Supplement I:

Variational Theory of Wave Kinetics

(Whitham)

→ Where does N ⟺ Action emerge from?

→ adiabatic invariance ⟺ conservation law

Noether's Thm? → symmetry!

Phase symmetry of wave train underlies action conservation and wave kinetics

→ derivation of wave kinetics via variational principle.

→ Variational Theory (Whitham)

Now, consider system, like ideal MHD, which can be described in terms of a displacement $\underline{\xi}$ such that

$$\underline{\xi} = \text{re}\left\{ A e^{i\phi} + A^* e^{-i\phi} \right\}$$

then relevant wave equation can be derived from:

$$\delta S = \delta \int dt \int d\underline{x} \; \mathcal{L}(\underline{\xi}) = 0$$

↳ Lagrangian density

Now, if write Lagrangian density in terms phase ϕ and amplitude a, have:

$$S = \int dt \int d\underline{x} \; \mathcal{L}(-\phi_t, \phi_{\underline{x}}, a)$$

where $\omega = -\phi_t = \partial\phi/\partial t$

$$\underline{k} = \phi_{\underline{x}} = \nabla\phi$$

→ this neglects <u>all</u> <u>corrections</u> to eikonal theory (WKB) i.e. all corrections to $\underline{k}$, ω, amplitude, etc.

→ L, above, corresponds to period averaged Lagrangian — ϕ indeterminate to constant

$\Rightarrow$ $\delta S \equiv \delta \int dt \int dx\, \mathcal{L}(-\phi_t, \phi_x, a)$

so have 2 variational equations:

1) $\delta S / \delta a = 0$

2) $\delta S / \delta \phi = 0$

Now, within scope of linear theory

$$\mathcal{L} = G(\omega, k)\, a^2$$

i.e. for MHD, can write:

$$\mathcal{L} = \frac{1}{2} \rho \dot{\underline{\xi}}^2 - \frac{1}{2} \rho \left[D(\underline{k}, \underline{x}, t) \right] \underline{\xi}^2$$

↳ ~ eikonal form of potential energy (from stiffness matrix)

and if $\underline{\xi} = \underline{A} e^{+i\phi} + \underline{A}^* e^{-i\phi}$

$$\mathcal{L} = \frac{1}{2} \rho \left(\frac{\partial \phi}{\partial t} \right)^2 |A|^2 - \frac{1}{2} \left[D(\nabla \phi, \underline{x}, t) \right] |A|^2$$

is concrete form of avg. Lagrangian.

so $G(\omega, k) = \frac{1}{2}\rho\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left[D(\nabla\phi, \underline{x}, t)\right]\right]|A|^2$

Now, i) $\Rightarrow$ $\frac{\partial S}{\partial a} = 0$

$$\Rightarrow \boxed{G(\omega, k) = 0}$$

but: $G = \omega^2 - \left[D(\underline{k}, \underline{x}, t)\right]^2 = 0$

is just dispersion relation!

ii) $\Rightarrow$ $\delta S/\delta\phi = 0$

so

$$\delta S = \int dt \int dx \left\{ \frac{\partial \mathcal{L}}{\partial(-\phi_t)} \delta(-\phi_t) + \frac{\partial \mathcal{L}}{\partial(\phi_{\underline{x}})} \delta\phi_{\underline{x}} \right\}$$

$$= \int dt \int dx \left\{ \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(-\phi_t)}\right) - \frac{\partial}{\partial \underline{x}} \cdot \left(\frac{\partial \mathcal{L}}{\partial \phi_{\underline{x}}}\right) \right\} \delta\phi$$

$\Rightarrow$ $\delta S = 0$ $\Rightarrow$

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \underline{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0}$$

so have: $\mathcal{L} = G(\omega, \underline{k})\, a^2$

$$\Rightarrow \quad G(\omega, \underline{k}) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

Now $\quad G(\omega, \underline{k}) = 0$

$$\Rightarrow \quad \frac{\partial G}{\partial \omega} d\omega + \frac{\partial G}{\partial \underline{k}} d\underline{k} = 0$$

$$\therefore \quad \underline{V}_{gr} = \frac{d\omega}{d\underline{k}} = -\frac{\partial G/\partial \underline{k}}{\partial G/\partial \omega}$$

$\left(\text{i.e. } G(k, \omega) = 0 \quad dG = 0 = \frac{\partial G}{\partial \omega} d\omega + \frac{\partial G}{\partial \underline{k}} \cdot d\underline{k}\right)$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{\partial G}{\partial \omega} a^2\right) + \underline{\nabla} \cdot \left[-\frac{\partial G/\partial \underline{k}}{\partial G/\partial \omega} \frac{\partial G}{\partial \omega} a^2\right] = 0$$

and so $\quad N \equiv \frac{\partial G}{\partial \omega} a^2$

$$\frac{\partial N}{\partial t} + \nabla \cdot \left(\underline{V}_{gr} N\right) = 0$$

(N not yet action ....)

→ Now, can further note for G invariant to time trans.

⇒ energy is conserved.

so, ∃ energy conservation equation (from Noether's Thm ⟺ symmetry).

Now, note have:

can proceed by working with:

$$\partial \mathcal{L}/\partial a = 0 \quad \Rightarrow \quad G(\omega, \underline{k}) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \underline{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}} \qquad \left(\frac{\partial}{\partial t}\frac{\partial \phi}{\partial \underline{x}} = \frac{\partial}{\partial \underline{x}}\frac{\partial \phi}{\partial t}\right)$$

$$\underline{\nabla} \times \underline{k} = 0 \qquad (\underline{k} = \underline{\nabla}\phi)$$

⇒ have $\frac{\partial}{\partial t}(\omega \mathcal{L}_\omega - \mathcal{L}) + \nabla\cdot\left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$

check: $\omega \frac{\partial \mathcal{L}_\omega}{\partial t} - \frac{\partial \mathcal{L}}{\partial t} + \underline{\nabla}\cdot\left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}}\right) \cdots + \frac{\partial \mathcal{L}}{\partial \omega}\frac{\partial \omega}{\partial t}$

$\overset{?}{=} 0$

$$= \frac{\partial \mathcal{L}}{\partial \omega}\frac{\partial \omega}{\partial t} + \omega\,\nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) - \nabla\cdot\left(\omega\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) - \frac{\partial \mathcal{L}}{\partial t}$$

$$= \cancel{\omega\,\underline{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right)} - \cancel{\omega\,\underline{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right)} - \frac{\partial \mathcal{L}}{\partial \underline{k}}\cdot\underline{\nabla}\omega - \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \omega}\frac{\partial \omega}{\partial t}$$

$$= +\frac{\partial \mathcal{L}}{\partial \underline{k}}\cdot\frac{\partial \underline{k}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \omega}\frac{\partial \omega}{\partial t} - \frac{\partial \mathcal{L}}{\partial t}$$

$= 0$ ✓.

constructed form of energy eqn:

$$\frac{\partial}{\partial t}\left(\omega\frac{\partial \mathcal{L}}{\partial \omega} - \mathcal{L}\right) + \underline{\nabla}\cdot\left(-\omega\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

but $\mathcal{L} = 0$, so energy density of wave is

$(G(\omega, \underline{k}) = 0)$

$$\boxed{\mathcal{E} = \omega\frac{\partial \mathcal{L}}{\partial \omega}}$$

above states energy conservation

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \omega} = \frac{\mathcal{E}}{\omega} \equiv \text{Action density!}}$$

$\Rightarrow$ <u>now</u> N is action density!

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\partial \epsilon}{\partial \omega} a^2\right) + \underline{\nabla} \cdot \left[-\frac{\partial \epsilon / \partial \underline{k}}{\partial \epsilon / \partial \omega} \frac{\partial \epsilon}{\partial \omega} a^2 \right] = 0\\ \Rightarrow\ &\frac{\partial N}{\partial t} + \underline{\nabla} \cdot \left[\underline{V}_{gr} N \right] = 0\end{aligned}}$$

$$\omega N = \mathcal{E}$$

Now note if write Vlasov-like equation:

$$\frac{\partial N}{\partial t} + \underline{V}_{gr} \cdot \underline{\nabla} N - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

Liouville $\Rightarrow$

$$\frac{\partial N(\underline{k}, \underline{x}, t)}{\partial t} + \nabla \cdot \left[\underline{V}_{gr} N \right] + \frac{\partial}{\partial \underline{k}} \cdot \left[-\frac{\partial \omega}{\partial \underline{x}} N \right] = 0$$

and $\int d\underline{k}$, with assumption of narrow spread in $\underline{k}$
for N

$\Rightarrow$

$$\frac{\partial}{\partial t} N(\underline{x}, t) + \underline{\nabla} \cdot \left[\underline{V}_{gr} N \right] = 0$$

Recall:

→ Hamiltonian structure of eikonal theory, etc. ⇒

$$\frac{\partial}{\partial t}\rho(\underline{k},\underline{x},t) + \underline{v}_{gr}\cdot\underline{\nabla}\,\rho(\underline{k},\underline{x},t) - \frac{\partial\omega}{\partial\underline{x}}\cdot\underline{\nabla}_k\,\rho(\underline{k},\underline{x},t) = 0$$

→ Physical arguments suggest $\rho = \frac{\mathcal{E}}{\omega} = N$

wave action density

→ Variational Approach

$$S = \int dt \int d^3x\, \mathcal{L} \quad , \qquad \mathcal{L} = G(\omega,\underline{k})\,a^2$$

$$\omega = -\partial\phi/\partial t = -\phi_t$$

$$\delta S = 0 \qquad \underline{k} = \underline{\nabla}\phi = \phi_{\underline{x}}$$

but two parameters varied $\left\langle \begin{matrix} a \\ \phi \end{matrix} \right.$

$\delta S/\delta a = 0 \quad \Rightarrow \quad G(\omega,\underline{k}) = 0 \quad \rightarrow$ dispersion relation

$$\delta S/\delta\phi = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\omega}\right) - \frac{\partial}{\partial\underline{x}}\cdot\left(\frac{\partial\mathcal{L}}{\partial\underline{k}}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial G}{\partial\omega}a^2\right) - \frac{\partial}{\partial\underline{x}}\cdot\left(\frac{\partial G}{\partial\underline{k}}a^2\right) = 0$$

and time translation symmetry and $G = 0$ ⇒

$$\mathcal{E} = \omega \frac{\partial \epsilon}{\partial \omega} a^2 \Rightarrow N = \frac{\mathcal{E}}{\omega} = \frac{\partial \epsilon}{\partial \omega} a^2$$

and $$\frac{\partial \epsilon}{\partial \underline{k}} a^2 = \underline{V}_{gr} N$$

→ Helpful Reminder:

recall, for electrostatic plasma waves

if $\epsilon(\omega, \underline{k}) = 0 \Rightarrow$ dispersion relation

then $$\mathcal{E}_k = \frac{\partial}{\partial \omega}(\omega \epsilon)\Big|_{\omega_k} \frac{|E_k|^2}{8\pi}$$

$$= \omega_k \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega_k} \frac{|E_k|^2}{8\pi}$$ → wave energy density

$\therefore$ $$N_k = \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega_k} \frac{|E_k|^2}{8\pi}$$

and $$\underline{\Gamma}_k = -\frac{\partial \epsilon}{\partial \underline{k}}\Big|_{\omega_k} \frac{|E_k|^2}{8\pi}$$ → wave energy density flux

$$= \underline{V}_{gr} N_k$$

since $G(\underline{k}, \omega) = 0$, so along rays

$$dG = d\omega \frac{\partial G}{\partial \omega} + d\underline{k} \cdot \frac{\partial G}{\partial \underline{k}} = 0$$

$$d\omega/d\underline{k} = -\left(\partial G/\partial \underline{k}\right)/\left(\partial G/\partial \omega\right)$$

etc.

so have Vlasov-like eqn. in $\underline{x}, \underline{k}$ phase space

$$\frac{\partial N}{\partial t} + \underline{v}_g \cdot \frac{\partial N}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

and continuity-type eqn. in $\underline{x}$ space:

$$\frac{\partial N}{\partial t} + \underline{\nabla} \cdot [\underline{v}_{gr} N] = 0$$

Observe:

- → order of derivatives matters, but Liouville helps
- → continuity-type eqn. for packets
- → useful to note that total derivative of $\underline{k}$, following packet

$$\frac{d\underline{k}}{dt} = \frac{\partial \underline{k}}{\partial t} + \underline{v}_{gr} \cdot \frac{\partial \underline{k}}{\partial \underline{x}}$$

$$= -\left(\frac{\partial \omega}{\partial \underline{x}}\right) + \underline{v}_{gr} \cdot \frac{\partial \underline{k}}{\partial \underline{x}}$$

$\left\{ \begin{array}{l} \text{if } \omega = \Omega(\underline{k}, \underline{x}, t) \\ \text{from } \epsilon = 0 \end{array} \right.$

$$= -\frac{\partial \Omega}{\partial \underline{k}} \cdot \frac{\partial \underline{k}}{\partial \underline{x}} - \frac{\partial \Omega}{\partial \underline{x}} + \underline{v}_{gr} \cdot \frac{\partial \underline{k}}{\partial \underline{x}} = -\frac{\partial \Omega}{\partial \underline{x}}$$

$$= -(\partial \omega / \partial x)_{\underline{k}}$$

no conflict with $\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}}$