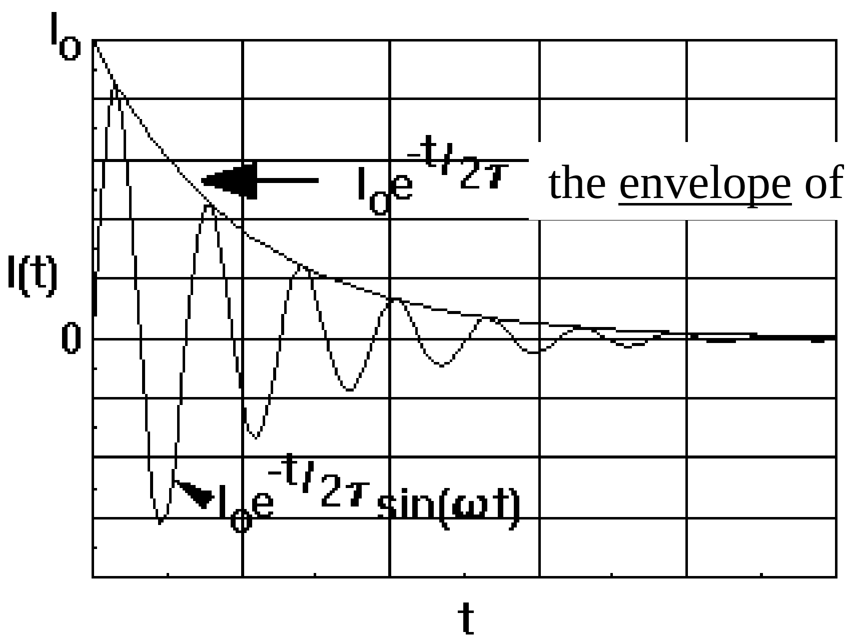
$$I = I_0 e^{\alpha t} \qquad \alpha = -\frac{1}{2\tau} \pm \sqrt{\frac{1}{4\tau^2} - {\omega_0}^2}$$

The Underdamped Oscillator

$$\alpha = -\frac{1}{2\tau} \pm i\omega \qquad \text{where} \qquad \omega = \sqrt{{\omega_0}^2 - \frac{1}{4\tau^2}} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$



$$I(t) = I_0 e^{-t/2\tau} e^{\pm i\omega t} \longrightarrow I(t) = I_0 e^{-t/2\tau} \sin \omega t$$



the envelope of the oscillation

the decay time of the LRC series circuit

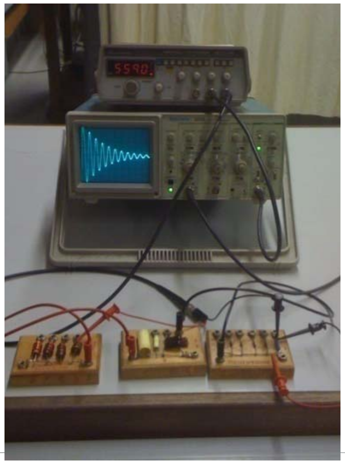
$$t_D = 2\tau$$

$$\tau \equiv \frac{L}{R}$$

The Quality factor

$$Q = 2\pi \frac{\text{Energy of Oscillation}}{\text{Energy lost in one cycle}}$$

$$Q = \omega_0 \tau = \omega_0 \frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$



change of oscillation frequency

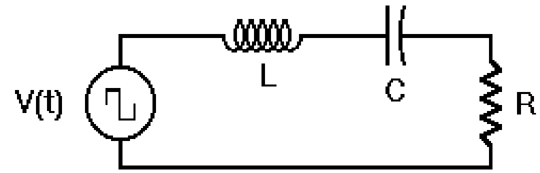
$$\left(\frac{\omega_0}{\omega}\right)^2 = 1 + \frac{1}{4Q^2}\left(\frac{\omega_0}{\omega}\right)^2$$

for large Q $\quad \omega \approx \omega_0$ $\qquad Q = \pi n$

n is umber of oscillations in one decay time $\quad n = \frac{t_D}{T} = \frac{\omega t_D}{2\pi} = \frac{\omega \tau}{\pi}$

oscillation period $\quad T = 2\pi / \omega$

Energy



capacitor $E_C = \frac{1}{2} C V^2$

inductor $E_L = \frac{1}{2} L I^2$

consider an <u>undamped oscillator $R = 0$</u>

$I = I_0 \sin \omega_0 t$

$E_L = \frac{1}{2} L I_0^2 \sin^2 \omega_0 t$

$V_C = \int \frac{1}{C} I dt = -\frac{I_0}{\omega_0 C} \cos \omega_0 t$

$E_C = \frac{1}{2} \frac{I_0^2}{\omega_0^2 C} \cos^2 \omega_0 t$

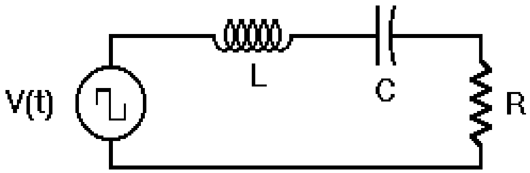
the total energy in the circuit $E_C + E_L = \frac{1}{2} I_0^2 \left(L \sin^2(\omega_0 t) + \frac{\cos^2(\omega_0 t)}{C / LC} \right)$

$= \frac{L I_0^2}{2}$

$\omega_0 = \sqrt{\frac{1}{LC}}$

<u>energy is conserved and oscillates between L and C</u>

Energy



$\underline{R \neq 0}$

the stored energy is eventually lost because the power loss I^2R in the resistor

the amount of energy which the resistor removes each cycle of oscillation

$$E_{loss/cycle} = \int_0^T I^2 R dt = I_0{}^2 R \int_0^T \sin^2(\omega t) dt = I_0^2 \frac{RT}{2}$$

$$Q = 2\pi \frac{\text{Energy of Oscillation}}{\text{Energy lost in one cycle}} \longrightarrow Q = 2\pi \frac{\frac{1}{2} L I_0{}^2}{\frac{1}{2} R I_0{}^2 T} = 2\pi f \frac{L}{R} = \omega\tau$$

Summary

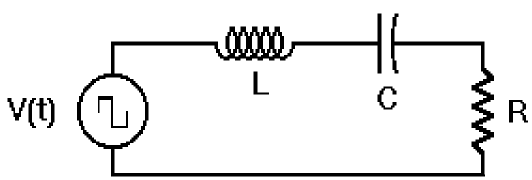
time dependence $I(t) = I_0 e^{-t/2\tau} \sin \omega t \quad \text{for } \tau = L/R$

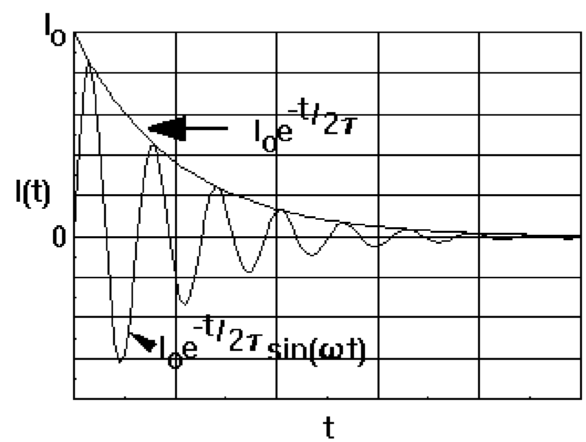
frequency $\omega = \sqrt{\omega_0{}^2 - \frac{1}{4\tau^2}} \quad \text{for } \omega_0 = \frac{1}{\sqrt{LC}}$

decay time $t_D = 2\tau = 2\frac{L}{R}$

quality factor $Q = \omega_0 \tau = \omega_0 \frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$

critical damping $R_{critical} = 2\omega_0 L = 2\sqrt{\frac{L}{C}}$

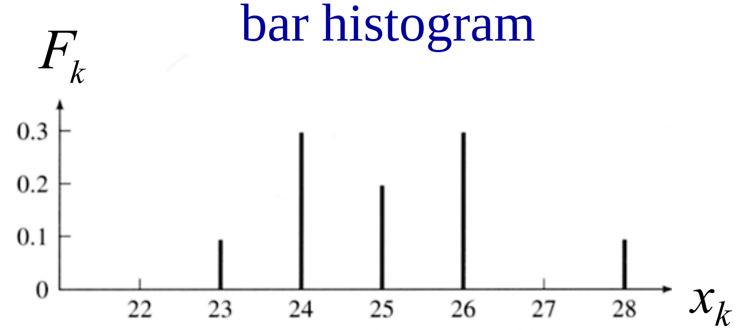




Histograms and Distributions

10 measurements:
26, 24, 26, 28, 23, 24, 25, 24, 26, 25

bar histogram



different values x_k	23	24	25	26	27	28
number of occurrences n_k	1	3	2	3	0	1

$$F_k = \frac{n_k}{N}$$ fraction of measurements that gave the result x_k

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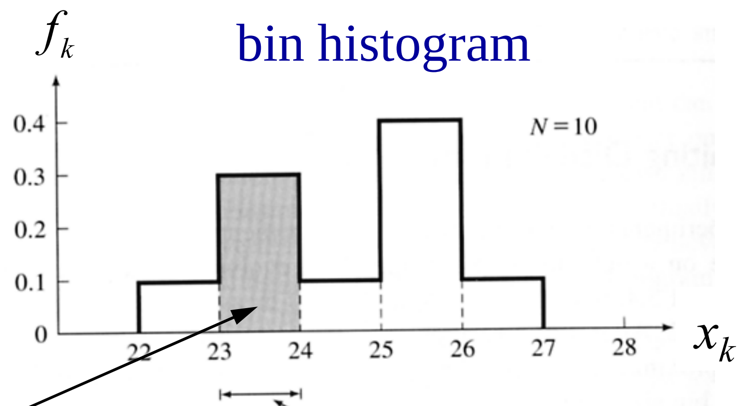
bin histogram



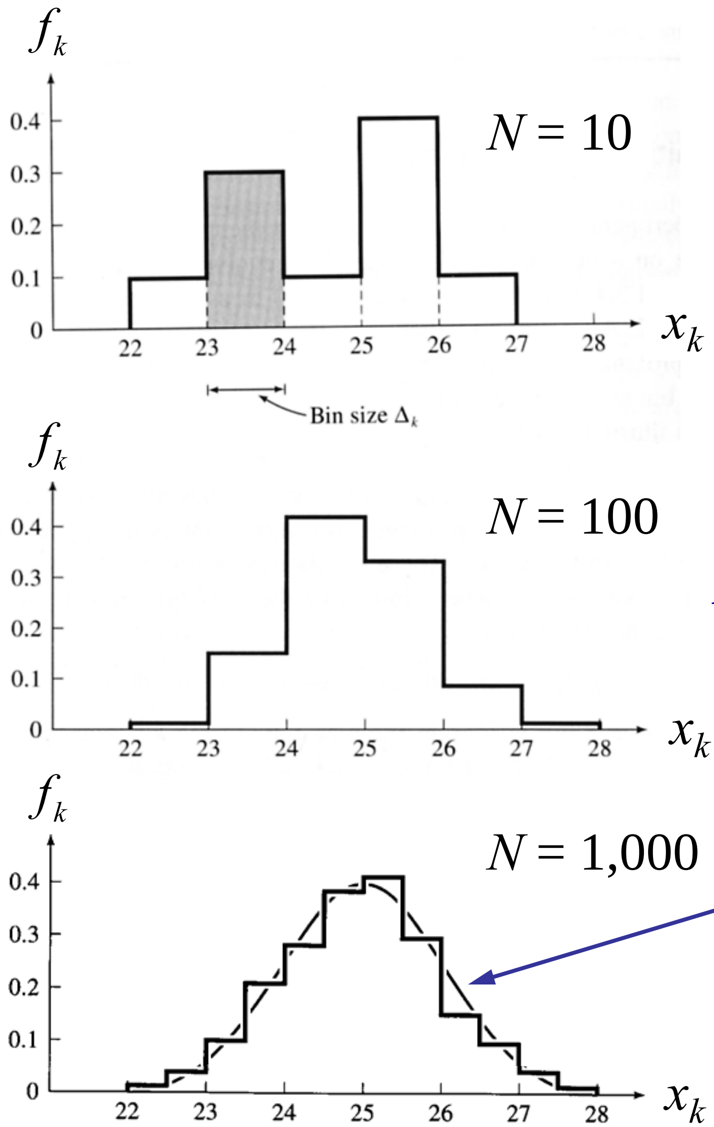
10 measurements:
26.4, 23.9, 25.1, 24.6, 22.7, 23.8, 25.1, 23.9, 25.3, 25.4

bins for x_k	22-23	23-24	24-25	25-26	26-27
n_k	1	3	1	4	1

$$f_k \Delta_k = \frac{n_k}{N}$$ fraction of measurements in k-th bin

$f_k \Delta_k$ = the area of the k-th rectangle
has the same significance
as the height F_k of the k-th bar in a bar histogram

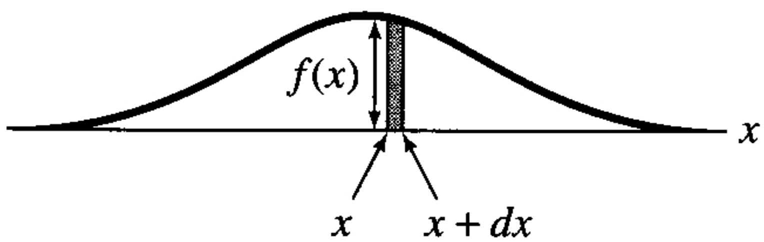
Limiting Distributions

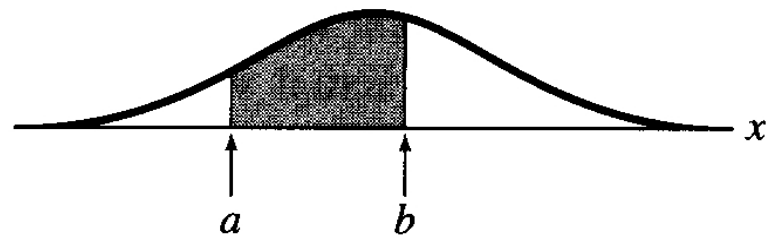


as the number of measurements approaches infinity, their distribution approaches some definite continuous curve

this curve is called the limiting distribution, *f(x)*

Limiting Distributions





$f(x)\,dx$ = fraction of measurements that fall between x and $x+dx$
= probability that any measurement will give an answer between x and $x+dx$

$\int_a^b f(x)dx$ = fraction of measurements that fall between $x=a$ and $x=b$
= probability that any measurement will give an answer between $x=a$ and $x=b$

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$ normalization condition