General Formula for Error Propagation

$$q = q(x, y)$$

$$q_{best} = q(x_{best}, y_{best})$$

partial derivative of q with respect to x

$$q(x_{best} + \delta x, y_{best} + \delta y) = q(x_{best}, y_{best}) + \frac{\partial q}{\partial x}\delta x + \frac{\partial q}{\partial y}\delta y$$

extreme values of q: $$q(x_{best}, y_{best}) \pm \left(\left|\frac{\partial q}{\partial x}\right|\delta x + \left|\frac{\partial q}{\partial y}\right|\delta y\right)$$

$$\delta q = \left|\frac{\partial q}{\partial x}\right|\delta x + \left|\frac{\partial q}{\partial y}\right|\delta y$$

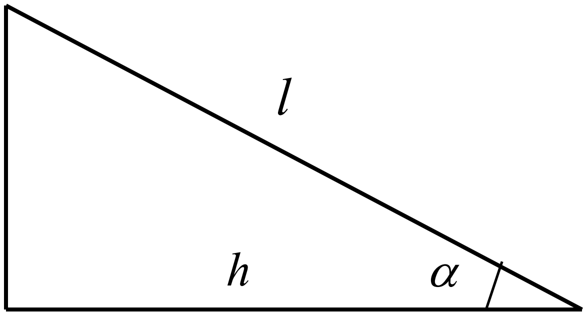
overestimated for independent δx and δy

$$\boxed{\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \left(\frac{\partial q}{\partial y}\delta y\right)^2}}$$

main formula for error propagation
always use this formula

for independent random errors δx and δy

Example



$l = 10 \pm 0.1$ m

$\alpha = 20 \pm 3^0$

Find h.

$$h = l \cdot \cos\alpha = 10 \cdot \cos 20^0 = 10 \cdot 0.94 = 9.4 \text{ m}$$

$$\delta h = \sqrt{\left(\frac{\partial h}{\partial l}\delta l\right)^2 + \left(\frac{\partial h}{\partial \alpha}\delta\alpha\right)^2}$$

always use radians when calculating the errors on trig functions

$$\frac{\partial h}{\partial l} = \cos\alpha$$

$$\delta\alpha = 3^0 = \frac{2\pi \text{ rad}}{360^0} \cdot 3^0 = 0.05 \text{ rad}$$

$$\frac{\partial h}{\partial \alpha} = l \cdot (-\sin\alpha)$$

$$\delta h = \sqrt{(\cos\alpha \cdot \delta l)^2 + (l \cdot (-\sin\alpha) \cdot \delta\alpha)^2} = \sqrt{(0.94 \cdot 0.1)^2 + (10 \cdot [-0.34] \cdot 0.05)^2} = 0.2 \text{ m}$$

$$h = 9.4 \pm 0.2 \text{ m}$$

Notations

$\sqrt{a^2 + b^2} = a \oplus b$ shorthand notation for quadratic sum

quadratic sum = addition in quadrature

for independent random errors $\delta x \leftrightarrow \sigma_x$

σ = (sigma) = Standard Deviation

Statistical analysis

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Period of pendulum

uncertainty

error propagation

statistical analysis

two methods to find uncertainty

The mean

$x_1,\ x_2,\ \ldots\ ,\ x_N$ N measurements of the quantity x

$x_{best} = \bar{x}$ the best estimate for x → the average or mean

$$\boxed{\bar{x} = \frac{x_1 + x_2 + \ldots + x_N}{N} = \frac{\sum x_i}{N}}$$

$$\sum_{i=1}^{N} x_i = \sum_{i} x_i = \sum x_i = x_1 + x_2 + \ldots + x_N$$

sigma notation

common abbreviations

The standard deviation

$d_i = x_i - \bar{x}$ deviation of x_i from $\bar{x}$

$$\sigma_x = \sqrt{\frac{1}{N}\sum d_i^2} = \sqrt{\frac{1}{N}\sum (x_i - \bar{x})^2}$$

average uncertainty of the measurements $x_1, \ldots, x_N$

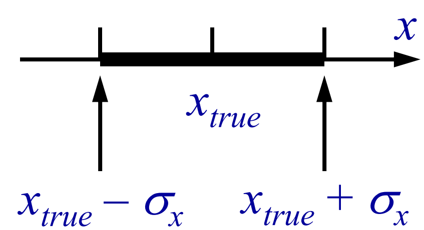
$$\boxed{\sigma_x = \sqrt{\frac{1}{N-1}\sum (x_i - \bar{x})^2}}$$

standard deviation of x

RMS (route mean square) deviation

uncertainty in any one measurement of $x \longrightarrow$ $\underline{\delta x = \sigma_x}$

68% of measurements will fall in the range $x_{true} \pm \sigma_x$



The standard deviation of the mean

$$\boxed{\begin{aligned}\sigma_{\bar{x}} &= \frac{\sigma_x}{\sqrt{N}} \\ \delta x &= \sigma_{\bar{x}}\end{aligned}}$$

uncertainty in $\bar{x}$
is the standard deviation of the mean

based on the N measured values $x_1, \ldots, x_N$ we can state our final answer for the value of x:

$$(\text{value of } x) = x_{best} \pm \delta x$$

$$x_{best} = \bar{x} \qquad \delta x = \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

$$\boxed{(\text{value of } x) = \bar{x} \pm \sigma_{\bar{x}}}$$

Example

We make measurements of the period of a pendulum 3 times and find the results:
$T = 2.0$, 2.1, and 2.2 s.
(a) What is the mean period?
(b) What is the RMS error (the standard deviation) in the period?
(c) What is the error in the mean period (the standard deviation of the mean)?
(d) What is the best estimate for the period and the uncertainty in the best estimate.

$$\boxed{\bar{x} = \frac{1}{N}\sum x_i} \quad \boxed{\sigma_x = \sqrt{\frac{1}{N-1}\sum\left(x_i - \bar{x}\right)^2}} \quad \boxed{\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}} \quad \boxed{\left(\text{value of } x\right) = \bar{x} \pm \sigma_{\bar{x}}}$$

$$\bar{T} = \frac{1}{N}\sum T_i = \frac{1}{3}\sum(2 + 2.1 + 2.2) = \underline{2.1\ \text{s}}$$

$$\sigma_T = \sqrt{\frac{1}{N-1}\sum(T_i - \bar{T})^2} = \sqrt{\frac{1}{2}\left[(2-2.1)^2 + (2.1-2.1)^2 + (2.2-2.1)^2\right]} = \sqrt{\frac{1}{2}\left[0.1^2 + 0.1^2\right]} = \underline{0.1\ \text{s}}$$

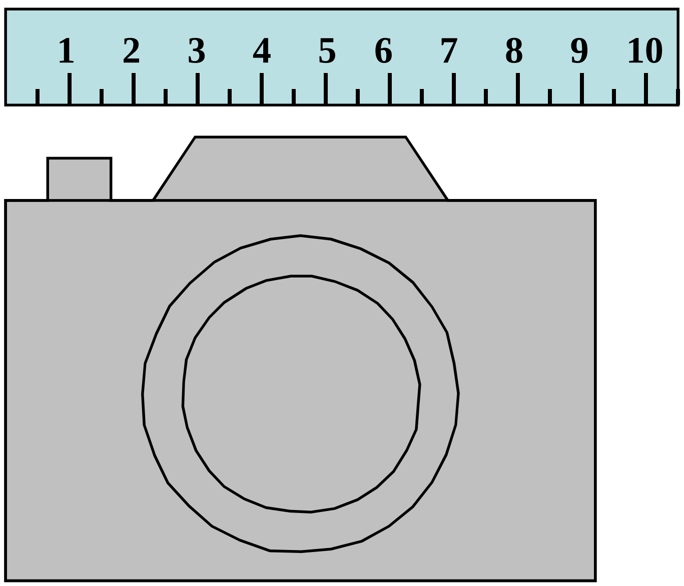
$$\sigma_{\bar{T}} = \frac{\sigma_T}{\sqrt{N}} = \frac{0.1}{\sqrt{3}} = 0.057735\ \text{s} \rightarrow \underline{0.06\ \text{s}}$$

$$T = \bar{T} \pm \sigma_{\bar{T}} = \underline{2.10 \pm 0.06\ \text{s}}$$

Systematic errors

$$\delta x = \sqrt{\left(\delta x_{ran}\right)^2 + \left(\delta x_{sys}\right)^2}$$

random component   systematic component



Experiment 1: Resistor–capacitor (RC) Circuits

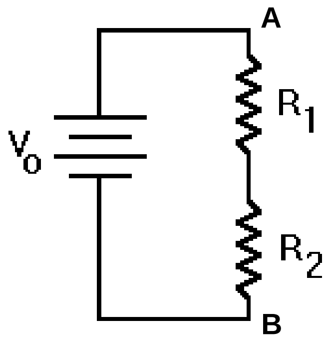
Goal: examine a simple circuit consisting of one capacitor and one resistor

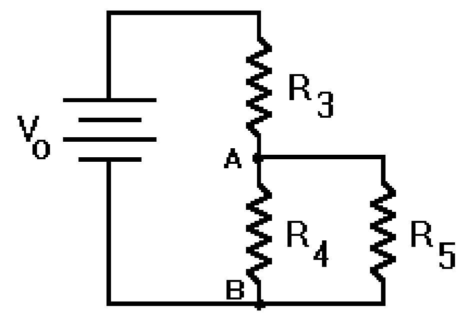
1. Determine the capacitor discharge decay time τ by applying a constant voltage (also called DC or direct current) to the circuit.

2. Determine τ by applying alternating current (AC) and varying the frequency of the current.

compare results to calculated value

circuit elements: resistors, capacitors, and inductors





resistors R_1 and R_2 in series

resistors R_4 and R_5 in parallel

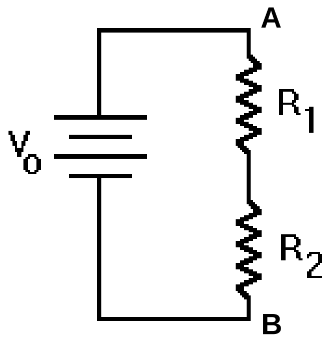
Two circuit elements are <u>in series</u> if all of the current flowing through one also flows through the other.

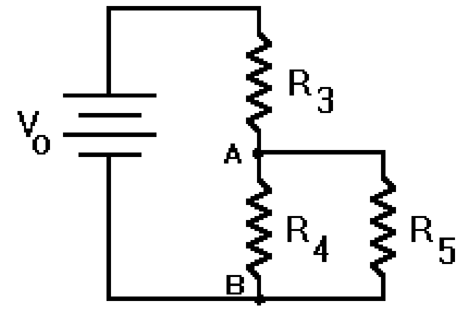
$I_1 = I_2$

Two circuit elements are <u>in parallel</u> if they are connected to the same nodes. The potential difference (voltage drop) across all elements connected in parallel must be the same.

$V_4 = V_5$

a <u>node</u>: a section of a circuit that is at constant voltage (e.g. a piece of wire)





Ohm's law: The current through a conductor between two points is directly proportional to the potential difference or voltage across the two points, and inversely proportional to the resistance between them.

$$I = V/R$$

Kirchhoff's current law (Kirchhoff's first rule): At any node (junction) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node.
By convention, every current flowing towards the node is positive and every current flowing away is negative
Follows from the conservation of electric charge.

$$\Sigma\, I_{\mathrm{k}} = 0$$

Kirchhoff's voltage law (Kirchhoff's second rule): The directed sum of the electrical potential differences around any closed circuit must be zero.
Follows from the conservation of energy (voltage is the energy per unit charge).

$$\Sigma\, V_{\mathrm{k}} = 0$$

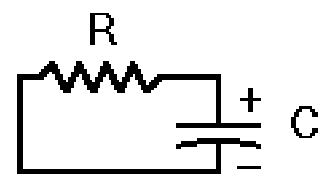
$$R_{\mathrm{AB}} = R_1 + R_2$$

$$R_{\mathrm{AB}} = \left(1/R_4 + 1/R_5\right)^{-1} = R_4 * R_5 / \left(R_4 + R_5\right)$$

1. The Discharging (Charging) RC Circuit

A capacitor can be charged by connecting its terminals to a battery. $Q_C = CV_C$

If the capacitor is disconnected from the battery, it will retain Q_C and V_C.



If a resistor is connected to a charged capacitor, charge will flow through the resistor until the potential difference between the two terminals goes to zero.

Kirchhoff's law: $V_R + V_C = 0$

$$V_R = I_R R$$

$$Q_C = CV_C$$

$$I_R = \frac{-1}{RC} Q_C$$

If one applies a constant voltage to this circuit then the voltage source will be an additional constant term in the sum.

$$\frac{dQ_C}{dt} = I_R = \frac{-1}{RC} Q_C \xrightarrow{Q=Q_0 \text{ at } t=0} \int_{Q_0}^{Q} \frac{dQ}{Q} = \frac{-1}{RC} \int_0^t dt$$

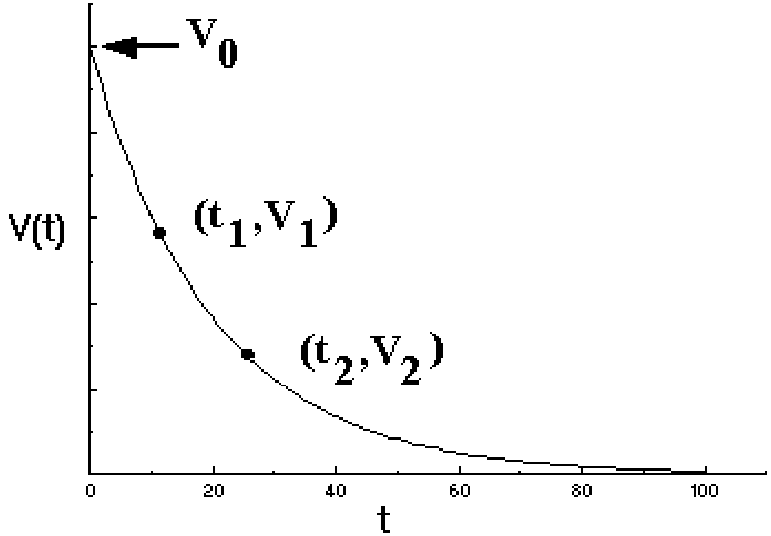
$$\ln\left(\frac{Q}{Q_0}\right) = \frac{-1}{RC} t$$

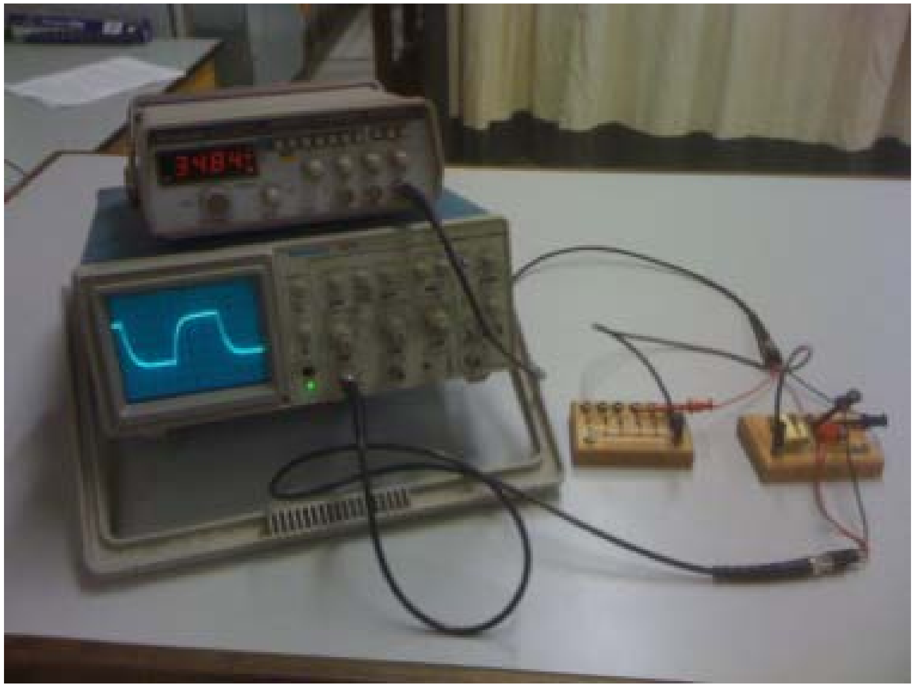
$$\boxed{\begin{array}{l} Q = Q_0 e^{-t/\tau} \\ \text{where} \quad \tau = RC \end{array}}$$

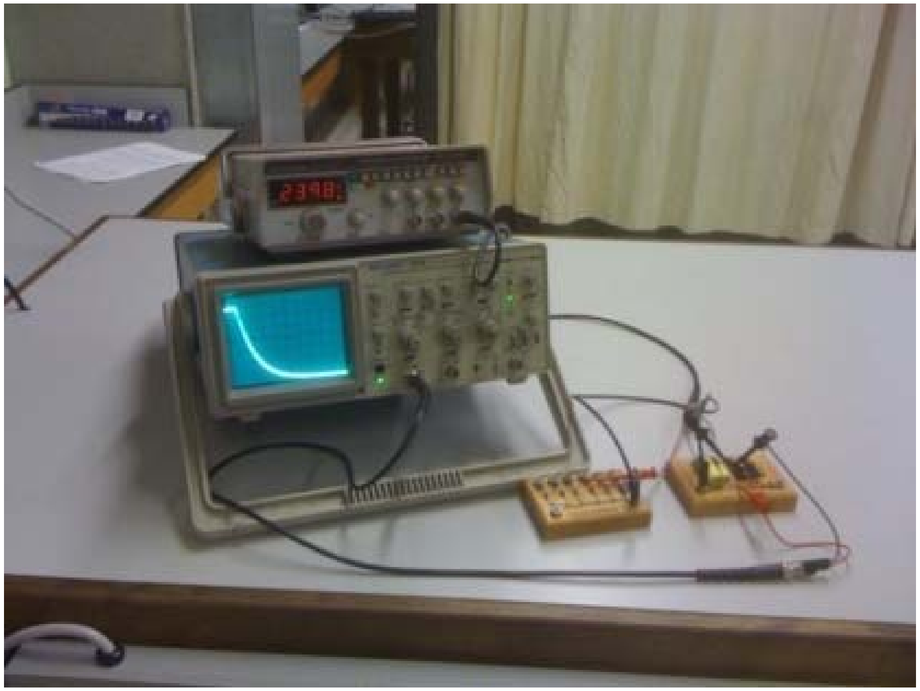
$$I_R = \frac{dQ}{dt} = Q_0 \left(\frac{-1}{\tau}\right) e^{-t/\tau}$$

$$\underline{V_R} = I_R R = Q_0 R \left(\frac{-1}{\tau}\right) e^{-t/\tau} = \frac{-Q_0}{C} e^{-t/\tau} = \underline{V_0 e^{-t/\tau}}$$

$$V(t) = V_0 e^{-t/\tau} \quad \text{where} \quad \tau = RC$$







$$V_1 = V_0 e^{-t_1/\tau} \text{ and } V_2 = V_0 e^{-t_2/\tau}$$

$$\frac{V_1}{V_2} = e^{(t_2 - t_1)/\tau}$$

$$\tau = \frac{t_2 - t_1}{\ln(V_1 / V_2)}$$

Impedance

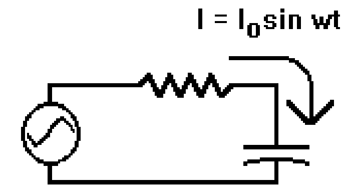
The complex generalization of resistance is <u>impedance</u> $Z = \dfrac{\tilde{V}}{\tilde{I}}$

Z is complex (of the form $a + ib$ where a and b are real numbers)

$\tilde{V} = V_0 e^{i\omega t}$ $\tilde{I} = I_0 e^{i(\omega t + \phi)}$ ϕ is the amount by which $\tilde{V}$ and $\tilde{I}$ are out of phase

$$Z_R = R; \qquad Z_C = \frac{1}{i\omega C}; \qquad Z_L = i\omega L$$

2. RC Circuit in the Frequency Domain



RC circuit with signal generator

Kirchhoff's law: $\tilde{V}_S + \tilde{V}_C + \tilde{V}_R = 0$

$$\tilde{V}_S - \tilde{I}(Z_R + Z_C) = 0$$

$$Z_R + Z_C = R + \frac{1}{i\omega C}$$

$$= R\left(1 - i\frac{1}{\omega RC}\right)$$

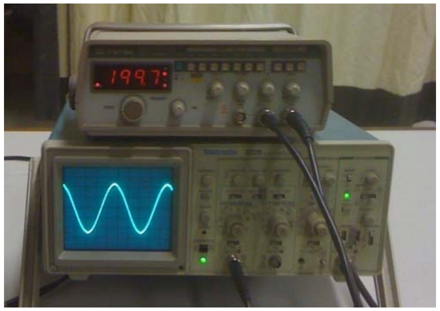
$$\tilde{V} = V_0 e^{i\omega t}$$

$$\tilde{I} = I_0 e^{i(\omega t + \phi)}$$

$$= \underbrace{R\sqrt{1 + \frac{1}{(\omega RC)^2}}}_{\text{the ratio of the amplitude of } V \text{ to } I} \exp\left(\underbrace{i\arctan\left[\frac{-1}{\omega RC}\right]}_{\text{the phase offset between } V \text{ and } I}\right) \longrightarrow V_0 = I_0 R\sqrt{1 + \frac{1}{(\omega RC)^2}}$$

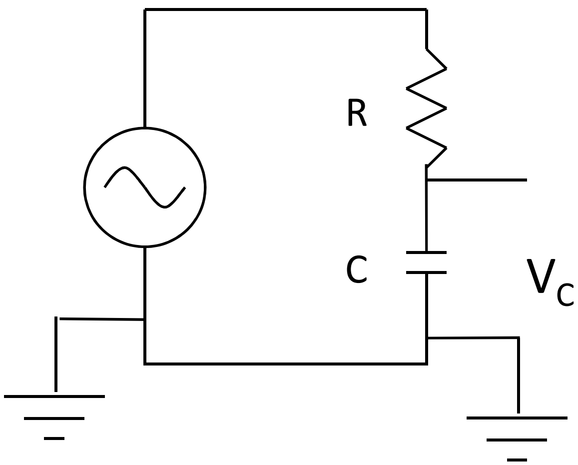
$$|V_C| = |Z_C| I_0$$

$$= \left(\frac{1}{\omega C}\right)\left(\frac{V_0}{R}\sqrt{\frac{(\omega RC)^2}{(\omega RC)^2 + 1}}\right) = \frac{V_0}{\sqrt{1 + (\omega RC)^2}} = \frac{V_0}{\sqrt{1 + (\omega\tau)^2}}$$

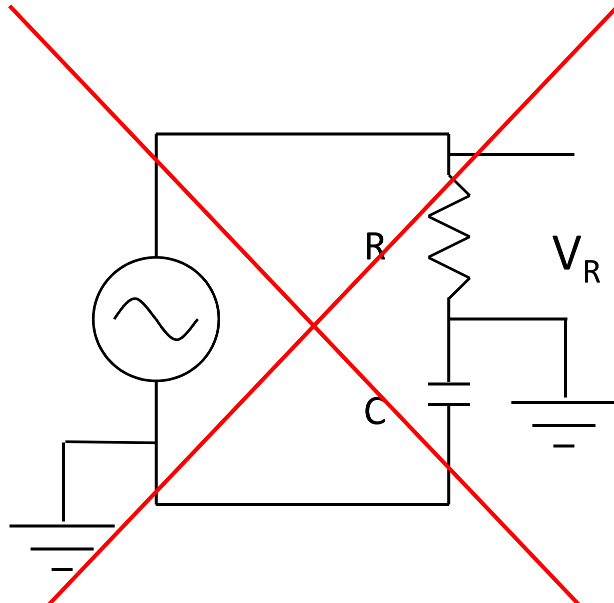


Important notes

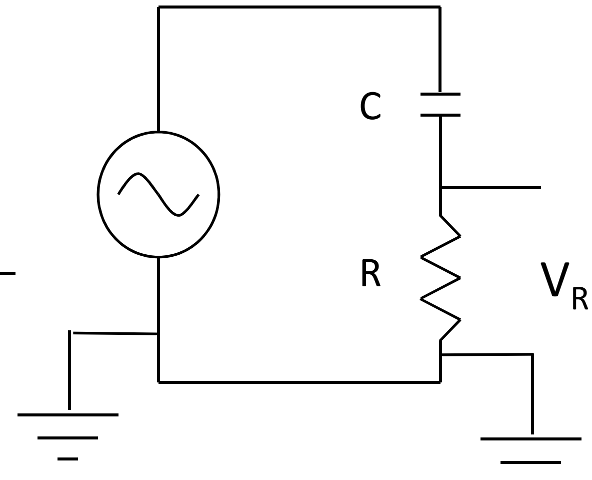
Both the generator and the oscilloscope will have one end which fixes the voltage to the <u>common ground</u> voltage. **Place the grounds properly.**



V_C is measured properly

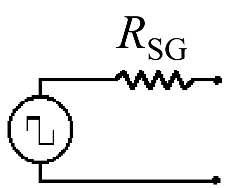


V_R is measured incorrectly.
Circuit is 'shorted'.



V_R is measured properly

Include resistance from the signal generator (50 Ω with an uncertainty of 5%).



Signal generator with internal resistance

$$R_{total} = R_R + R_{SG}$$