I. Stability and Waves

5) Wave Energy and Momentum

Seek develop general 'Poynting' theorem for plasma waves, i.e. a relation of form:

$$\frac{\partial W}{\partial t} + \underline{\nabla} \cdot \underline{S} + Q = 0$$

$W \leftrightarrow$ Wave Energy Density
$\underline{S} \leftrightarrow$ Wave Energy Density Flux / Momentum
$Q \leftrightarrow$ Dissipation

n Electromagnetism:

$$W = (E^2 + H^2)/8\pi$$
$$\underline{S} = \frac{c}{4\pi}(\underline{E} \times \underline{H})$$

Consider electrostatic fluctuations (simplicity) $\Rightarrow$

$$\frac{dW}{dt} = \frac{1}{8\pi} \text{re} \left(\underline{E}^* \cdot \frac{d\underline{D}}{dt} \right) = 0$$

$\rightarrow$ total rate change energy

Consider: $\underline{E} = \underline{E}_0(t, \underline{x}) e^{i(\underline{k}_0 \cdot \underline{x} - \omega_0 t)}$

slow space-time variation
$t \leftrightarrow$ build-up of energy
$\underline{x} \leftrightarrow$ spread of initially local perturbation

$\therefore$ for α $(\alpha \ll \omega_0)$ s/t slow $\left\{\begin{array}{l}\text{time}\\ \text{space}\end{array}\right.$

$\underline{q}$ $(|\underline{q}| \ll k_0)$

$\Rightarrow$

$$\underline{E} = \sum_{\alpha, \underline{q}} \underline{E}_{0\alpha, \underline{q}} \exp\left[i(\underline{k}_0 + \underline{q}) \cdot \underline{x} - i(\omega_0 + \alpha) t\right]$$

$$\boxed{\begin{array}{l} \dfrac{d\underline{D}}{dt} = \sum\limits_{\alpha, \underline{q}} f(\omega_0 + \alpha, \underline{k}_0 + \underline{q}) e^{i(\underline{q} \cdot \underline{x} - \alpha t)} \underline{E}_{\alpha, \underline{q}} e^{i(\underline{k}_0 \cdot \underline{x} - \omega_0 t)} \\ f(\underline{k}, \omega) = -i\omega \epsilon(\underline{k}, \omega) , \quad D(\underline{k}, \omega) = \epsilon(\underline{k}, \omega) E_{\underline{k}, \omega} \end{array}}$$

Then, expand in α, $\underline{q}$:

$$\frac{d\underline{D}}{dt} = \sum_{\alpha, \underline{q}} \left[-i\omega \epsilon(\underline{k}, \omega) + \alpha \frac{\partial}{\partial \omega} (-i\omega\epsilon) \bigg|_{k_0, \omega_0} + \underline{q} \cdot \frac{\partial}{\partial \underline{k}} (-i\omega\epsilon) \bigg|_{k_0, \omega_0} \right] e^{i(\underline{q} \cdot \underline{x} - \alpha t)} \underline{E}_0 e^{i(\underline{k}_0 \cdot \underline{x} - \omega_0 t)}$$

$$= \left[-i\omega\epsilon\, \underline{E}_0(t, \underline{x}) + \frac{\partial}{\partial \omega} (\omega\epsilon) \bigg|_{k_0, \omega_0} \frac{\partial \underline{E}_0(t, \underline{x})}{\partial t} - \frac{\partial}{\partial \underline{k}} (\omega\epsilon) \cdot \underline{\nabla} \underline{E}_0(t, \underline{x}) \right] \exp\left[i\underline{k}_0 \cdot \underline{x} - i\omega_0 t\right]$$

$$\frac{dW}{dt} = \frac{1}{8\pi}\, re \left(\underline{E}^* \cdot \frac{d\underline{D}}{dt} \right)$$

$$= \frac{\partial}{\partial t}\left[\left.\frac{\partial}{\partial \omega}(\omega \epsilon)\right|_{\underline{k}_0, \omega_0} \frac{|\underline{E}_0|^2}{8\pi} \right]_{\underline{k}_0, \omega_0}$$

$$- \underline{\nabla} \cdot \left[\frac{\partial}{\partial \underline{k}}(\omega \epsilon) \frac{|\underline{E}_0|^2}{8\pi} \right]_{\underline{k}_0, \omega_0} + \left. \omega \epsilon_{IM}(k, \omega) \frac{|\underline{E}_0|^2}{8\pi} \right|_{\underline{k}_0, \omega_0}$$

<u>so</u>

$$W = \left.\frac{\partial}{\partial \omega}(\omega \epsilon)\right|_{\underline{k}_0, \omega_0} \left(|\underline{E}_0|^2 / 8\pi \right) \rightarrow \text{wave energy}$$

$$\underline{S} = -\left.\frac{\partial}{\partial \underline{k}}(\omega \epsilon)\right|_{\underline{k}_0, \omega_0} \left(|\underline{E}_0|^2 / 8\pi \right) \rightarrow \text{wave energy density flux}$$

$$Q = \omega \epsilon_{IM} \left(|\underline{E}_0|^2 / 8\pi \right) \rightarrow \text{energy dissipation rate}$$

For EM wave :

$$\rightarrow W \rightarrow \left.\frac{\partial}{\partial \omega}(\omega \epsilon)\right|_{\underline{k}_0, \omega_0} \left(|\underline{E}|^2 / 8\pi \right) + \left.\frac{\partial}{\partial \omega}(\omega \mu)\right|_{\underline{k}_0, \omega_0} \left(|\underline{H}|^2 / 8\pi \right)$$

etc.

$$\rightarrow S \rightarrow S + \frac{c}{4\pi}\left(\underline{E} \times \underline{H} \right)$$

Note:

i) At wave resonance, $\epsilon(\underline{k}_0, \omega_0) = 0$

$$W = \omega_{\underline{k}} \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\underline{k}_0, \omega_{\underline{k}}} \left(|E_0|^2 / 8\pi\right)$$

$$\underline{S} = -\omega_{\underline{k}} \left.\frac{\partial \epsilon}{\partial \underline{k}}\right|_{\underline{k}, \omega_{\underline{k}}} \left(|E_0|^2 / 8\pi\right)$$

$$Q = \omega_{\underline{k}} \left.\epsilon_{IM}\right|_{\underline{k}, \omega_{\underline{k}}} \left(|E_0|^2 / 8\pi\right)$$

ii) $$\underline{V}_{gr_{\underline{k}}} = \underline{S} / W$$

$$= -\left(\partial \epsilon / \partial \underline{k}\right)_{\omega_{\underline{k}}} / \partial \epsilon / \partial \omega_{\omega_{\underline{k}}}$$

Alternatively, along wave path:

$$d\epsilon = \frac{\partial \epsilon}{\partial \omega} d\omega + \frac{d\epsilon}{d\underline{k}} \cdot d\underline{k} = 0$$

$$\underline{V}_{gr} = -\partial \epsilon / \partial \underline{k} \,/\, \partial \epsilon / \partial \omega \Big|_{\omega_{\underline{k}}}$$

Physics of Wave Energy / Momentum

i.) $W = \frac{\partial}{\partial\omega}(\omega\epsilon)\Big|_{k_0,\omega_0} |E_0|^2/(8\pi)$

$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$, for cold plasma

$W = \left(1 + \frac{\omega_p^2}{\omega^2}\right)|E_0|^2/(8\pi)$

$= W_{Field} + W_{\substack{sloshing \\ Energy}}$

$\underset{\sim}{\text{'Wave'}}$ = Field + Particle Mtm.

i.e. $\frac{\partial\tilde{V}}{\partial t} = \frac{q}{m}E$

$n_0\frac{1}{2}m|\tilde{V}|^2 = \frac{n_0}{2}\frac{q^2}{m^2}\frac{m}{\omega^2}|E|^2$

$= \frac{1}{8\pi}\frac{\omega_p^2}{\omega^2}|E|^2$

ii.) $\underline{S} = -\omega\frac{\partial\epsilon}{\partial\underline{k}}|E_0|^2/8\pi$

$\epsilon = 1 - \omega_p^2/\omega^2 - k^2V_{th}^2$ (Warm Plasma

$\Rightarrow \underline{S} = +\omega\,\omega_p^2\,2\underline{k}V_{th}^2/(\omega^2 - k^2V_{th}^2)^2 \Rightarrow \underset{\text{(Compressional WV)}}{\underline{S} \sim \underline{k}}$.

.) If cold, collisional plasma

$$\epsilon = 1 - \omega_p^2/\omega(\omega + i\nu)$$

$$\cong 1 - \frac{\omega_p^2(\omega - i\nu)}{\omega(\omega^2+\nu^2)}$$

$$\epsilon_{IM} = \omega_p^2\nu/\omega(\omega^2+\nu^2)$$

$$Q = \frac{\omega_p^2\nu}{\omega^2+\nu^2}\frac{|E|^2}{4\pi} \qquad Q \sim \nu.$$

i) Positive / Negative Energy Waves

$$W = (|E|^2/8\pi)\ \omega_k \partial\epsilon/\partial\omega|_{\omega_k}$$
$$= (|E_k|^2/8\pi)\ \partial(\omega\epsilon)/\partial\omega$$

Contrast

→ cold plasma $\quad \epsilon = 1 - \frac{\omega_p^2}{\omega^2}$

$$W_k = (|E_k|^2/8\pi)(1 + \omega_p^2/\omega_k^2)$$
$$= |E_k|^2/4\pi$$

- $W_k > 0$ $\Rightarrow$ need put energy into oscillator to excite motion

- kinetic energy $\rightarrow \frac{1}{2} m |V|^2$
  Potential $\rightarrow |E|^2/8\pi$ (electrostatic)

equal in simple oscillator.

$\rightarrow$ Beam - Plasma System

$$\begin{cases} \underline{V} = V_0 \hat{z} + \tilde{\underline{V}} \\ 1D \end{cases}$$

$$\frac{\partial \tilde{V}}{\partial t} + V_0 \frac{\partial}{\partial x} \tilde{V} = + \frac{q}{m} E$$

$$\frac{\partial \tilde{n}}{\partial t} + V_0 \frac{\partial \tilde{n}}{\partial x} = - n_0 \underline{\nabla} \cdot \tilde{\underline{V}}$$

$$\epsilon = 1 - \omega_p^2 / (\omega - k V_0)^2$$

$$\omega = k V_0 \pm \omega_p$$

$$W_k = \omega_k \left. \frac{\partial \epsilon}{\partial \omega} \right|_{\omega_k} \left(|E_k|^2 / 8\pi \right)$$

$$= (k V_0 \pm \omega_p) \frac{2 \omega_p^2}{(\omega - k V_0)^3} \left(|E_k|^2 / 8\pi \right)$$

$$= (k V_0 \pm \omega_p) \frac{2 \omega_p^2}{(\pm \omega_p)^3} \left(|E_k|^2 / 8\pi \right)$$

$$W_N = \pm \frac{(kV_0 \mp \omega_p)}{\omega_p} |E_{1s}|^2 / (4\pi)$$

Note: i.) + root $\omega = kV_0 + \omega_p$

$\Rightarrow$ + energy wave

ii.) − root $\omega = +kV_0 - \omega_p$

$$W_N = \frac{(\omega_p - kV_0)}{\omega_p} |E_{1s}|^2 / (4\pi)$$

$\omega_p > kV_0$

$\Rightarrow$ + energy wave for

− for $\omega_p < kV_0$

Negative Energy Wave:

→ excited by extraction of energy from system (i.e. slowing down beam)

→ occurs in 'active' medium $\Rightarrow V_0$

→ destabilized by dissipation

i.e. $\frac{\partial W_N}{\partial t} + \nabla \cdot S_N + Q_N = 0$

$\therefore$ if $S_{\underline{k}} \approx 0$ (radiative damping can destabilize negative energy wave)

$\gamma_{\underline{k}} = -Q_{\underline{k}}/W_{\underline{k}}$

$\Rightarrow$ $W_{\underline{k}} > 0$, $Q_{\underline{k}} > 0 \Rightarrow \gamma_{\underline{k}} < 0$ (usual)

$W_{\underline{k}} < 0$, $Q_{\underline{k}} > 0 \Rightarrow \gamma_{\underline{k}} > 0 \Rightarrow$ instability!

V) Instabilities in Fluid-Plasma Systems.

Bunching instabilities
$\rightarrow$ Jeans.

Instability:

$\rightarrow$ instantaneous growth of perturbation about an equilibrium

$\rightarrow$ indicative of perturbation extracting free energy from eqbrm. (i.e. beam)

$\rightarrow$ limited by
(quasi-linear, nonlinear theory)
$\rightarrow$ relaxation of free energy source (i.e. beam slows down)
$\rightarrow$ dissipation

Instability classifications:

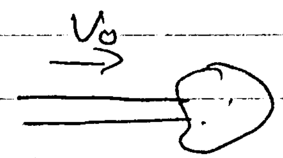
$\rightarrow$ absolute: grows in all reference frames

→ convective: grows in co-moving frame

e.g. ① Beam – Plasma System

Consider → cold, low density beam ($\omega_{pb} \ll \omega_p$)

→ cold fluid plasma



$$\epsilon = 1 - \frac{\omega_{pb}^2}{(\omega - kV_0)^2} - \frac{\omega_p^2}{\omega^2}$$

$\epsilon(k,\omega) \cong 0 \Rightarrow$ resonance (i.e. wave)

$$\therefore \quad 1 = \frac{\omega_p^2}{\omega^2} + \frac{\omega_{pb}^2}{(\omega - kV_0)^2}$$

$$\omega = kV_0 + \delta$$

$$\Rightarrow \quad 1 = \frac{\omega_p^2}{(kV_0)^2} + \frac{\omega_{pb}^2}{\delta^2}$$

$$\delta^2\left(1 - \frac{\omega_p^2}{(kV_0)^2}\right) = \omega_{pb}^2$$

$$\delta^2 = \omega_{pb}^2 \Big/ \left(1 - \frac{\omega_p^2}{(kV_0)^2}\right)$$

$\delta^2 < 0 \Rightarrow$ instability *

$\Leftrightarrow \omega_p > k V_0$

need:
$k V_0 < \omega_p$ so
$\delta / k V_0 \ll 1$

$\epsilon < 0 \Rightarrow$ bunching

$\Rightarrow$ perturbation grows !

Physical Interpretation:

- $\epsilon_{pl} = 1 - \frac{\omega_p^2}{\omega^2}$

$k^2 \epsilon(\underline{k}, \omega)\, \phi_{\underline{k},\omega} = 4\pi \rho_{ext}(\underline{k}, \omega)$

$\epsilon(\underline{k}, \omega) < 0 \Rightarrow$ bunching occurs
$\Rightarrow$ perturbations fed by beam causing more bunching

$\omega \sim k V_0 + \delta$

$\therefore \epsilon < 0 \Rightarrow k V_0 < \omega_p$.

+ energy pl. wave ⊕
⊖ energy beam wave.

- also $\omega_p > k V_0 \sim \omega \gg \omega_{pb}$

$k V_0 \gg \omega_{pb} \rightarrow$ beam perturbation is negative energy

(dissipation)

--) Stability – bunching

→ Stability is Basic Constraint on Galactic Structure
i.e. key test of equilibrium viability

→ Basic instability ↔ Jeans instability

Jeans instability → { gravitational collapse
vs.
pressure (acoustic)

⇓

{ sets basic space-time scales

Analysis → ∞, stationary, slab
i.e.
$P_0, \rho_0 \rightarrow$ constant
$\underline{V}_0 = 0$

⇒

$\underline{\nabla}\phi_0 = 0$ (Euler eqn.)

but

$\nabla^2\phi_0 = 4\pi\rho_0$

⇒

inconsistency unless $\rho_0 = 0$!

"Jeans Swindle"

→ perturbation equations describe perturbed scalar only, not ϕ_0

→ assume $\phi_0 = 0$

why is this semi-believable ???

→ scale separation

→ rotation (i.e. equilibrium as gravity vs. centrifugal force).

i.) Hydrodynamic Jeans Inst.

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho \underline{v}) = 0 \qquad \nabla^2\phi = 4\pi\rho G$$

$$\rho \frac{d\underline{v}}{dt} = -\nabla p - \rho\nabla\phi$$

⇒ linearizing:

$$\frac{\partial\tilde{\rho}}{\partial t} + \rho_0 \nabla\cdot\tilde{\underline{v}} = 0$$

$$\frac{\partial\tilde{\underline{v}}}{\partial t} = -\frac{\nabla\tilde{p}}{\rho_0} - \nabla\tilde{\phi} \quad ; \quad \tilde{p} = c_s^2\tilde{\rho}$$

$$\nabla^2\tilde{\phi} = 4\pi\tilde{\rho}G$$

⇒

$$\frac{\partial}{\partial t}\nabla\cdot\tilde{\underline{v}} = -\nabla^2 c_s^2\frac{\tilde{\rho}}{\rho_0} - \nabla^2\tilde{\phi}$$

$$= -c_s^2\nabla^2\frac{\tilde{\rho}}{\rho_0} - 4\pi G\tilde{\rho}$$

$$\frac{\partial^2\tilde{\rho}}{\partial t^2} - c_s^2\nabla^2\tilde{\rho} - 4\pi G\rho_0\tilde{\rho} = 0$$

$$\Rightarrow \tilde{\rho} = \sum_{\underline{k},\omega} \tilde{\rho}_{\underline{k},\omega}\, e^{i(\underline{k}\cdot x - \omega t)}$$

$$\left(-\omega^2 + k^2 c_s^2 - 4\pi G\rho_0\right)\tilde{\rho}_{\underline{k},\omega} = 0$$

$$\Rightarrow \boxed{\omega^2 = k^2 c_s^2 - 4\pi G\rho_0}$$

[Dispersion relation for Jeans Instability]

i.e. $k^2 c_s^2 > 4\pi G\rho_0 \rightarrow$ stability $(\omega = \omega_r)$

$\qquad\quad < 4\pi G\rho_0 \rightarrow$ instability $(\omega = i\gamma)$

$$\omega^2 = \underbrace{k^2 c_s^2}_{\text{pressure waves (sound)}} - \underbrace{4\pi G\rho_0}_{\text{gravitational collapse}}$$

Clearly:

- stability for $k > k_J = \left(\frac{4\pi G\rho_0}{c_s^2}\right)^{1/2}$

$$\underbrace{(2\pi/k_J)^2}_{\text{Jeans length}} = \lambda_J^2 = \frac{\pi c_s^2}{G\rho_0}$$

$\lambda > \lambda_J \Rightarrow$ unstable

$\lambda < \lambda_J \Rightarrow$ stable

— Jeans instability is "negative compressibility"

i.e. can write:

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} = C_{eff}^2 \nabla^2 \tilde{\rho} \Rightarrow \frac{\partial^2 \tilde{\rho}_k}{\partial t^2} = -k^2 C_{eff_k}^2 \tilde{\rho}_k$$

$$C_{eff_k}^2 = C_s^2 - \frac{4\pi G \rho_0}{k^2} \quad ; \quad k < k_J \Rightarrow C_{eff}^2 < 0$$

$k > k_J \Rightarrow C_{eff}^2 > 0.$

— Jeans Mass $\equiv$ mass within sphere of λ_J

$$M_J = \frac{4\pi}{3} \rho_0 \left(\frac{\lambda_J}{2}\right)^3 = \frac{1}{6} \pi \rho_0 \left(\frac{\pi C_s^2}{G\rho_0}\right)^{3/2}$$

Misc. - Waves

Read: P. A. Sturrock, J. Appl. Phys. 31 (1960) 2052

(compare Jeans)

→ Bunching Instability
Beam-Plasma

- consider beam-plasma system

$$1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - kV_0)^2} = 0 \qquad n_b \ll n_0$$

⇒ ⊕ energy plasma oscillation
& ⊖ energy beam modes V_0 → active medium
interaction ⇒ instability

$n_b \ll n_0$ ⇒ need $\omega \sim kV_0 + \delta$

$$1 - \frac{\omega_p^2}{(kV_0)^2} - \frac{\omega_{pb}^2}{\delta^2} \cong 0$$

⇒

$$\delta^2 = \omega_{pb}^2 \Big/ 1 - \frac{\omega_p^2}{(kV_0)^2}$$

$$= \omega_{pb}^2 \Big/ \epsilon(k,\omega)_p$$

$\omega = kV_0$

$(kV_0)^2 < \omega_p^2$ ⇒ $\delta^2 < 0$ ⇒ instability

$\epsilon < 0$

- note: → $\epsilon(k, kv_0) < 0$ converts oscillation to instability

→ contrast: $\epsilon < 0 \rightarrow$ instability

i.e. converts ~~repulsion to~~ attraction (c.f. HW)

$\epsilon = 1 + \frac{1}{k^2 \lambda_D^2} > 0 \rightarrow$ (stable) simple screening

→ recall HW $\epsilon = 1 - \frac{\omega_p^2/2}{(\omega - kv_0)^2} - \frac{\omega_p^2/2}{(\omega + kv_0)^2}$

$(kv_0)^2 < \omega_p^2$ for instability

(similar but $n_b = n$)

- beam - plasma is type of reactive instability.

→ Types of Instability

- reactive → beam - plasma
- negative energy + dissp'n → beam + dissp'n.
- fluid ⇒ Rayleigh - Taylor
- inverse dissipation → bump-on-tail
  → CDIA.

the unconstricted arc is approximately proportional to current while peak current densities are independent of current. Then, according to Eq. (3), the pressure available to produce streaming is proportional to (current), and according to Eq. (5) the velocities resulting will be then proportional to (current)$^{1/2}$. Since the cross-sectional area is directly proportional to (current), this suggests that total flows should increase as (current)$^{3/2}$.

This simple picture is modified by the fact that as current increases in this range, temperatures also increase, so that velocity and heat flow should increase somewhat faster, while mass flow should increase slower than suggested by the foregoing. The data show this trend.

In conclusion it can be said that the magnetically produced streaming in the high current arc plays an important role in the over-all behavior of the arc and makes a very considerable contribution to the heat and mass transfer of the arc. The mechanism for the heat transfer appears to be analogous to that observed in flames.

I would like to thank H. N. Olsen and O. H. Nestor for supplying unpublished data for this work.

SIMPLE ANALOG EXPERIMENT DEMONSTRATING ARC PUMPING

A two-dimensional analog experiment was run in mercury to demonstrate the pumping which occurs in a divergent current path. A flat dish was filled to a depth of $\frac{1}{2}$ cm with mercury, and a small area electrode and large area electrode were connected to a generator to simulate the geometry existing in the arc (Fig. 6). With a current of 500 amp passing through the mercury, a vigorous streaming of mercury away from the small area electrode, with peak velocities of 5 cm/sec, was observed. A white powder on the surface shows the stream lines and velocities in a photograph exposed $\frac{1}{5}$ sec. A card was placed on the surface of the mercury and iron filings sprinkled on the surface to show the magnetic field lines. This was photographed and the two pictures superimposed in printing.

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In What Sense Do Slow Waves Carry Negative Energy?*

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It has been found in the theory of electron tubes that, according to the "small-amplitude power theorem," the fast and slow space-charge waves carry positive and negative energy, respectively. Similar analysis of different systems leads to similar results, leading one to conjecture that there is some sense in which one might assert that, for a wide class of dynamical systems, slow waves carry negative energy. In a one-dimensional model, "slow" and "fast" waves in a moving propagating medium refer to waves of which the phase velocity does or does not change sign, respectively, on transforming from the moving frame to the stationary frame. Small-amplitude disturbances of any dynamical system may be described by a quadratic Lagrangian function, from which one may form the canonical stress-tensor, elements of which are quadratic functions of the variables which appear in the linearized equations of motion. For any pure wave in this system, the energy density E and the momentum density P, as they appear in the canonical stress tensor, are related to the frequency ω and wave number k by $E=J\omega$, $P=Jk$, where $2\pi J$ is the action density. The rules for Galilean transformation now show that the energy densities, as measured in the stationary frame, of fast and slow waves have positive and negative sign, respectively, if (as is usually the case) the energy densities of both waves are positive in the moving frame. Similar arguments explain the signs of the energy density of the two "synchronous" waves which arise in the analysis of transverse disturbances of an electron beam in a magnetic field.

I. INTRODUCTION

ONE of the most illuminating and useful concepts in the theory of microwave tubes is the so-called "small-amplitude power theorem"[1–3] which was first given, in a very restricted form, by L. J. Chu.[4] It was found that, in simple cases, it is possible to ascribe to the particles of a modulated electron beam a "kinetic power," the formula for which involves only terms which appear in the linearized equations for the system, and which, when added to the Poynting flux of the associated electromagnetic field, is properly conserved. In more complicated cases, interaction terms arise. Certain simple but acceptable models for electron beams make it possible to analyze an arbitrary disturbance of a free beam into a "fast wave" and a "slow wave," the phase velocities of which are greater and less, respectively, than the particle velocity. The kinetic power of the fast wave is positive, that of the slow wave is negative; since the group velocities have the direction of the beam velocity, one must conclude that the corresponding energy density of the fast wave is positive, whereas that of the slow wave is negative.

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[1] W. H. Louisell and J. R. Pierce, Proc. I.R.E. 43, 425–427 (1955).

[2] H. A. Haus, *Noise in Electron Devices*, edited by L. D. Smullin and H. A. Haus (John Wiley & Sons, Inc., New York, 1959), pp. 77–153.

[3] P. A. Sturrock, Ann. Phys. 4, 306–324 (1958).

[4] L. J. Chu, paper presented at the Institute of Radio Engineers Electron Devices Conference, University of New Hampshire, June, 1951.

The small-amplitude power theorem has been the subject of some controversy which might be dispersed if it were generally agreed that the power which, in the framework of this theorem, is ascribed to the particle motion is not necessarily the correct or "physical" kinetic power of the beam, and that the usefulness of the theorem does not rest upon the equivalence of these two quantities. In solving the equations of motion, terms such as v_1 appear (where v is velocity) which are of first order in the amplitude, and also higher-order terms such as v_2 which is of second order in the amplitude. Terms such as v_2 must normally be obtained by solving nonlinear forms of the equations. It is important to note that such nonlinear terms are not determined uniquely by the linear terms: one may either complete the specification in an arbitrary way or, as is appropriate in electron-tube problems, by examining the way in which the wave is set up. It follows at once that we cannot expect to assert that the physical energy of a slow wave is negative, only that the energy of a slow wave *generated in a specified way* is negative.

The above point may be clarified by consideration of an ideal experiment. Let us accept that an appropriate coupler, excited in a certain way, will give rise to a slow wave on an electron beam and will, in the process, extract energy from the beam; this is one argument used by Pierce[5] to demonstrate that slow waves carry negative energy. Now consider a more complicated coupler in which the rf energy extracted from a beam in setting up the slow wave is converted to dc and then used to accelerate the beam. Such a coupler excites a slow wave with the same "small-amplitude" parameters, but in this case we should ascribe zero physical power to the slow wave since the coupler has neither added power to nor removed power from the beam. The analysis of Walker,[6] which aims at demonstrating the equivalence of the small-amplitude power theorem with the "physical" power theorem, contains an undetermined constant, the presence of which represents the impossibility of determining second-order quantities uniquely from first-order quantities. It is commonly believed that the negative-energy attribute of slow waves is peculiar to systems in which the vibratory motion is parallel to the dc velocity. Pierce[7] makes this assertion but points to what appears to be a counter-example: the experiment performed by C. C. Cutler, C. F. Chapman, and W. E. Mathews[8] on coupled torsional vibrations of the rims of two bicycle wheels rotating at different speeds. The instability of this system lends weight to the belief that slow waves in a moving medium capable of transverse vibrations again has negative energy in some sense, although one can see that the physical energy of any such disturbance must be positive.

As we have seen in discussing space-charge waves, we should not expect the physical energy of slow waves to be negative, although this may be so in particular propagating systems when the wave is excited in a particular way. We should therefore not be deterred from looking for a sense in which a slow wave can carry negative energy even in a system such as that considered in the previous paragraph. Indeed, the fact that one would wish to ascribe such an energy to a wave which is determined only in linear approximation requires that we look for an appropriate generalization of the small-amplitude power theorem rather than investigate the physical power of a class of propagating systems.

That such a generalization exists has been pointed out elsewhere.[3] It is possible to set up a small-amplitude energy theorem for any dynamical system, that is, for any system which may be described by an action principle. The Lagrangian function describing such a system may be expanded in a series of homogeneous polynomials in the dynamical variables representing the disturbance of the system from its quiescent state:

$$L=L^{(0)}+L^{(1)}+L^{(2)}+\cdots. \tag{1}$$

Since the term $L^{(0)}$ is independent of the dynamical variables, it may be ignored. Since the quiescent state, described by setting all dynamical variables equal to zero, is a solution of the Euler-Lagrange equations, we may ignore $L^{(1)}$ also. The lowest-order nonvanishing term is therefore $L^{(2)}$ which yields the linearized equations for the system. The fact that we have found a Lagrangian function to describe the "linear" system makes it possible to obtain, by standard procedures, conservation theorems for this system. It has been shown[3] that one may assign a complete stress tensor to the small-amplitude disturbances of an arbitrary electrodynamical system: this leads to the familiar small-amplitude power theorem as a special case. We shall show that it is this generalization of the small-amplitude energy theorem, applicable to any dynamical system, which enables us to assert that all slow waves carry negative energy.

II. THE SMALL-AMPLITUDE STRESS TENSOR

Consider a continuous dynamical system described by the action principle

$$\delta\int \mathcal{L}d^3x dt=0, \tag{2}$$

where the Lagrangian density $\mathcal{L}$ is expressible as

$$\mathcal{L}=\mathcal{L}\left(\phi_\alpha, \frac{d\phi_\beta}{dt}, \frac{d\phi_\gamma}{dx_r}, t, x_s\right), \tag{3}$$

in terms of the dynamical variables $\phi_\alpha(x_r,t)$. We write $x_r(r=1, 2, 3)$ for the spatial variables and reserve the partial differential sign for functional differentiation as in $\partial\mathcal{L}/\partial t$. We may now introduce the following

[5] J. R. Pierce, Bell System Tech. J. 33, 1343–1372 (1954).
[6] L. R. Walker, J. Appl. Phys. 26, 1031–1033 (1955).
[7] J. R. Pierce, J. Appl. Phys. 25, 179–183 (1954).
[8] W. E. Mathews, Proc. I.R.E. 39, 1044–1051 (1951).

variables which are canonically conjugate to ϕ_α

$$\Pi_{\alpha,t}=\frac{\partial\mathfrak{L}}{\partial(d\phi_\alpha/dt)},\quad \Pi_{\alpha,r}=\frac{\partial\mathfrak{L}}{\partial(d\phi_\alpha/dx_r)}. \tag{4}$$

Then the Euler-Lagrange equations[9] derivable from (2) are

$$\frac{d\Pi_{\alpha,r}}{dt}+\sum_r\frac{d\Pi_{\alpha,r}}{dx_r}=\frac{\partial\mathfrak{L}}{\partial\phi_\alpha}. \tag{5}$$

We may now form from the Lagrangian function the canonical stress tensor[10] which has the following components

$$\left.\begin{aligned} T_{tt}&=\sum_\alpha \Pi_{\alpha,t}\frac{d\phi_\alpha}{dt}-\mathfrak{L},\\ T_{tr}&=\sum_\alpha \Pi_{\alpha,r}\frac{d\phi_\alpha}{dt},\\ T_{rt}&=\sum_\alpha \Pi_{\alpha,t}\frac{d\phi_\alpha}{dx_r},\\ T_{rs}&=\sum_\alpha \Pi_{\alpha,s}\frac{d\phi_\alpha}{dx_r}-\mathfrak{L}\delta_{rs}.\end{aligned}\right\} \tag{6}$$

It is convenient to introduce the following symbols:

$$E=T_{tt},\quad S_r=T_{tr},\quad P_r=-T_{rt}; \tag{7}$$

E is the energy density, S_r the energy-flow (or "power") vector, P_r the momentum density, and $-T_{rs}$ the momentum flow tensor. We may verify from (5) that the following relations are satisfied

$$\frac{dE}{dt}+\sum_r\frac{dS_r}{dx_r}=-\frac{\partial\mathfrak{L}}{\partial t}, \tag{8}$$

$$\frac{dP_r}{dt}=\sum_s\frac{dT_{rs}}{dx_s}+\frac{\partial\mathfrak{L}}{\partial x_r}. \tag{9}$$

We see from (8) that if the Lagrangian function does not depend explicitly on time, energy is conserved; similarly, we see from (9) that if the Lagrangian function does not depend explicitly on any spatial coordinate, the corresponding component of momentum is conserved.

We now wish to consider wave propagation in such a continuous dynamical system. We suppose the system to be time-independent and uniform in one or more spatial coordinates. We may remove other coordinates from the problem by an appropriate normal-mode analysis. We now consider a wave-like solution of the equations, for which every dynamical variable is expressible as a function of the combination $\sum_r k_rx_r-\omega t$ of period 2π in this argument. In the particular case which is of interest to us (that the Lagrangian function is quadratic in its arguments), these periodic functions will be circular functions.

It has been shown elsewhere[11] that, for such a wave propagating in such a medium, the mean values of the energy density and momentum density are related to a quantity $2\pi J$, the "action density," in the following way

$$E=J\omega,\quad P_r=Jk_r. \tag{10}$$

The quantity J is obtained by introducing a phase angle κ into the expression for the wave function, for instance by replacing ωt by $\omega t+\kappa$, and then evaluating the expression

$$J=\frac{1}{2\pi}\int d\kappa\sum_\alpha \pi_{\alpha,t}\frac{\partial\phi_\alpha}{\partial\kappa}. \tag{11}$$

The relations (10) involving the wave energy density and momentum density, which are identical in form with the familiar relations of quantum mechanics between energy and frequency, momentum, and wave vector,[12] enable us to establish a sense in which slow waves carry negative energy.

III. SLOW WAVES AND NEGATIVE ENERGY

In order to obtain an appropriate generalization of the idea of "fast" and "slow" waves, we consider a *convected* propagating medium. From now on, we consider only one spatial coordinate z. We introduce primed quantities, such as z', for quantities referred to a frame of reference which is convected along with the medium. We retain unprimed quantities for measurements with respect to a fixed frame, and suppose that the medium is moving with velocity v in the z direction. Then the time and space coordinates of the two frames are related as follows

$$t=t',\quad z=z'+vt', \tag{12}$$

so that frequencies and wave numbers are related as follows

$$\omega=\omega'+vk',\quad k=k'. \tag{13}$$

We now consider the energy and momentum densities in the two frames. According to (10), the following relations should hold

$$\left.\begin{aligned} E&=J\omega, & P&=Jk,\\ E'&=J'\omega', & P'&=J'k'.\end{aligned}\right\} \tag{14}$$

The usual rules for transformation of a stress tensor on

[9] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1950), Sec. 11–2.
[10] L. Landau and E. Lifschitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company Inc., Reading, Massachusetts, 1942), Sec. 4–7.
[11] P. A. Sturrock, "Field-theory analogs of the Lagrange and Poincaré invariants," Microwave Laboratory Report M.L. 689 (Stanford University, Stanford, California, 1960).
[12] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Company, Inc., New York, 1955), p. 17.

going to a moving coordinate system [13] require that

$$P=P', \quad E=E'+vP'. \tag{15}$$

The relations (15) are indeed compatible with (13) and (14), and show that

$$J=J'. \tag{16}$$

If we denote by u the phase velocity of a wave, so that

$$u=\omega/k, \quad u'=\omega'/k', \tag{17}$$

the second of the relations (15) may be rewritten as

$$E=\frac{u}{u'}E'. \tag{18}$$

Now consider two waves, with the same wave number, which propagate in opposite directions with respect to the moving reference frame; then $u'>0$ for the "forward" wave and $u'<0$ for the "backward" wave. If $v>|u'|$, both waves appear to be going forward in the fixed frame of reference; one is a "fast wave," traveling faster than the convected medium, and the other is a "slow wave," which travels slower than the medium. It we assume that, when looked at from the comoving frame, the medium looks the same for a wave traveling to the right as for a wave traveling to the left, E' will have the same value for both waves, if they have the same amplitude. Hence we see that the energy of the fast wave E_f and the energy of the slow wave, E_s, will be given by

$$E_f=\frac{v+u'}{u'}E'. \quad E_s=-\frac{v-u'}{u'}E'. \tag{19}$$

We see that the fast and slow waves do indeed have energies of opposite signs with respect to the fixed coordinate system. It will frequently, but not invariably, be true that E' gives the correct expression for the physical energy density in the frame moving with the medium; in this case, E' must be positive. We then see from (19) that *the fast wave carries positive energy and the slow wave carries negative energy*. It is interesting to note from the second of relation (19) that if the convected velocity is not great enough to convert the backward wave of the moving frame into a forward wave of the fixed frame, then the slow wave (which is now a backward wave) has positive energy.

IV. DISCUSSION

It must be emphasized that the relations (10), which make it possible to assign negative energy to slow waves in a general way, hold for the energy and momentum of a wave as defined by the small-amplitude stress tensor. In the case that the exact equations for the system are linear, it is not in general true that the canonical stress tensor is identical with the physical stress tensor. Nevertheless, the mean values of these quantities are identical[11] under conditions which lead to the action relation (10); moreover, it may happen that certain contributions to the canonical stress tensor can be directly related to physically significant quantities—such as Poynting flux.

We see from (10) that evaluation of the small-amplitude energy and momentum densities is a simpler process than evaluation of the corresponding nonlinear quantities, since all components may be derived from the one quantity J. Formula (11) for J is usually simpler to evaluate than corresponding direct expressions for E and P_r. Indeed, we may write down simple expressions for the remaining terms S_r and T_{rs} of the stress tensor. If, as we are here assuming, $\partial\mathfrak{L}/\partial t$ and $\partial\mathfrak{L}/\partial x_r$ vanish, we may use the properties of group velocity[11,14] to establish from (8) and (9) the following relations

$$S_r=E\frac{\partial\omega}{\partial k_r}, \quad T_{rs}=-P_r\frac{\partial\omega}{\partial k_s}. \tag{20}$$

Hence, by combining (10) and (20), we may write down the following expressions relating the sixteen components of the stress tensor to the action density

$$\left.\begin{aligned} E&=J\omega & S_r&=J\omega\partial\omega/\partial k_r, \\ P_r&=Jk_r, & T_{rs}&=-Jk_r\partial\omega/\partial k_s. \end{aligned}\right\} \tag{21}$$

There are a few further points which should be noted concerning the relationship of the small-amplitude[11] stress tensor, the canonical stress tensor and the physical stress tensor. In setting up the canonical stress tensor for small-amplitude disturbances of electrodynamical systems,[3] the usual difficulty was found to arise, that formulas for components of the tensor were gauge-dependent. It was therefore expedient to modify the canonical tensor by adding a term which did not impair the conservation relations (8) and (9). The necessary transformation is of a type[11] which does not invalidate the relation (21).

The negative energy carried by slow space-charge waves explains the operation of traveling-wave tubes[5]; it also explains the difficulty of removing noise from the slow wave of an electron beam.[2] In looking for a mechanism for removing this noise, one might direct attention to the physical energy represented by the slow wave of a beam but this would be inadvisable. The problem of removing noise is simply the problem of coupling different types of electrodynamic systems, a problem which may be discussed by means of the linearized equations. Study of the small-amplitude energy theorem therefore provides information about this coupling problem; study of the physical stress tensor, on the other hand, provides information also about nonlinear effects of the wave equations and coupling mechanisms which are irrelevant to the problem of noise removal.

[13] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* 1 (McGraw-Hill Company, Inc., New York, 1953), pp. 98–100.

[14] L. Brillouin, *Wave Propagation in Periodic Structures* (McGraw-Hill Company, Inc., New York, 1946).

It has been noticed[15] that the formulas for kinetic power of an electron beam can yield the correct expression for the physical power lost by an electron beam in an electron tube, and it is interesting to see when and why this is so. Suppose that the beam enters the interaction region with power $S_{b,i}$ and that the input coupler introduces an electromagnetic field with power $S_{f,i}$; suppose also that the beam leaves the interaction region with power $S_{b,f}$ and that the output coupler removes field power $S_{f,f}$. The equation of conservation of energy requires that

$$S_{b,i}+S_{f,i}=S_{b,f}+S_{f,f}. \tag{22}$$

We first interpret (22) as relating the "physical" powers involved. However, we may set up an analogous relation between the powers assigned to the beam and field by the small-amplitude energy theorem

$$S_{b,i}'+S_{f,i}'=S_{b,f}'+S_{f,f}'. \tag{23}$$

(If the beam is initially unmodulated, $S_{b,i}'=0$.) In the usual statement of the small-amplitude energy theorem for electron tubes,[1–3] the expression for the power of an electromagnetic field alone gives correctly the physical power carried by this field; hence

$$S_{f,i}'=S_{f,i}, \quad S_{f,f}'=S_{f,f}. \tag{24}$$

We note that interaction terms in the expressions for energy flow do not appear in our equations since we are evaluating the power carried by various components of the system outside of the interaction region. We now see from (22), (23), and (24) that

$$S_{b,f}-S_{b,i}=S_{b,f}'-S_{b,i}', \tag{25}$$

which states that the physical power lost by the beam is equal to the power loss as evaluated by the small-amplitude power theorem. We can see that it is generally true that if an electrodynamic system interacts with an "external" field for a finite length of space or time, the small-amplitude formulas give correctly the loss of power or energy by this system.

It is interesting to return to consideration of transverse torsional waves in a moving medium. The analysis of Sec. III would lead us to assign negative canonical energy to slow waves in such a system. This is compatible with results of the experiment of Cutler, Chapman, and Mathews.[8] However, Pierce[7] has stated that "an analysis shows that when a torsional wave on a fixed rod is coupled purely by couples about the axes to the slow torsional wave on a parallel rod moving axially with the respect to the first, no gaining wave results." Pierce resolves the discrepancy between this statement and the experiment referred to by pointing out that the interaction mechanism in the experiment involves longitudinal forces. If, on the other hand, one looks for a resolution of this paradox within the framework of the small-amplitude energy theorem, one is led to conjecture that the mathematical model considered by Pierce was not a valid model of a dynamical system in that the equations were not derivable from a Lagrangian function.

In conclusion, let us consider briefly the theory of transverse-field electron tubes. It has been shown[16] that the motion of a filamentary electron beam in a longitudinal dc magnetic field may be analyzed into four waves. One pair of these waves, which Siegman terms cyclotron waves, is similar to space-charge waves in that one is "fast" and carries positive energy and the other is "slow" and carries negative energy. This is as we should expect. The other pair is termed "synchronous waves" since its phase velocity is equal to the dc beam velocity. Of these, it is found that one carries positive energy and the other negative energy, but it is not immediately obvious from our theory why this should be so.

The synchronous waves have the form of right-hand and left-hand helices convected with the beam velocity. Hence, in the comoving frame, these waves have zero frequency and hence zero canonical energy. Evaluation of energy in the laboratory coordinate system therefore turns upon evaluation of the momentum of the two waves, which will be the same in the laboratory system and in the comoving coordinate system. However, the presence of the magnetic field makes the medium anisotropic so that we cannot assert that the action densities of the two waves in the comoving frame should be equal if their amplitudes are equal. This anisotropy may be removed by going to the Larmor frame,[17] which rotates with half the cyclotron frequency. Hence the two waves, which were of the form

$$x+iy=re^{\pm ik(z-vt)} \tag{26}$$

in the original coordinate system, have the form

$$x'+iy'=e^{-i\omega_L t'\pm ikz'}, \tag{27}$$

in the comoving Larmor frame if the appropriate transformation is written as

$$x+iy=(x'+iy')e^{i\omega_L t}, \quad z=z'+vt', \quad t=t'. \tag{28}$$

If the field is so directed that ω_L is positive, the waves characterized by plus and minus signs may be termed "antirotating" and "corotating." We expect both waves to have energy of the same sign in the Larmor frame so that the action densities which we should assign to both waves have the same sign. The momentum in the comoving Larmor frame is $\pm Jk$; this is the same in the comoving nonrotating frame and in the stationary frame. Hence, from (26), the energy densities of the two waves in the stationary frame are $\pm Jvk$. If, as we should expect, $J>0$, we see that the antirotating wave has positive energy and the corotating wave has negative energy. This is in agreement with Siegman's analysis.

[15] M. Chodorow, private communication.

[16] A. E. Siegman, J. Appl. Phys. 31, 17–26 (1960).

[17] H. Goldstein, *Classical Mechanics* (McGraw-Hill Company, Inc., New York, 1950), section 5–8.

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Two-stream instability, wave energy, and the energy principle

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A generalized Poynting theorem for a system of uniform electron beams is obtained. Two examples of the two-stream instability with beams of equal density are used to discuss the relation between negative wave energy and negative potential energy, which arises in the energy principle of ideal magnetohydrodynamics. In the first example, $v_{10}>v_{20}$, while in the second example, $v_{20}=-v_{10}$, where $v_{10,20}$ are the equilibrium beam velocities. Both cases can be interpreted in terms of the energy density arising from the generalized Poynting theorem. The first instability is due to the coupling of negative and positive energy waves at a frequency $k(v_{10}+v_{20})/2$. The second instability is due to the coupling of the same two perturbations, but at zero frequency. In this case, there is no oscillatory (wave) energy, but the beam electrons still make a negative contribution to the total energy. [DOI: 10.1063/1.2768016]

I. INTRODUCTION

Two-stream instability is an example of a general class of instabilities that can occur in a conservative system. Such instabilities have been referred to as reactive.[1]

Two-stream instabilities and the related beam-plasma instabilities are, of course, very well known. However, they continue to be of interest in a variety of situations. For example, Startsev and Davidson[2] recently gave a part analytical, part numerical analysis of a two-stream instability for a longitudinally compressing charged particle beam. In another recent study, a Vlasov-Poisson simulation of electron beam interaction was also described by Silin, Sydora, and Sauer.[3]

Reactive instabilities can be understood through the use of conservation theorems,[4–6] which lead to the concept of negative energy waves. A few years after the introduction of this concept, the subject of ideal magnetohydrodynamic (MHD) stability theory was advanced by the discovery of the energy principle.[7] This principle requires the potential energy to be negative. A reactive instability occurs when two linear waves couple (or coalesce) at a critical frequency and where one of the modes has negative wave energy. On the other hand, for a stationary equilibrium, an ideal MHD instability occurs when two modes couple (coalesce) at zero frequency. For this case, there is no oscillatory (wave) energy, but the potential energy, as defined by the energy principle, must be negative. Although both negative wave energy and negative potential energy arise from conservation theorems, it is not clear what, if any, is the connection between the two.

The extension of the energy principle to nonstationary equilibria is relevant to contemporary problems, such as magnetic fusion, since equilibrium flows are widespread and of considerable interest. This extension is also relevant to other fields, for example solar physics, space physics, and astrophysics. Davidson[8] addressed this question for nondissipative flows in incompressible conservative systems that include ideal MHD.

In this paper, two-stream instability is revisited in order to illustrate a number of basic points concerning reactive instabilities. In addition, a link is noted between the concepts of a negative energy wave and negative potential energy, used in the analysis of ideal MHD instabilities. In conservative systems possessing free energy, the existence of negative energy waves is demanded by the appropriate conservation theorem.[5] Similarly, the ideal MHD energy principle for stationary equilibrium can be obtained from a generalized Poynting theorem.[9]

The reactive instabilities arising from the interaction between cold electron beams, although much simpler than ideal MHD instabilities of magnetically confined plasmas, allow some insight into the link between the two. For two beams, with equilibrium velocities v_{10} and v_{20} ($v_{10}>v_{20}$), two-stream instability occurs at a frequency $k(v_{10}+v_{20})/2$. The instability arises when the negative energy slow space-charge wave on the faster beam couples (coalesces) with the positive energy fast space-charge wave on the slower beam. For the special case in which the beams have equal and opposite velocities, $v_{10}+v_{20}=0$, the instability occurs at zero frequency, as in the case of ideal MHD.

In Sec. II, the generalized Poynting theorem for electrostatic fluctuations of a system of cold electron beams is obtained. This allows the identification to be made of the wave energy density for the unperturbed beam modes. It is demonstrated in Sec. III how the expression for the small signal energy is able to account for instability for both finite values of the frequency and for the special case in which Re $\omega=0$. In both cases, solutions of the dispersion relation are obtained that demonstrate explicitly the coalescence (or coupling) of the relevant beam modes. A summary and conclusions are given in Sec. IV.

II. THE GENERALIZED POYNTING THEOREM

Consider a system of j-electron beams, each of uniform density n_{j0} and velocity $v_{j0}=v_{j0}\mathbf{z}$, where $\mathbf{z}$ is the unit vector along the z axis. There is no equilibrium magnetic field. The analysis is restricted to one-dimensional, electrostatic perturbations to the uniform equilibrium. The linearized equations for the beams are

$$\left(\frac{\partial}{\partial t}+v_{jo}\frac{\partial}{\partial z}\right)v_{j1z}=-\frac{e}{m_e}E_{1z}, \tag{1}$$

$$\frac{\partial n_{j1z}}{\partial t}+\frac{\partial}{\partial z}(n_{j0}v_{j1z}+n_{j1}v_{j0})=0, \tag{2}$$

$$J_{1z}+\varepsilon_0\frac{\partial E_{1z}}{\partial t}=0, \tag{3}$$

$$\text{where } J_{1z}=-e\sum_j (n_{jo}v_{j1z}+n_{j1}v_{jo}). \tag{4}$$

Multiply Eq. (1) by $m_e n_{jo}v_{j1z}^*$ and the complex conjugate equation by $m_e n_{jo}v_{j1z}$ and add, where the * denotes the complex conjugate. Now, multiply Eq. (1) by $m_e n_{j1}^* v_{jo}$ and the complex conjugate equation by $m_e n_{j1} v_{jo}$ and again add. Combining the resulting pair of equations gives

$$\begin{aligned}&\frac{\partial}{\partial t}(n_{jo}m_e|v_{j1z}|^2)+\frac{\partial}{\partial z}(n_{jo}m_e v_{jo}|v_{j1z}|^2)+m_e v_{jo}^2 n_{j1}^*\frac{\partial v_{j1z}}{\partial z}\\&+m_e v_{jo}^2 n_{j1}\frac{\partial v_{j1z}^*}{\partial z}+m_e v_{jo}n_{j1}^*\frac{\partial v_{j1z}}{\partial t}+m_e v_{jo}n_{j1}\frac{\partial v_{j1z}^*}{\partial t}\\&=-e(n_{jo}v_{j1z}^*+n_{j1}^*v_{jo})E_{1z}-e(n_{jo}v_{j1z}+n_{j1}v_{jo})E_{1z}^*.\end{aligned} \tag{5}$$

Similarly, multiply Eq. (2) by $m_e v_{jo}v_{j1z}^*$ and its complex conjugate by $m_e v_{jo}v_{j1z}$ and add, giving

$$\begin{aligned}&m_e v_{jo}\left(v_{j1z}^*\frac{\partial n_{j1}}{\partial t}+v_{j1z}\frac{\partial n_{j1}^*}{\partial t}\right)+m_e v_{jo}\frac{\partial}{\partial z}(n_{jo}v_{j1z}v_{j1z}^*)\\&+m_e v_{jo}v_{j1z}^*\frac{\partial}{\partial z}(v_{jo}n_{j1})+m_e v_{jo}v_{j1z}\frac{\partial}{\partial z}(n_{j1}^*v_{jo})=0.\end{aligned} \tag{6}$$

Now use the relation

$$\frac{\partial}{\partial z}(v_{j1z}^*n_{j1})=v_{j1z}^*\frac{\partial n_{j1}}{\partial z}+n_{j1}\frac{\partial v_{j1z}^*}{\partial z} \tag{7}$$

and its complex conjugate, to obtain

$$\begin{aligned}v_{j1z}^*\frac{\partial n_{j1}}{\partial z}+v_{j1z}\frac{\partial n_{j1}^*}{\partial z}&=\frac{\partial}{\partial z}(v_{j1z}^*n_{j1}+v_{j1z}n_{j1}^*)-n_{j1}^*\frac{\partial v_{j1z}}{\partial z}\\&-n_{j1}\frac{\partial v_{j1z}^*}{\partial z}.\end{aligned} \tag{8}$$

Combining Eqs. (6) and (8),

$$\begin{aligned}&m_e v_{jo}\left(v_{j1z}^*\frac{\partial n_{j1}}{\partial z}+v_{j1z}\frac{\partial n_{j1}^*}{\partial t}\right)+m_e v_{jo}\frac{\partial}{\partial z}(n_{jo}v_{j1z}v_{j1z}^*)\\&+m_e v_{j0}^2\frac{\partial}{\partial z}(v_{j1z}^*n_{j1}+v_{j1z}n_{j1}^*)-m_e v_{jo}^2 n_{j1}\frac{\partial v_{j1z}^*}{\partial z}\\&-m_e v_{jo}^2 n_{j1}^*\frac{\partial v_{j1z}}{\partial z}=0.\end{aligned} \tag{9}$$

Now add Eqs. (5) and (9) giving

$$\begin{aligned}&\frac{\partial}{\partial t}[n_{jo}m_e|v_{j1z}|^2+m_e v_{jo}(n_{j1}^*v_{j1z}+n_{j1}v_{j1z}^*)]\\&+\frac{\partial}{\partial z}[2m_e n_{jo}v_{jo}|v_{j1z}|^2+m_e v_{jo}^2(v_{j1z}^*n_{j1}+v_{j1z}n_{j1}^*)]\\&=-e(n_{jo}v_{j1z}^*+n_{j1}^*v_{jo})E_{1z}-e(n_{jo}v_{j1z}+n_{j1}v_{jo})E_{1z}^*.\end{aligned} \tag{10}$$

Summing Eq. (10) over all the j species of electron beams and taking account of Eq. (4), Eq. (10) can be written

$$\begin{aligned}&\frac{\partial}{\partial t}\sum_j\left\{\frac{1}{2}n_{jo}m_e|v_{j1z}|^2+\frac{1}{2}m_e v_{jo}(v_{j1z}n_{j1}^*+v_{j1}^*n_{j1})\right\}\\&+\frac{\partial}{\partial z}\left\{\sum_j\left[m_e n_{jo}|v_{j1z}|^2+\frac{1}{2}m_e v_{jo}(v_{j1z}^*n_{j1}\right.\right.\\&\left.\left.+v_{j1z}n_{j1}^*)\right]v_{jo}\right\}=\frac{1}{2}(J_{1z}^*E_{1z}+J_{1z}E_{1z}^*).\end{aligned} \tag{11}$$

Multiplying Eq. (3) by E_{1z}^* and its complex conjugate equation by E_{1z} yields

$$\frac{1}{2}(J_{1z}E_{1z}^*+J_{1z}^*E_{1z})=-\frac{1}{2}\varepsilon_o\frac{\partial}{\partial t}|E_{1z}|^2. \tag{12}$$

The generalized Poynting theorem for one-dimensional perturbations to an equilibrium system consisting of j-electron beams follows from Eqs. (11) and (12),

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\sum_j\left[\frac{1}{2}n_{jo}m_e|v_{j1z}|^2+\frac{1}{2}m_e v_{jo}(n_{j1}^*v_{j1z}+n_{j1}v_{j1z}^*)\right]\right.\\&\left.+\frac{1}{2}\varepsilon_o|E_{1z}|^2\right\}+\frac{\partial}{\partial z}\left\{\sum_j\left[m_e n_{jo}|v_{j1z}|^2\right.\right.\\&\left.\left.+\frac{1}{2}m_e v_{jo}(v_{j1z}^*n_{j1}+v_{j1z}n_{j1}^*)\right]v_{jo}\right\}=0.\end{aligned} \tag{13}$$

The equation expresses the conservation of small signal energy for perturbations to the above system of electron beams. The first term is the time rate of change of the energy density and the second term is the energy flow. For electrostatic fluctuations and cold beams, the energy flow is due to the equilibrium beam velocities. Since the equilibrium is uniform, averaging over a period of oscillation (or a wavelength) gives

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\frac{1}{2}\varepsilon_o|E_{1z}|^2+\sum_j\left[\frac{1}{2}n_{jo}m_e|v_{j1z}|^2+\frac{1}{2}m_e v_{jo}(n_{j1}^*v_{j1z}\right.\right.\\&\left.\left.+n_{j1}v_{j1z}^*)\right]\right\}=0.\end{aligned} \tag{14}$$

Denoting the expression inside the curly brackets in Eq. (14) by ξ, the small signal energy density, the equation becomes

$$\frac{\partial \xi}{\partial t} = 0. \tag{15}$$

The constancy of ξ implies that instability can only occur when the system permits some of the contributions to ξ to be negative.[5]

III. SMALL SIGNAL ENERGY AND INSTABILITY

The physical content of Eq. (14) is illustrated in this section with two specific examples of two-stream instability. Assuming that all perturbed (small signal) quantities vary as $\exp i(kz-\omega t)$, Eqs. (1) and (2) give

$$v_{j1z} = \frac{iq_j}{m_j}\frac{E_{1z}}{(\omega - kv_{jo})}, \tag{16}$$

$$n_{ji} = \frac{ikn_{jo}q_jE_{1z}}{m_j(\omega - kv_{jo})^2}. \tag{17}$$

With the aid of Eqs. (16) and (17), ξ can be written as

$$\xi = \frac{1}{4}\varepsilon_o|E_{1z}|^2\left\{1 + \sum_j\left[\frac{\omega_{pj}^2}{(\omega - kv_{jo})^2} + \frac{2kv_{jo}\omega_{pj}^2}{(\omega - kv_{jo})^3}\right]\right\}. \tag{18}$$

The first example of two-stream instability to be considered is when both streams have the same density and stream 1 moves faster than stream 2,

Case a) $\omega_{p1} = \omega_{p2} = \omega_p$; $v_{10} > v_{20}$.

For simplicity, the subscript "0" on the equilibrium beam velocities is dropped so that $v_1 > v_2$. The dispersion relation for this case can be found in most plasma physics text books, see, e.g., Stix,[10]

$$1 - \frac{\omega_p^2}{(\omega - kv_1)^2} - \frac{\omega_p^2}{(\omega - kv_2)^2} = 0. \tag{19}$$

The unstable solutions for this equation are well known and are given, for example, in Ref. 10, where the solution for maximum growth is given as

$$\omega = \frac{k(v_1 + v_2)}{2} \pm \frac{i\omega_p}{2}. \tag{20}$$

In Ref. 10, it was pointed out that the stable solutions of Eq. (19) reduce to the fast and slow space-charge waves on the two beams, which propagate independently for large k. However, these waves also play a key role in the unstable range of wave numbers, as will now be demonstrated.

The result given in Eq. (20) can be obtained by recognizing that the space-charge waves on each beam will perturb each other. The dispersion relation given by Eq. (19) can be written as

$$(\omega - kv_1 - \omega_p)(\omega - kv_1 + \omega_p) \times(\omega - kv_2 - \omega_p)(\omega - kv_2 + \omega_p) = \omega_p^4. \tag{21}$$

This form is physically revealing, since it suggests that the instability can be described in terms of the coupling of the fast and slow space-charge waves carried by the two beams. The fast space-charge waves are given by $\omega=kv_{1,2}+\omega_p$ and the slow space-charge waves by[11] $\omega=kv_{1,2}-\omega_p$. By substituting the fast space-charge wave frequencies into Eq. (18), it can be seen that the energy density of the fast waves will always be positive since $\omega-kv_{1,2}>0$. On the other hand, substituting the slow space-charge wave frequencies into Eq. (18) shows that the slow waves carry negative energy, since $\omega-kv_{1,2}<0$.

Since $v_1>v_2$, instability can be expected when the slow wave on beam 1 couples to the fast wave on beam 2. The coupling condition is

$$kv_1 - \omega_p = kv_2 + \omega_p. \tag{22}$$

In order to demonstrate this, assume a solution of Eq. (21) of the form

$$\omega = kv_1 - \omega_p + \delta\omega. \tag{23}$$

Substituting Eq. (23) into Eq. (21), making use of the coupling condition, Eq. (22), and neglecting $\delta\omega$ in nonresonant terms, Eq. (21) reduces to

$$(\delta\omega)^2 \approx -\frac{\omega_p^2}{4}. \tag{24}$$

Hence, Eqs. (22)–(24) yield

$$\omega = \frac{k(v_1 + v_2)}{2} \pm \frac{i\omega_p}{2} \tag{25}$$

in agreement with the exact solution given by Eq. (20). The wave number at the threshold for instability can be obtained from the condition that $\xi=0$. Using the coupling condition, Eq. (22), the frequency $\omega=kv_1-\omega_p=kv_2+\omega_p$ is substituted into Eq. (18) to give

$$\xi = \frac{1}{4}\varepsilon_0|E_{1z}|^2\left[3 - \frac{2k(v_1 - v_2)}{\omega_p}\right]. \tag{26}$$

Hence, the wave number at the threshold for instability resulting from the quadratic approximation is $k=3\omega_p/2(v_1-v_2)$. For comparison, the exact value[10] is $k=2\sqrt{2}\omega_p/(v_1-v_2)$. The wave number corresponding to the maximum growth rate is $\sqrt{3}\omega_p/(v_1-v_2)$. The coupling condition, Eq. (22), which is expected to correspond to maximum growth, gives the value $k=2\omega_p/(v_1-v_2)$, which is in fair agreement.

Case b) $\omega_{p1} = \omega_{p2} = \omega_p$; $v_1 = v$, $v_2 = -v$.

This is a rather special case in which the two beams again have equal densities but equal and opposite velocities. The dispersion relation for this case is

$$1 - \frac{\omega_p^2}{(\omega - kv)^2} - \frac{\omega_p^2}{(\omega + kv)^2} = 0. \tag{27}$$

The solution can be obtained from the exact result given by Eq. (20) by putting $v_1=v$, $v_2=-v$, and is

$$\omega = \pm i\frac{\omega_p}{2}. \tag{28}$$

Note that for this special case, Re $\omega=0$, which is analogous to ideal MHD instability. This result can again be obtained in terms of the coupling of the space-charge waves. It is clear that for this case there can only be coalescence of roots, and therefore instability, at zero frequency. The coupling condition corresponding to Eq. (22) is now

$$kv - \omega_p = -kv + \omega_p. \tag{29}$$

Assuming

$$\omega = kv - \omega_p + \delta\omega \tag{30}$$

and making use of Eq. (29), Eq. (21) yields the result given in Eq. (28).

Although there is no oscillatory energy for this case, since Re $\omega=0$, the energy expression given by Eq. (18) still holds and must take the value $\xi=0$, at the threshold for instability. Substituting $v_1=v$, $v_2=-v$ in Eq. (18), it can be seen that there are now two negative contributions to ξ,

$$\xi = \frac{1}{4}\varepsilon_o|E_{1z}|^2\left\{1 + \frac{\omega_p^2}{(\omega-kv)^2} + \frac{2kv\omega_p^2}{(\omega-kv)^3} + \frac{\omega_p^2}{(\omega+kv)^2} - \frac{2kv\omega_p^2}{(\omega+kv)^3}\right\}. \tag{31}$$

Imposing the threshold condition $\omega=0$, Eq. (31) yields

$$\xi_{th} = \frac{1}{4}\varepsilon_o|E_{1z}|^2\left(1 - \frac{2\omega_p^2}{k^2v^2}\right). \tag{32}$$

This result gives $k=\sqrt{2}\omega_p/v$ as the wave number at the instability threshold, which is in agreement with the value obtained from the solution of the dispersion relation, given by Eq. (27). The wave number corresponding to maximum growth is given by the exact solution of the dispersion relation[10] and is $k=\sqrt{3}\omega_p/2v$. For comparison, the resonance condition [Eq. (29)] gives ω_p/v, in reasonable agreement. It should be emphasized that the negative contributions to ξ in Eq. (31) still relate to the kinetic energy of the beam electrons. However, at zero frequency these contributions are equivalent to $\delta W<0$ in the case of ideal MHD instability.

IV. SUMMARY AND CONCLUSIONS

The generalized Poynting theorem for a system of cold, uniform electron beams with no equilibrium magnetic field has been derived. This provides the basis for a discussion of reactive instabilities, which result from the coalescence of two roots of the linear dispersion relation. The well known two-stream instability suggests a possible connection between negative small signal energy and negative potential energy ($\delta W<0$) characteristic of ideal MHD instability, where Re $\omega=0$, and for which there is no oscillatory energy.

A reactive instability occurs when two wave modes of the linear system couple at a critical frequency, where one of the modes carries positive energy and the other negative energy. Ideal MHD instabilities are a special case of reactive instability, and occur when two linear modes couple at zero frequency for a stationary equilibrium. The ideal MHD energy principle is restricted to stationary plasmas. Since equilibrium flows are common to many situations, it would be useful to have a corresponding result to the energy principle for nonstationary plasmas.

Two-stream instability has been discussed for two cases. In the first, two beams of equal density have drift speeds v_1 and v_2 with $v_1>v_2$. Instability is shown to occur when the negative energy slow space-charge wave carried by the faster beam couples with the positive energy fast space-charge wave carried by the slower beam. The requirement that the frequencies of the two modes should be equal gives the coupling (coalescence) condition, which allows a quadratic approximation to the dispersion relation, yielding the exact values of the frequency and maximum growth rate. A value for threshold wave number for the instability is obtained from the condition that the total energy is zero at the critical (coupling) frequency. This is compared with the exact result.

In the second case, the two beams of equal density have equal and opposite drift speeds $v_1=v$ and $v_2=-v$. Coalescence can only occur in this case at zero frequency, analogous to an ideal MHD instability where there is no oscillatory energy but where $\delta W<0$. For the second example, it is again shown how the coupling of two beam modes reduces the dispersion relation to a quadratic approximation, which nevertheless yields the exact growth rate. The expression for the nonoscillatory energy density yields the threshold wave number from the condition that the total energy is zero at threshold. Again, the wave number for maximum growth is compared with the wave number obtained from the coupling condition.

It is worth noting an interesting distinction between the two cases. For the finite frequency instability ($v_1>v_2$), there is only one negative contribution to the total energy, which comes, of course, from the negative energy wave on the faster beam. On the other hand, for the case with equal and opposite flows ($v_1+v_2=0$), both beams give negative contributions to the total energy.

Since the generalized Poynting theorem leads to a consistent interpretation of both oscillatory and nonoscillatory reactive instabilities for a simple two-beam example, it is suggested that the corresponding generalized Poynting theorem for ideal MHD for a nonstationary plasma would yield useful information through the identification of negative energy waves. For nonstationary equilibria, ideal MHD instabilities will occur at finite frequencies. It should be emphasized that the analysis given in this paper applies to the case in which the beam velocity is much greater than the thermal spread of the beam.

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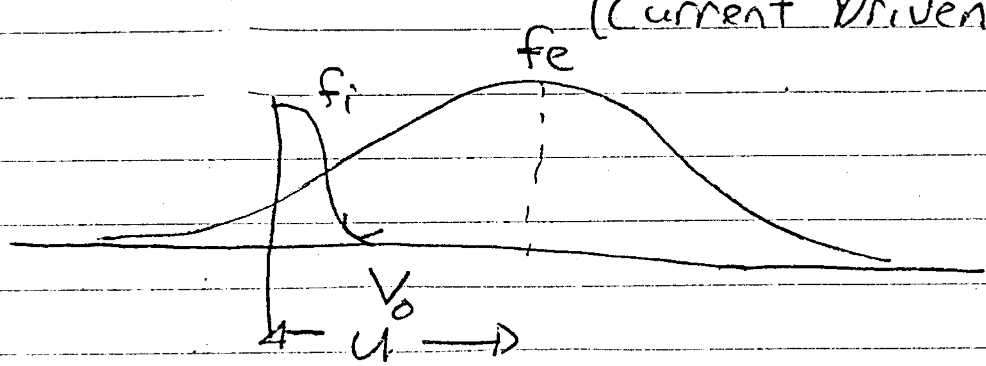
→ Vlasov Theory of Instability

→ $\partial \langle f \rangle / \partial v > 0 \Rightarrow$ growth!

→ Classic example: Ion Acoustic Instability
(Current Driven)



→ electrons: shifted Maxwellian / carry current

→ ions: stationary Maxwellian

Key Point: - unstable waves where $f_e' > 0$
yet $|f_i'|$ small $(Q \sim \varrho^2)$

- wave taps electron free energy
(due current!), yet avoids
ion dissipation

$\therefore \quad V_{T_i} < \frac{\omega}{k} < u \quad \Rightarrow$ range unstable
waves

- calculate:

$$\nabla^2\tilde{\phi} = -4\pi n_0 q\left(\int \tilde{f}_i\, dv - \int \tilde{f}_e\, dv\right)$$

For f_e, f_i:

$(q = -|e|)$

$$\frac{\partial \tilde{f}_e}{\partial t} + v\frac{\partial \tilde{f}_e}{\partial x} = -\frac{q}{m_e}\hat{E}\frac{\partial \langle f_e\rangle}{\partial v}$$

Now, as $kv > \omega$ → electrons adiabatic

$$\left[\text{For Maxwellian: } v\frac{\partial \tilde{f}}{\partial x} = +\frac{q}{m_e}\frac{\partial\tilde{\phi}}{\partial x}\frac{-v}{v_{Th}^2}\langle f\rangle \qquad \tilde{f} = -\frac{q\tilde{\phi}\langle f\rangle}{T_e}\right]$$

so take $\tilde{f}_e = \underbrace{-\frac{q\tilde{\phi}}{T_e}\langle f\rangle}_{\text{Boltzmann Response}} + \underbrace{\tilde{g}}_{\text{perturbation (non-adiabatic)}}$

$$\frac{\partial}{\partial t}\left(-\frac{q\tilde{\phi}}{T_e}\langle f\rangle + \tilde{g}\right) + v\left(\frac{-q}{T_e}\frac{\partial\tilde{\phi}}{\partial x}\right)\langle f\rangle + v\frac{\partial\tilde{g}}{\partial x}$$

$$= +\frac{q}{m_e}\left(\frac{\partial\tilde{\phi}}{\partial x}\right) - \frac{(v-u_0)}{v_{Th}^2}\langle f\rangle$$

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)\tilde{g} = u_0\frac{\partial}{\partial x}\frac{q\tilde{\phi}}{T_e}\langle f\rangle + \frac{\partial}{\partial t}\frac{q\tilde{\phi}}{T_e}\langle f\rangle$$

$$\Rightarrow$$

$$-i(\omega-kv)\,\tilde{g}_{k,\omega} = i(ku_0-\omega)\,\frac{q\tilde{\phi}_{k\omega}}{T_e}\langle f\rangle$$

$$\tilde{g}_{k,\omega} = -\frac{(ku_0-\omega)}{(\omega-kv)}\,\frac{q\tilde{\phi}_{k,\omega}}{T_e}\langle f\rangle$$

so

$$\frac{\tilde{n}_e}{n_0} = -\frac{q\tilde{\phi}}{T_e}\left[1+\int dv\,\langle f\rangle\,\frac{(ku_0-\omega)}{(\omega-kv)}\right]$$

$$= -\frac{q\tilde{\phi}}{T_e}\left[1+\int dv\,\langle f\rangle\,(ku_0-\omega)\left(\frac{P}{\omega-kv}-i\pi\delta(\omega-kv)\right)\right]$$

$$= -\frac{q\tilde{\phi}}{T_e}\left[1-\frac{i\pi}{|k|v_{the}}(ku_0-\omega)\,\langle f\rangle\Big|_{\frac{\omega}{k}}\right]$$

$q=-|e|$

$(\omega < ku_0)$

- → lowest order contribution to P vanishes
- → sign {dissipative / non-adiabatic} part varies with $k\left(u_0-\frac{\omega}{k}\right)$

For $\tilde{n}_i$: $\omega > k v_{Th_i}$ $\qquad q = |e|$

$$-i(\omega - kv)\tilde{f}_{i_{k,\omega}} = \frac{-q}{m_i} E_{k\omega} \frac{\partial \langle f \rangle}{\partial v}$$

$$\frac{\tilde{n}_i}{n_0} = \frac{q}{m_i}\left[\int \langle f \rangle \frac{kv}{v_{Th}\,\omega}\left(1 + \frac{kv}{\omega} + \cdots\right) - i\pi \frac{k}{|k|}\frac{(\omega/k)}{v_{Th_i}^2}\langle f_i \rangle\Big|_{\omega/k}\right]\tilde{\phi}_{k\omega}$$

$$\frac{\tilde{n}_i}{n_0} = \frac{q}{m_i}\frac{k^2 \cancel{v_{Ti}^2}}{\omega^2 \cancel{v_{Ti}^2}}\tilde{\phi}_{k,\omega} - i\pi \frac{q}{m_i}\frac{\omega}{|k| v_{Ti}\, v_{Ti}^2}\langle f_i \rangle_{\omega/k}\,\tilde{\phi}_{k,\omega}$$

$$= \frac{k^2 c_s^2}{\omega^2}\left(\frac{q\tilde{\phi}_{k,\omega}}{T_e}\right) - i\pi \frac{T_e}{T_i}\frac{\omega}{|k| v_{Ti}}\langle f_i \rangle_{\omega/k}\left(\frac{q\tilde{\phi}_{k,\omega}}{T_e}\right)$$

$\to$ reactive piece is ion-acoustic wave response

$\to$ dissipative piece vanishes as $T_i \to 0$

$$k^2 \lambda_{De}^2 \ll 1 \Rightarrow \frac{\tilde{n}_i}{n_0} = \frac{\tilde{n}_e}{n_0}$$

$$\cancel{\frac{+|e|}{T_e}}\,\Phi_{k,\omega}\left[1-\frac{i\pi}{|k|v_{The}}(k u_0-\omega)\,\langle \bar{f}_e\rangle\Big|_{\omega/k}\right]$$

$$=\left[\frac{k^2 c_s^2}{\omega^2}-i\pi\,\frac{T_e}{T_i}\,\frac{\omega}{|k|v_{Thi}}\,\langle \bar{f}_i\rangle\Big|_{\omega/k}\right]\cancel{\frac{e\Phi_{k,\omega}}{T_e}}$$

$$\frac{k^2 c_s^2}{\omega^2}=1-i\pi\underbrace{\left(\frac{(k u_0-\omega)}{|k|v_{Te}}\langle \bar{f}_e\rangle\Big| -\frac{T_e}{T_i}\frac{\omega}{|k|v_{Ti}}\langle \bar{f}_i\rangle\Big|\right)}_{\delta}$$

$$\omega^2(1-i\delta)=k^2 c_s^2$$

$$\omega^2=k^2 c_s^2(1+i\delta)/1+\delta^2$$

$\Rightarrow$ $\delta<1$:

$$\omega^2=k^2 c_s^2(1+i\delta)$$

$$\omega=k c_s\left(1+\frac{i\delta}{2}\right)$$

$$\omega=k c_s+\frac{i\pi}{2}\left[\frac{k(u_0-c_s)}{|k|v_{Te}}\langle \bar{f}_e\rangle\Big|_{c_s}-\frac{T_e}{T_i}\frac{k c_s}{|k|v_{Ti}}\langle \bar{f}_i\rangle\Big|_{c_s}\right]$$

so, need:

→ $u_0 > C_s$ for instability (i.e. availability free energy)

→ $T_e \gg T_i$ to avoid ion - Landau Damping

i.e.

$$\omega = kC_s + \frac{i\pi}{2}\left[\frac{k}{|k|}\left(u_0/v_{Te} - \sqrt{m_e/m_i}\right)\exp\left[-\left(\frac{C_s-u_0}{v_{Te}}\right)^2\right] - \frac{k}{|k|}\left(\frac{T_e}{T_i}\right)^{3/2} e^{-T_e/T_i}\right]$$

- instability taps free energy in electron current

- $u_0 > C_s \Rightarrow (ku_0 - \omega) > 0$

$$\underset{\substack{\downarrow \\ \text{energy released}}}{ku_0} > \underset{\substack{\downarrow \\ \text{heating}}}{kv_{Res.}}$$

i.e. shift must make $\left.\langle f_e\rangle'\right|_{\omega/k} > 0$

- dynamically → current slowed
$\langle f_e\rangle$ plateaus

→ ions heated