Topic I → $\left\{\begin{array}{l}\text{Thermal Equilibrium} \\ \text{Plasma}\end{array}\right.$ 6.

(I.)

→ Thermal Equilibrium Plasma

- simplest possible dynamics question

⇒ What is spectrum of thermal equilibrium fluctuations in plasma?

- answer → determined by balance between

→ emission and absorption

≡

→ Fluctuation ⇌ Dissipation

What is key physics of each?

⇒ Physics of F.D.T.

Generic Consideration

Consider some simple examples:

- particle undergoing Brownian force in fluid

particle in fluid at temp T

$$m\frac{d\underline{v}}{dt} = \underbrace{-\gamma m \underline{v}}_{\text{Stokes drag}} + \underbrace{\tilde{\underline{f}}}_{\text{thermal fluctuations}}$$

- $\tilde{f}$ → random (statistical) so uncorrelated in time

→ correlation time

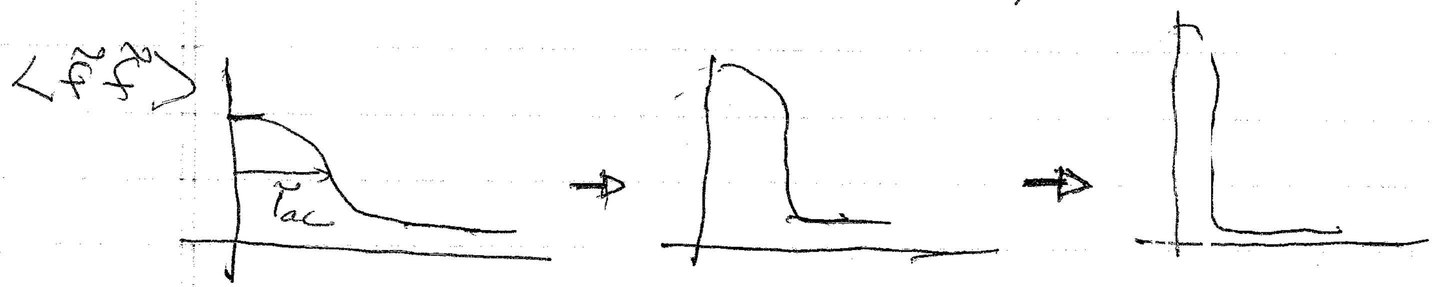
$$\langle \tilde{f}(t_1)\,\tilde{f}(t_2) \rangle = 2\tilde{f}_0^{\,2}\,\tau_c\,\delta(t_1 - t_2)$$

What is τ_c ? → spectral auto-correlation time
(self-coherence)

→ measures self-correlation of random force.

i.e. if stationary,

$$\langle \tilde{f}(0)\tilde{f}(\tau)\rangle = \langle \tilde{f}(t_1)\tilde{f}(t_2)\rangle$$



⇒ for "white noise"
$\tau \ll$ all other time scales

now,

$$\frac{d\tilde{v}}{dt} + \gamma \tilde{v} = \frac{\tilde{f}}{m}$$

$$\tilde{v}(t) = e^{-\gamma t}\tilde{v}(0) + \int_0^t dt' \, e^{-\gamma(t-t')}\frac{\tilde{f}(t')}{m}$$

Now $|\tilde{v}|^2 = e^{-2\gamma t}|\tilde{v}(0)|^2 + \cancel{\langle \tilde{v}\tilde{f}\rangle}$ cross terms

$$+ \int_0^t dt' \, e^{-\gamma(t-t')}\frac{\tilde{f}(t')}{m}\int_0^t dt'' \, e^{-\gamma(t-t'')}\frac{\tilde{f}(t'')}{m}$$

$$|\tilde{v}|^2 = e^{-2\gamma t}|\tilde{v}(0)|^2 + \int_0^t dt' \int_0^t dt'' \, e^{-\gamma(t-t')} e^{-\gamma(t-t'')} \frac{\tilde{f}^*(t')\,\tilde{f}(t'')}{m^2}$$

$$\langle |\tilde{v}|^2 \rangle = e^{-2\gamma t}\langle |\tilde{v}(0)|^2 \rangle + \int_0^t dt' \int_0^t dt'' \, e^{-\gamma(t-t')} e^{-\gamma(t-t'')} \frac{\langle \tilde{f}^*(t')\,\tilde{f}(t'')\rangle}{m^2}$$

ensemble, statistical average

$$= e^{-2\gamma t}\langle |\tilde{v}(0)|^2 \rangle + \int_0^t dt' \int_0^t dt'' \, e^{-\gamma(t-t'')} e^{-\gamma(t-t')} \frac{2\tilde{f}_0^2\tau_c}{m^2}\delta(t'-t'')$$

(2 for symmetrization)

$$= e^{-2\gamma t}\langle |\tilde{v}(0)|^2 \rangle + \int_0^t dt' \, e^{-2\gamma(t-t')} \frac{2\tilde{f}_0^2\tau_c}{m^2}$$

$$= e^{-2\gamma t}\langle |\tilde{v}(0)|^2 \rangle + e^{-2\gamma t}\frac{2\tilde{f}_0^2\tau_c}{m^2}\frac{1}{2\gamma}\left(e^{2\gamma t}-1\right)$$

$$= e^{-2\gamma t}\langle |\tilde{v}(0)|^2 \rangle + \frac{\cancel{2}\tilde{f}_0^2\tau_c}{\cancel{2}\gamma m^2}\left(1-e^{-2\gamma t}\right)$$

so for t large ($\gamma t \gg 1$)

$$\boxed{\langle \tilde{V}^2 \rangle \cong \frac{\tilde{f}_0^2 \tau_c}{\gamma m^2}}$$

but $\quad m \frac{\langle \tilde{V}^2 \rangle}{2} \equiv T \quad \rightarrow$ bath of T!

$$\Rightarrow \qquad T \cong \frac{\tilde{f}_0^2 \tau_c}{2 \gamma m}$$

$$\boxed{\frac{\tilde{f}_0^2 \tau_c}{m^2} = \gamma \frac{T}{m}}$$

$\rightarrow$ Fluctuation - dissipation theorem.

i.e. $\rightarrow$ given
- noise $(\tilde{f}_0^2 \tau_c)$
- damping (γ)
- temperature (T)

must have:

$$\boxed{(\text{noise}) = (\text{damping})\, T}$$

$\rightarrow$ given 2 of 3 $\Rightarrow$ deduce third!

$$\frac{\partial}{\partial t}\tilde{v} + \gamma\tilde{v} = \frac{\tilde{f}}{m}$$

$\tilde{v}\,\times$

stationarity

$$\cancel{\frac{\partial}{\partial t}\frac{\langle\tilde{v}^2\rangle}{2}} + \gamma\langle\tilde{v}^2\rangle = \frac{\langle\tilde{f}\tilde{v}\rangle}{m}$$

$\gg \tau_c$

but $$\tilde{v}(t) = \cancel{e^{-\gamma t}\tilde{v}(0)} + \int_0^t dt'\, e^{-\gamma(t-t')}\frac{\tilde{f}(t')}{m}$$

$$\langle\tilde{v}^2\rangle = \frac{T}{m} = \frac{1}{\gamma}\left\langle \tilde{f}\int_0^t dt'\, e^{-\gamma(t-t')}\frac{\tilde{f}(t')}{m}\right\rangle$$

$$\langle\tilde{f}(t)\,f(t')\rangle = |\tilde{f}|^2\tau_c\,\delta(t-t')$$

$$\langle\tilde{v}^2\rangle = \frac{T}{m} = \frac{1}{\gamma}\frac{|\tilde{f}|^2\tau_c}{m}$$

$$\boxed{\langle\tilde{f}^2\rangle\tau_c = \gamma T}$$

→ equilibrium :

→ emission by noise

→ absorption by damping

⇒ balance matches T !

note: alternatively

$$(-i\omega + \gamma)\,\tilde{V}_\omega = \tilde{f}_\omega / m$$

$$|\tilde{V}_\omega|^2 = \frac{|\frac{\tilde{f}_\omega}{m}|^2}{(\omega^2+\gamma^2)}$$

White noise: spectral intensity flat

$$\int d\omega\, |\tilde{V}_\omega|^2 = \frac{2T}{m} = \frac{|\tilde{f}|^2}{m^2}\int \frac{d\omega}{\omega^2+\gamma^2}$$

$$= \frac{|\tilde{f}_\omega|^2/m^2}{\gamma}$$

$$\frac{|\tilde{f}_\omega|^2}{m^2} = 2\gamma \frac{T}{m}$$

→ same

→ factors ⟺ normalization

note

$$\underbrace{|\tilde{v}_\omega|^2}_{\text{response spectral density}} = \frac{\overbrace{|\tilde{f}_\omega|^2/m^2}^{\text{noise spectral density}}}{\omega^2+\gamma^2}$$

↳ damping

$$= \frac{|\tilde{f}_\omega|^2/m^2}{|r(\omega)|^2}$$

↳ response function [damping ⟺ width]

i.e. $\frac{T}{m} = \int \frac{|\tilde{q}|^2}{|r(\omega)|^2} d\omega$

if oscillator:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{\tilde{f}}{m}$$

$$|\tilde{x}_\omega|^2 = \frac{|\tilde{f}_\omega|^2/m^2}{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}$$

note: $$T/m = \int \frac{|\tilde{a}|^2}{|r_r(\omega)|^2 + |r_{IM}(\omega)|^2} d\omega$$

if → $|\tilde{a}(\omega)|^2$ broad
→ $r(\omega)$ has lines, so

$$r_r(\omega) = (\omega - \omega_0) \frac{\partial r}{\partial \omega}$$

$$T/M = \int \frac{|\tilde{a}|^2 \, d\omega}{(\omega-\omega_0)^2 \left(\frac{\partial r}{\partial \omega}\right)^2 + |r_{IM}(\omega_0)|^2}$$

$$= |\tilde{a}(\omega_0)|^2 \int \frac{d\omega / |r_{IM}(\omega_0)|^2}{\left[\frac{(\omega-\omega_0)^2}{|r_{IM}(\omega_0)|^2} \left|\frac{\partial r_n}{\partial \omega}\right|^2 + 1\right]}$$

$$\cong \frac{|\tilde{a}(\omega_0)|^2}{|r_{IM}(\omega_0)| \left|\frac{\partial r_n}{\partial \omega}\right|_{\omega_0}}$$

$$\underbrace{|\tilde{a}(\omega_0)|^2}_{\text{noise}} = \left(\underbrace{r_{IM}}_{\text{dissp}} \underbrace{T/m}_{\text{Temp}}\right)\Big|_{\omega_0} \left|\frac{\partial r_n}{\partial \omega}\right|_{\omega_0}$$

→ Fluctuations set by { noise
[damping
collective modes]

response

i.e. $\omega \simeq \omega_0$

natural frequency

→ $2\left(\frac{1}{2} k x^2\right) = 2\left(\frac{m\omega_0^2}{2} x^2\right) = T$

sets condition

Lesson: → Thermal equilibrium spectrum
set by
- collective modes
- damping

} resonances

- noise

→ F-D Thm links these,
explicitly

→ For plasma, thermal equilibrium
requires understanding
→ noise
→ collective modes
→ damping

⇒ Test Particle Model (TPM)

(1) point: plasma → particles, → fields supports both

emission, absorption

(2) need calculate → collective modes, damping } response

→ emission ⟺ noise

TPM → how related?

For modes:

- need plasma model
- i.e. set of equations
  - Liouville → follow N particles
  - Boltzmann → $f(x, v, t)$

i.e. $$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{e}{m} \underline{E} \cdot \frac{\partial f}{\partial \underline{v}} = C(f)$$

↳ collision operator

$$\nabla^2 \phi = -4\pi n_0 \sum \int dv\, f$$

- Vlasov $\to$ $C(f) \to 0$.

- need develop theory of plasma models

$$\text{models} \begin{cases} \text{modes} \\ \text{noise} \\ \text{damping} \end{cases} + \quad T \ (\text{temp})$$

= equilibrium spectrum

but, before embarking on systematic theory of models, useful to explore:

$\to$ simple theory of plasma waves/modes.

i.e. collective response resonances.

I. Kinetic Equations and BBGKY Hierarchy

Now have established:

1 - nature of fluctuations near equilibrium (FDT)

2 - basic modes, dynamics $\left\{ \begin{array}{l} \underline{\text{but}} \to \text{damping} \} ! \\ \text{collisionless} \} ! \end{array} \right.$

⇒ now need kinetic equation on firm theoretical foundation

∴ <u>BBGKY</u> <u>Hierarchy</u> $\left\{ \begin{array}{l} \text{Bogoliubov, Born} \\ \text{Green, Kirkwood, Yvon} \end{array} \right.$

<u>i.e.</u> from

(a) Liouville (Klimontovich) ⟺ $N = \sum_i \delta(\underline{x} - \underline{x}_i(t))\, \delta(\underline{v} - \underline{v}_i(t))$

→ exact but useless $\left\{ \begin{array}{l} \sim 10^{23} \\ \text{particles} \end{array} \right.$

to

(b) f_s ⟺ $f_s = f(\Gamma_1, \Gamma_2, \ldots \Gamma_s)$

$$\frac{\partial}{\partial t} f_s + \ldots = S(f_{s+1}) \to \text{coupled hierarchy}$$

~ (e)

to Boltzmann ⟺ truncated 1 body eqn.

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{q}{m} \underline{E} \cdot \frac{\partial f}{\partial \underline{v}} = C[f(1,2)]$$

$f = f(\underline{x}, \underline{v}, t)$ $= C[f(1)\, f(2)]$

→ { 1 particle distrib.
density of phase space
fluid →

to

(d) Vlasov ⟺ collisionless Boltzmann

continuity equation ⟺ phase space fluid.

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{q}{m} \underline{E} \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

$df/dt = 0$
phase space fluid
density conserved

N.B. Only Boltzmann and Vlasov Equations are "useful", in any sense

to

(e) Fluid equations ⟺ moments of Boltzmann Eqn
— truncation problem.

How? → Dilateness / Weak Correlation

$$\rightarrow \boxed{T \gg \frac{e^2}{r} \sim e^2 n^{1/3}}$$

thermal energy
>> e.s. energy

→ low ... thermal energy ⟺ weak correlation i.e. drop h.o. correlations

$\Rightarrow 1/n\lambda_D^3 \ll 1 \Rightarrow$ fundamental ordering parameter

$1/n \left(\frac{T/m}{4\pi n e^2/m}\right)^{3/2} \sim \frac{n^{1/2}}{T^{3/2}}$

i.e. for $\left\{\begin{array}{l}\text{weakly correlated plasma:} \\ \text{collisional}\end{array}\right.$ $\Rightarrow \left\{\begin{array}{l}\text{basic} \\ \text{scale} \\ \text{ordering}\end{array}\right.$

$$b_{eff} < \bar{r} < l_{mfp} < l_{system}$$

b_{eff}: effective force range

$\bar{r}$: mean interparticle spacing

l_{mfp}: mean free path

l_{system}: system size

recall:

$\frac{e^2}{b} \sim v_{Th}^2$

$b_{eff} \cong e^2/T \sim \lambda_D/(n\lambda_D^3)$

$\bar{r} \cong 1/n^{1/3} \sim \lambda_D/(n\lambda_D^3)^{1/3}$

$l_{mfp} \cong \frac{v_{Th}}{\nu_c} \sim \frac{v_{Th}}{\frac{\omega_p}{(n\lambda_D^3)}} \sim \lambda_D (n\lambda_D^3)$

$l_{system} = L$ geometry

Then clearly have:

$b_{eff} < \bar{r} < l_{mfp}$

Why care? → from Liouville → Boltzmann

→ electrostatics ⇒ N body, Hamiltonian system

$\nabla_\Gamma \cdot \underline{V}_{\text{phase space}} = 0$ (Liouville's Thm.)

i.e. $\dfrac{\partial}{\partial \underline{q}_i} \cdot \dot{\underline{q}}_i + \dfrac{\partial}{\partial \underline{p}_i} \cdot \dot{\underline{p}}_i = 0$ $\left.\begin{matrix} q \\ p \end{matrix}\right\}$ generalized coordinates, momenta

rpt./summed

so if

$$N' = \sum_{i=1}^{N} \delta(\underline{x} - \underline{x}_i(t))\, \delta(\underline{v} - \underline{v}_i(t))$$

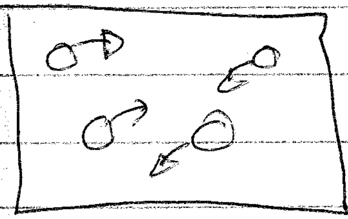
→ N body distribution
→ exact distribution ⟺ conserved

i.e. $\dfrac{\partial}{\partial t} N' + \nabla_\Gamma \cdot (\underline{V}_\Gamma N) = 0$

$\Rightarrow \dfrac{\partial N'}{\partial t} + \underline{V}_\Gamma \cdot \nabla_\Gamma N' = 0$ Liouvillian conserved along N-body trajectories but → N' not useful → $6N$ dimension

→ need collective description → phase space fluid evolution

i.e.



→ Liouville Eqn. evolves N "peas" individually

better: $\begin{cases} \text{crush / smear peas} \Rightarrow \text{"pea soup"} \\ \text{no gaps} \Leftrightarrow 1/n\lambda_D^3 \ll 1 \end{cases}$

⇒ evolve density of pea soup → f!

i.e. seek equation for $f(x,v,t)$ evolution.

? How from dN'/dt → Vlasov Equation.

→ Consider N particle system
- Hamiltonian
- $V(1,2) = q_1 q_2 / r_{1,2}$ → 2 body interaction potential

then for N particle distribution function f^N:

$$\frac{\partial}{\partial t} f^N + \sum_{i=1}^{N} \left(\dot{\underline{x}}_i \cdot \frac{\partial f^N}{\partial \underline{x}_i} + \dot{\underline{p}}_i \cdot \frac{\partial f^N}{\partial \underline{p}_i} \right) = 0$$

$\Rightarrow$

$$\frac{\partial f^N}{\partial t} + \sum_{i=1}^{N} \left(\underline{v}_i \cdot \frac{\partial f^N}{\partial \underline{x}_i} - \frac{\partial f^N}{\partial \underline{p}_i} \sum_{j<i} \frac{\partial V_{ij}}{\partial \underline{x}_i} \right) = 0$$

since $\dot{\underline{x}}_i = \underline{v}_i$

$$\dot{\underline{p}}_i = -\partial \sum_{j<i} V_{ij} \Big/ \partial \underline{x}_i$$

now construct hierarchy by integrating out points, i.e.

$$\Gamma = (\underline{x}, \underline{p})$$

$f(t, \underline{x}, \underline{p}) = \int d\Gamma_2 \, d\Gamma_3 \ldots d\Gamma_N \, f^N \rightarrow$ 1 pt distribution

$f(t, \underline{x}_1, \underline{p}_1, \underline{x}_2, \underline{p}_2) = \int d\Gamma_3 \ldots d\Gamma_N \, f^N \rightarrow$ 2 pt. distribution

$f^{(2)}, f^{(3)}$ etc. fully correlated only.

$\Rightarrow$

$$\frac{\partial f^{(1)}}{\partial t} + \underline{v}_1 \cdot \underline{\nabla}_1 f^{(1)} + \frac{\varepsilon}{m} \underline{E} \cdot \frac{\partial f^{(1)}}{\partial \underline{v}_1} = (N-1) \int d\Gamma_2 \frac{\partial V_{12}}{\partial \underline{x}_1} \cdot \frac{\partial f^{(2)}}{\partial \underline{p}_1}$$

f^N is fully correlated

N-1 binary pairs

interaction

2 particle distribution

similarly,

$$\frac{\partial f^{(2)}}{\partial t} + \underline{V}_1 \cdot \frac{\partial}{\partial \underline{x}_1} f^{(2)} + \underline{V}_2 \cdot \frac{\partial}{\partial \underline{x}_2} f^{(2)} + \frac{q}{m} \underline{E}(1) \cdot \frac{\partial f^{(2)}}{\partial \underline{V}_1}$$

$$+ \frac{q}{m} \underline{E}(2) \cdot \frac{\partial f^{(2)}}{\partial \underline{V}_2}$$

$$= \underbrace{(N-2)}_{\substack{\#\\(N-2)\\\text{pairs}}} \int d\Gamma_3 \left[\frac{\partial f^{(3)}}{\partial \underline{p}_1} \cdot \frac{\partial V_{1,3}}{\partial \underline{r}_1} + \frac{\partial f^{(3)}}{\partial \underline{p}_2} \cdot \frac{\partial V_{2,3}}{\partial \underline{r}_2} \right]$$

so

$f^{(1)}$ in terms $f(1,2) = f^{(2)}$

$f^{(2)}$ in terms $f(1,2,3) = f^{(3)}$

$\vdots$

$\Rightarrow$ coupled hierarchy $\Leftrightarrow$ nonlinearity of Liouville eqn.

$N \rightarrow$ fields $\Rightarrow$ orbits

To truncate:

have $b_{eff} < \bar{r} < l_{mfp}$

so $\frac{v_{th}}{b_{eff}} > \frac{v_{Th}}{\bar{r}} > \underbrace{\frac{v_{Th}}{l_{mfp}}}_{\nu_{coll.}}$

Now, $d\Gamma_3 = d\Gamma_{3x}\, d\Gamma_{3v}$

$$\int d\Gamma_3 \frac{\partial f^{(3)}}{\partial \underline{p}} \sim \underbrace{\frac{1}{V}}_{\substack{\text{spatial}\\ \text{normalization}\\ \sim 1/\bar{r}^3}} \frac{\partial f^{(2)}}{\partial \underline{p}} \underbrace{A(3)}_{\text{fctn.}}$$

→ contains spatial dependence of interaction

Now, claim

$$\underset{\text{for } f^{(2)}}{RHS} \sim \frac{\partial V}{\partial r} \frac{\partial f^{(2)}}{\partial p} N \frac{b^3}{\bar{r}^3}$$

→ effective "active volume" for scattering (n.b. screening helps)

→ volume normalization

Now, $N-2 \approx N$, as $N \gg 1$.

$$N \int d\Gamma_3 \frac{\partial f^{(3)}}{\partial p} \frac{\partial V}{\partial r} \sim \cancel{N} \frac{\partial V}{\partial r} \frac{b^3}{\cancel{N}\,\bar{r}^3} \frac{\partial f^{(2)}}{\partial p}$$

$$\cong \frac{b^3}{\bar{r}^3} \frac{\partial V}{\partial r} \frac{\partial f^{(2)}}{\partial r}$$

so

$$\mathrm{RHS}/\mathrm{LHS} \sim b^3/\bar{r}^3$$

$$\sim 1/(n\lambda_D^3)^2 \ll 1.$$

so

$$\frac{d}{dt} f^{(2)}(t, \Gamma_1, \Gamma_2) \cong 0$$

→ consistent with notion of particles interacting weakly.

→ diluteness ⟺ "molecular chaos" → factorization

i.e. $f^{(2)}(1,2,t)\Big|_{t=0} = f^{(1)}(1,t)\Big|_{t=0} f^{(1)}(2,t)\Big|_{t=0}$

i.e. particles initially uncorrelated.

so — if $f^{(2)}$ initially uncorrelated $\Rightarrow$ remains so

— $1/(n\lambda_D)^3 \ll 1$ $\Rightarrow$ no growth of correlation

Now consider $f^{(1)}$ equation:

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\frac{\partial f}{\partial \underline{x}} + \frac{q}{m}\underline{E}\cdot\frac{\partial f}{\partial \underline{v}} = N\int d\Gamma_2\, \frac{\partial V_{1,2}}{\partial \underline{r}_1}\cdot\frac{\partial f^{(2)}}{\partial \underline{p}_1}$$

but diluteness:

So $f^{(2)} = f^{(1)}(1,t)\, f^{(2)}(2,t)$

$\Rightarrow$

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\frac{\partial f}{\partial \underline{x}} + \frac{q}{m}\underline{E}\cdot\frac{\partial f}{\partial \underline{v}} = N\int d\Gamma_2\, \frac{\partial V_{1,2}}{\partial \underline{r}_1}\cdot\frac{\partial f^{(2)}}{\partial \underline{p}_1}$$

$$= N\int d\Gamma_2\, \frac{\partial V_{1,2}}{\partial \underline{r}_1}\cdot\frac{\partial}{\partial \underline{p}_1}\left[f^{(1)}(1,t)\, f^{(1)}(2,t)\right]$$

$$\equiv \underbrace{C(f)}_{\text{collision operator}}$$

→ have arrived at <u>Boltzmann</u> <u>Eqn.</u>

→ evolves f in <u>collisional</u> plasma (with Poisson)

→ Vlasov Eqn.

→ along with Poisson Eqn,

$$-\nabla^2\phi = 4\pi q\int dv\, f$$

— describes collisionless plasma dynamics.

N.B. Careful:

→ even if fluctuations "collisionless"

→ use of Maxwellian for $\langle f \rangle$

$\Leftrightarrow$ collisions.

∴ → Need identify characteristic $\begin{cases}\text{collective} \\ \text{collisional}\end{cases}$ scales

→ plasma is continuum ⇒ characterize by collective modes (can calculate response).

II Plasma / Fluid Collective Modes, Response

why $\epsilon = 1$

a.) Cold Plasma $(T, P \Rightarrow 0)$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \underline{v}) = 0$$ → continuity

$$n m \frac{d\underline{v}}{dt} = n\left(q \underline{E} + \frac{\underline{v}}{c} \times \underline{B}\right)$$ → momentum balance

+ Maxwell Equations

For electromagnetic / electrostatic wave :

$$n = n_0 + \tilde{n}$$

$$\underline{v} = \cancel{\underline{v}_0} + \tilde{\underline{v}}$$

$$\underline{E} = \cancel{\underline{E}_0} + \tilde{\underline{E}}$$

$$B = \cancel{B_0} + \tilde{\underline{B}}$$

$\Rightarrow$ 1 species — ions stationary

$$\frac{\partial \tilde{n}}{\partial t} = -n_0 \, \underline{\nabla} \cdot \underline{\tilde{V}}$$

$$\frac{\partial \underline{\tilde{V}}}{\partial t} = \frac{q}{m} \underline{\tilde{E}} + \cancel{\frac{q \, \underline{\tilde{V}} \times \underline{\tilde{B}}}{mc}}$$

$$\underline{\nabla} \cdot \underline{\tilde{E}} = 4\pi q \tilde{n} \qquad \nabla \cdot \underline{\tilde{B}} = 0$$

$$\underline{\nabla} \times \underline{\tilde{B}} = \frac{4\pi}{c} \underline{\tilde{J}} + \frac{1}{c} \frac{\partial \underline{\tilde{E}}}{\partial t} \qquad \underline{\nabla} \times \underline{\tilde{E}} = -\frac{1}{c} \frac{\partial \underline{\tilde{B}}}{\partial t}$$

$$\underline{\tilde{J}} = n_0 q \underline{\tilde{V}}$$

$\Rightarrow$ Fourier Transforming $\qquad \underset{\text{etc}}{\underline{E}} = \sum_{\underline{k},\omega} \underline{E}_{\underline{k},\omega} e^{i(\underline{k}\cdot\underline{x} - \omega t)}$

$$\underline{k} \times \underline{\tilde{B}}_{\underline{k},\omega} = -\frac{4\pi n_0 q \, i}{c} \underline{\tilde{V}}_{\underline{k},\omega} - \frac{\omega}{c} \underline{\tilde{E}}_{\underline{k},\omega}$$

$$\frac{c\underline{k}}{\omega} \times (\underline{k} \times \underline{E}_{\underline{k},\omega}) = \frac{4\pi n_0 q^2}{m_e c} \underline{E}_{\underline{k},\omega} - \frac{\omega}{c} \underline{E}_{\underline{k},\omega}$$

$\Rightarrow$

$$\underline{k}(\underline{k} \cdot \underline{E}_{\underline{k},\omega}) - k^2 \underline{E}_{\underline{k},\omega} = \underbrace{\frac{4\pi n_0 q^2}{c^2 m_e}}_{\omega_p^2/c^2} \underline{E}_{\underline{k},\omega} - \frac{\omega^2}{c^2} \underline{E}_{\underline{k},\omega}$$

→ EM waves $(\underline{k}\cdot\underline{E}_{\underline{k}\omega}=0)$

$$k^2\underline{E}_{\underline{k}\omega}-\underline{k}(\underline{k}\cdot\underline{E}_{\underline{k}\omega}) = \frac{\omega^2}{c^2}\underline{E}_{\underline{k}\omega}-\frac{\omega_p^2}{c^2}\underline{E}_{\underline{k}\omega}$$

$\omega_p^2 = 4\pi n_0 q^2/m$ → plasma frequency
→ characteristic frequency for (non-neutralized) plasma oscillations
→ (ions stationary ⇒ $\omega \gg \omega_{p_i}$ $(\sim 1/M_i)$)

$$k^2 = \frac{\omega^2}{c^2}\left(1-\frac{\omega_p^2}{\omega^2}\right)$$

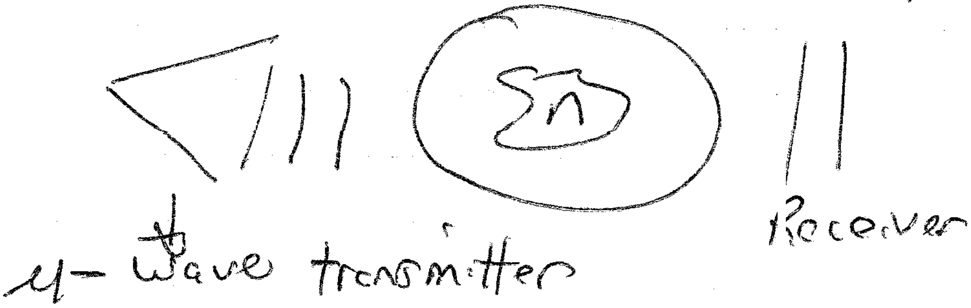
$$\omega^2 = \omega_p^2 + c^2k^2$$

Dispersion Relation for EM Waves in Unmagnetized plasma

→ $\epsilon(\omega) = 1-\omega_p^2/\omega^2$ — cold plasma dielectric (dispersive)

→ $\omega<\omega_p \Rightarrow k^2<0$ — ω_p is cut-off frequency

→ can diagnose density



→ Electrostatic Waves / oscillations (Langmuir Osc.)

$\underline{k} \cdot \underline{E} = k\underline{E}$ ⟷ alternatively obtain via $\left\{ \begin{array}{l} \text{Fluid eqns} \\ + \\ \text{Gauss Law} \end{array} \right.$

$\Rightarrow \quad 0 = \left[(\omega^2 - \omega_p^2)/c^2 \right] \underline{E}_{k,0}$

$\omega^2 = \omega_p^2$

- ions stationary ⟷ $\omega^2 \gg \omega_{pi}^2 \sim 1/M_i$
- non-propagating oscillation $\omega^2 = \omega_p^2$

b.) Warm Plasma Waves (Electrostatic)
(Langmuir Waves)

Now, introduce pressure

$$m \frac{\partial \tilde{\underline{v}}}{\partial t} = q \tilde{\underline{E}} - \frac{\underline{\nabla} \tilde{p}}{n_0}$$

$$\frac{\partial \tilde{n}}{\partial t} = -n_0 \underline{\nabla} \cdot \tilde{\underline{v}}$$

$$\underline{\nabla} \cdot \tilde{\underline{E}} = 4\pi q \tilde{n}$$

$\left\{ \begin{array}{l} p = p_0 (n/n_0)^\gamma \\ \quad - \text{adiabatic} \\ p = \tilde{n} T \\ \quad - \text{isothermal} \end{array} \right.$

↓

determines eqn. state from kin. Th.

, ∂w)

$$\frac{\partial^2 \tilde{n}}{\partial t^2} = -n_0 \left(\frac{q}{m} \nabla \cdot \underline{E} - \frac{\nabla^2 \tilde{p}}{n_0 m} \right)$$

$$= -\omega_p^2 \tilde{n} + \frac{T}{m} \nabla^2 \tilde{n}$$

$$\Rightarrow \frac{\partial^2 \tilde{n}}{\partial t^2} = -\underbrace{\omega_p^2 \tilde{n}}_{\text{plasma oscillation}} + \underbrace{\frac{T}{m} \nabla^2 \tilde{n}}_{\text{streaming induced by } \nabla p \text{ (a la' acoustics)}} \qquad \frac{T}{m} \equiv V_{the}^2$$

$$\omega^2 = \omega_p^2 + k^2 V_{Th}^2$$

$$\rightarrow \omega^2 = \omega_p^2 (1 + k^2 \lambda_D^2)$$

$$\lambda_D^2 \equiv V_{Th}^2 / \omega_p^2 \qquad \rightarrow \text{Debye Length}$$

i.e.

$$\nabla^2 \tilde{n} - \frac{1}{\lambda_D^2} \tilde{n} = \frac{1}{V_T^2} \frac{\partial^2 \tilde{n}}{\partial t^2} \qquad \Rightarrow \omega \to 0 \text{ recovers Screened Gauss' Law}$$

Recall Debye Length :

$$\nabla^2 \phi = \underbrace{4\pi \rho_{ind}}_{\text{screening response}} + \underbrace{4\pi q \delta(\underline{x} - \underline{x}_0)}_{\text{test charge in plasma}}$$

$$f = n_0 \exp\left[-\frac{mv^2}{T} \pm \frac{q\phi}{T}\right]$$

$$\rho_{ind} = n_0 q \exp[q\phi/T_e] - n_0 q \exp[-q\phi/T_i]$$

$$\cong \frac{\omega_{pe}^2}{4\pi V_{Te}^2}\phi + \frac{\omega_{pi}^2}{4\pi V_{Ti}^2}\phi$$

$$\Rightarrow \nabla^2\phi - \left(\frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2}\right)\phi = 4\pi q\,\delta(\underline{x} - \underline{x}_0)$$

Then $\omega \ll \omega_p$ $\Rightarrow$ plasma response is streaming to screen test charge

$\Rightarrow$ hence appearance Debye length

$\omega \gg \omega_p$ $\Rightarrow$ warm plasma oscillation (too fast to screen)

Note: cold plasma $(T=0)$ $\Rightarrow$ no energy to move to screen charge

→ Warm Plasma Wave combines $\begin{cases} \text{plasma oscillation} \\ \oplus \text{ acoustic wave} \end{cases}$

i.e. carries wave momentum

.) Ion Acoustic Wave

- so far, 'single species' dynamics

- consider now, ion acoustic wave, with
  $v_{Ti} < \frac{\omega}{k} < v_{Te}$

Recall, for warm electrons:

$$\frac{\partial \tilde{n}}{\partial t} + \cancel{\nabla \cdot (\tilde{n}\tilde{v})} = -n_0 \nabla \cdot \underline{\tilde{v}}$$

$$m_e \frac{\partial \tilde{v}}{\partial t} = -|e| \tilde{E} - T_e \frac{\nabla \tilde{n}}{n_0} \qquad (\tilde{p} = \tilde{n}\, T_e)$$

$$= +|e| \nabla \tilde{\phi} - T_e \frac{\nabla \tilde{n}}{n_0}$$

$\Rightarrow$

$$\frac{\partial^2 \tilde{n}}{\partial t^2} = -n_0 \left(\frac{|e|}{m_e} \nabla^2 \tilde{\phi} - \frac{v_{Te}^2}{n_0} \nabla^2 \tilde{n} \right)$$

$$\underbrace{\frac{\partial^2 \tilde{n}}{\partial t^2}}_{O(\omega^2)} - \underbrace{v_{Te}^2 \nabla^2 \tilde{n}}_{\substack{\text{electron compression} \\ O(k^2 v_{Te}^2)}} = -n_0 \frac{|e|}{m_e} \nabla^2 \tilde{\phi}$$

$\therefore$ for $k^2 V_{Te}^2 \gg \omega^2$

$$\frac{\tilde{n}}{n_0} = \frac{|e|\hat{\phi}}{T_e}$$

Note:

→ equivalent to limit where electron inertia negligible

i.e. $m_e \to 0$ $\quad (V_{Te}^2 \to \infty) \Rightarrow \frac{\tilde{n}}{n} = \frac{|e|\hat{\phi}}{T_e}$

→ could, in limit $k^2 V_{Te}^2 \gg \omega^2$, obtain from Boltzmann response

i.e. $E \to E - |e|\hat{\phi} \Rightarrow f_e = c \exp\left[-\frac{\left(\frac{mv^2}{2} - |e|\hat{\phi}\right)}{T}\right]$

$$\cong \left(1 + \frac{|e|\hat{\phi}}{T}\right) f_{0M}$$

For ions (cold)

$$\frac{\partial \tilde{n}}{\partial t} = -n_0 \underline{\nabla} \cdot \underline{\tilde{V}}$$

$$\frac{\partial \tilde{V}_i}{\partial t} = +\frac{|e|}{m_i} \tilde{E}$$

$$\frac{\partial^2 \tilde{n}}{\partial t^2} = +n_0 \frac{|e|}{m_i} \nabla^2 \tilde{\phi}$$

$$\frac{\hat{n}_{i\,k,\omega}}{n_0} = + \frac{|e|}{m_i} \frac{k^2}{\omega^2} \tilde{\phi}_{k,\omega}$$

$$\nabla^2 \phi = -4\pi n_0 |e| \left(\frac{\hat{n}_i}{n_0} - \frac{\hat{n}_e}{n_0} \right)$$

$$k^2 \hat{\phi}_{k,\omega} = +4\pi n_0 |e| \left(\frac{|e|}{m_i} \frac{k^2}{\omega^2} \hat{\phi}_{k,\omega} - \frac{|e|}{T_e} \hat{\phi}_{k,\omega} \right)$$

$$k^2 = \frac{\omega_{pi}^2 k^2}{\omega^2} - \underbrace{\frac{\omega_{pe}^2}{v_{Te}^2}}_{\to \lambda_{De}}$$

$$\Rightarrow \left(k^2 + 1/\lambda_{De}^2 \right) = \frac{\omega_{pi}^2}{\omega^2} k^2$$

$$\therefore \boxed{\begin{aligned} \omega^2 &= k^2 c_s^2 \Big/ \left(1 + k^2 \lambda_{De}^2\right) \\ c_s^2 &= T_e/m_i \end{aligned}}$$

Note:

→ Compare hydrodynamic acoustic wave:

$$\frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \nabla \cdot \tilde{v} \quad ; \quad \frac{\partial \tilde{v}}{\partial t} = -\frac{\nabla \tilde{p}}{\rho_0}$$

$$\tilde{p} = c_s^2 \tilde{\rho}$$

	Hydro	Ion-Acoustic
"Springiness"	Gas Pressure	T_e
Inertia	Gas Density/inertia	m_i

i.e. ion-acoustic wave is two component, hybrid oscillation

$$\rightarrow \left(k^2 + 1/\lambda_{De}^2\right) = \frac{\omega_{pi}^2}{\omega^2} k^2$$

$$\left(1 + 1/k^2\lambda_{De}^2\right) = \frac{\omega_{pi}^2}{\omega^2}$$

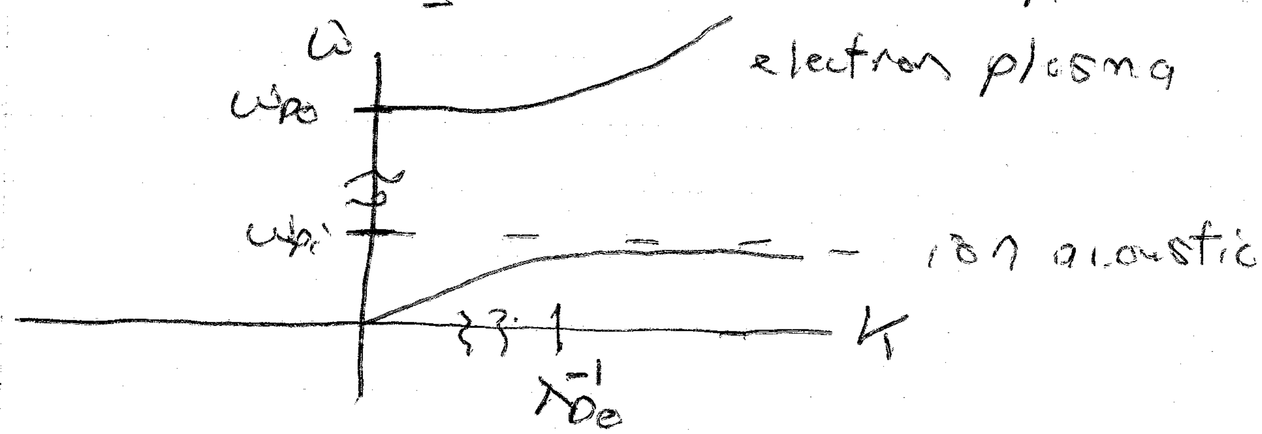
Debye shielding (T_e)

ion plasma oscillation (m_i)

Ion-acoustic wave as Debye-shielded ion plasma oscillation

Note : $k^2 \lambda_{De}^2 \gtrsim 1 \Rightarrow \omega^2 \to \omega_{pi}^2$

$\Rightarrow$



Basic modes (electrostatic) of un-magnetized plasma.

Basic Scales : $\begin{cases} \omega_{pe}, \ \omega_{pi} \\ \lambda_{De} \\ V_{Te}, \ C_s \end{cases}$

C.) Nonlinear Fluid Plasma Waves

$\Rightarrow$ Langmuir, Ion Acoustic Waves $\Rightarrow$ $\underline{1D}$
Compressional Waves

$\Rightarrow$ 1D Compressional Wave (Linear)

$\downarrow$ (steepening – finite amplitude)

Shock

collisional (standard) (Burgers) $\leftarrow$ Shock $\rightarrow$ collisionless (KdV)