Now seek:

$$C(k,\omega) = \int dv_1 \int dv_2 \, \langle \tilde{f}(1)\tilde{f}(2)\rangle_{k,\omega}$$

$$= \int dv_1 \int dv_2 \int d\underline{x}_- \, e^{-ikx_-} \int d\tau \, e^{i\omega\tau} \langle \tilde{f}(0)\,\tilde{f}(x_-,\tau)\rangle$$

→ where have used: $\left\{ \begin{array}{l} \text{homogeneity} \\ \text{stationarity} \end{array} \right.$

→ $v_\pm = (\underline{v}_2 \pm \underline{v}_1)/2$

So

$$\int d\underline{v} \langle \tilde{f}(1)\tilde{f}(2)\rangle_{k,\omega} = \frac{\langle f\rangle}{n} \int e^{i\omega\tau} e^{-ik\cdot x(\tau)} \, d\tau$$

$\underline{x}(\tau) = \underline{v}\tau$ → ballistic, u.p.o.

⇒

$$\boxed{\int d\underline{v} \langle \tilde{f}^2\rangle_{k,\omega} = 2\pi\,\delta(\omega - \underline{k}\cdot\underline{v})\,\frac{\langle f\rangle}{n}}$$

$2\pi\,\delta(\omega - \underline{k}\cdot\underline{v})$ → k,ω orbit propagator

$\frac{\langle f\rangle}{n}$ → weight function ↔ particle distribution

So $$\boxed{\underbrace{C(k,\omega)}_{\langle \hat{n}\hat{n}\rangle_{k,\omega}} = \int dv\, \frac{\langle f\rangle}{n}\, 2\pi\,\delta(\omega - k\cdot v)}$$ $\Rightarrow$ noise spectrum

so, can write in 1D: (hereafter)

$$C(k,\omega) = \frac{2\pi}{n|k|v_{Te}} \left\langle \bar{f}(\omega/kv_{Te})\right\rangle$$

$\hookrightarrow$ v_{Te} factor extracted

$\Downarrow$ discreteness noise has Maxwellian Doppler spectrum...

and

$$\boxed{\langle \hat{\phi}^2\rangle_{k,\omega} = \left(\frac{4\pi n_0|e|}{k^2}\right)^2 \frac{1}{n_0}\, \frac{2\pi}{|k|v_{Te}}\, \frac{\langle \bar{f}(\omega/kv_{Te})\rangle}{|\epsilon(k,\omega)|^2}}$$

$\therefore$ have obtained thermal equilibrium spectrum.

Can re-write as:

Maxwellian Dist: Source

$$\langle \phi^2 \rangle_{k,\omega} = n_0 \left(\frac{4\pi |e|}{k^2} \right)^2 \frac{2\pi}{|k| v_{Te}} \frac{\langle \bar{f}(\omega / k v_{Te}) \rangle}{|\epsilon(k,\omega)|^2}$$

particles

Coulomb Spectrum

Screening

→ Spectrum defined / determined by:

- eqbrm. particle emission spectrum $\sim e^{-\omega^2/k^2 v_{Te}^2}$
- collective response $\quad \epsilon \cong 1 - \frac{\omega_p^2}{\omega^2} + i\epsilon_{IM}$

$(\omega > k v_{Te})$

$\epsilon \cong 1 + 1/k^2 \lambda_D^2 + i\epsilon_{IM}$

$(\omega < k v_{Te})$

→ Can note:

1) collective response <u>strongest</u> at wave resonance $(kv \sim \omega \sim \omega_{pe})$

⇒ expect peak at natural frequency.

2) $\omega \gg \omega_p$ ($v \gg \omega_p/k$) $\Rightarrow$ noise source decouples from collective dynamics (i.e. plasma as vacuum)

$$\langle \hat{\phi}^2 \rangle_{k,\omega} \cong n_0 \left(\frac{4\pi |e|}{k^2} \right)^2 \frac{2\pi}{|k| v_{Te}} e^{-\omega^2/k^2 v_{Te}^2}$$

3) $\omega \ll \omega_p \Rightarrow$ static limit (Debye screening)

$$\langle \hat{\phi}^2 \rangle_{k,\omega} \cong n_0 (4\pi |e|)^2 \frac{2\pi}{|k| v_{Te}} e^{-\omega^2/k^2 v_{Te}^2} \Big/ \left(k^2 + 1/\lambda_D^2 \right)^2$$

$$\epsilon = 1 + \frac{\omega_p^2}{k} \int dv \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$$

So, can write Electric Field Spectrum:

$$\boxed{\frac{|\hat{E}_{k,\omega}|^2}{8\pi} = \frac{4\pi^2 n_0 |e|^2}{k^2 |k|} \left\{ \frac{F(\omega/k v_{Te})}{\left(1 - \frac{\omega_p^2}{\omega^2}\right)^2 + \left(\frac{\pi \omega_{pe}^2}{|k| k} F'\right)^2} \right\}}$$

$$|\epsilon_r|^2 + |\epsilon_{IM}|^2$$

$F' \sim$ slope

i.e. $F' = dF/d\left(\frac{\omega}{k}\right)$

Now, to make contact with usual expectations:

i.e. "$k_B T/2$ per degree-of-freedom"

$$W_k = \int \frac{d\omega}{2\pi} |\hat{E}_{k,\omega}|^2 / 8\pi$$

field energy per mode

Useful trick: Pole Approximation

$$\frac{1}{|\epsilon|^2} \cong \frac{1}{(\omega-\omega_k)^2 \left|\frac{\partial\epsilon}{\partial\omega}\right|^2 + |\epsilon_{IM}|^2}$$

$|\epsilon_{IM}|^2$ → sets width

$(\omega-\omega_k)^2$ → real freq. sets location

$$\cong \frac{1}{|\epsilon_{IM}|} \left\{ \frac{|\epsilon_{IM}|}{(\omega-\omega_k)^2 \left|\frac{\partial\epsilon}{\partial\omega}\right|^2 + |\epsilon_{IM}|^2} \right\}$$

$$\cong \frac{1}{|\epsilon_{IM}|} \left| (\partial\epsilon_r/\partial\omega)_{\omega_k} \right|^{-1} \delta(\omega-\omega_k)$$

$$\boxed{\frac{1}{|\epsilon|^2} \cong \pi\delta(\omega-\omega_k) \Big/ |\epsilon_{IM}(k,\omega_k)| \left|\frac{\partial\epsilon_r}{\partial\omega}\right|_{\omega_k}}$$

so, integrating in pole approximation:

$$W_k = \frac{m_e \omega_{pe}}{2|k|} \frac{F}{|F'|}$$

$$\cong \frac{m_e \omega_{pe}}{2|k|} \frac{\cancel{F}}{\frac{\omega_{pe}}{|k| v_{Te}^2} \cancel{F}} = \frac{T}{2}$$ ✓

in accord with "T/2 per degree" intuition.

→ if $k\lambda_D \gg 1$ $\left(\epsilon \to 1 + \frac{1}{k^2\lambda_D^2} \text{ limit}\right)$

$$W_k \cong \frac{T}{2} \, 1/k^2\lambda_D^2$$ → strong cut-off beyond λ_D.

so, for total energy density:

$$\left\langle \frac{E^2}{8\pi} \right\rangle = \int d\underline{k} \, W_{\underline{k}} = \int \frac{d\underline{k}}{(2\pi)^3} \frac{T/2}{(1+k^2\lambda_D^2)^{1/2}}$$ old

$$\sim \left(\frac{k_b T}{2}\right) k_{max}^3$$

$$\sim \left(\frac{n k_b T}{2}\right) \frac{1}{n\lambda^3}$$

so $$\frac{\langle E^2 \rangle}{8\pi} \sim \left(\frac{n k_B T}{2}\right) \frac{1}{n \lambda_D^3}$$

$\sim$ (Particle Kinetic Energy Density) / $n \lambda_D^3$

\# particles in Debye sphere

$$\Rightarrow \boxed{(FED) \sim \frac{(PKED)}{n \lambda_D^3}}$$

$1/n\lambda_D^3 \sim$ discreteness factor

→ To connect formally, to fluctuation-dissipation theorem:

note: $$\epsilon_{IM} = -\frac{\omega_p^2 \pi}{k|k|} \left.\frac{\partial \langle f \rangle}{\partial v}\right|_{\omega/k}$$

$$= \frac{2\pi\omega}{k^2 v_{Te}^2} \frac{\omega_p^2}{|k| v_{Te}} \langle \bar{f}(\omega/k) \rangle \quad , \quad \text{for Maxwellian } \langle f \rangle$$

so $$\langle \bar{f}(\omega/k) \rangle = k^2 v_{Te}^2 |k| v_{Te}\, \epsilon_{IM} \Big/ 2\pi\omega\, \omega_p^2$$

and have:

$$\langle \hat{\phi}^2 \rangle_{k,\omega} = \frac{2\pi n}{|k| v_{Te}} \left(\frac{4\pi |e|}{k^2} \right)^2 \frac{\langle \bar{f}(\omega/k) \rangle}{|\epsilon(k,\omega)|^2}$$

so, plugging in:

$$\langle \hat{\phi}^2 \rangle_{k,\omega} = \frac{8\pi \bar{T}}{k^2 \omega} \frac{\operatorname{Im}\epsilon}{|\epsilon|^2}$$

$\langle \tilde{U} \rangle \sim 5T$ form

and $$\boxed{\frac{\langle \hat{E}^2 \rangle_{k,\omega}}{8\pi} = \frac{T}{\omega} \frac{\operatorname{Im}\epsilon}{|\epsilon|^2}} \Leftrightarrow \text{Fluctuation-Dissipation Theorem}$$

(restates form of spectrum)

→ relates thermal fluctuations to dissipation in collective modes ($\operatorname{Im}\epsilon$)

→ obviously consistent (by construction) with physical picture of [illegible] b.) Some general comments:

→ Key element of T.P.M. is $\left\{\begin{array}{l}\text{causality}\\ \text{use of}\end{array}\right.$ linear $f^c_{k,\omega}$ or, equivalently, unperturbed orbits,

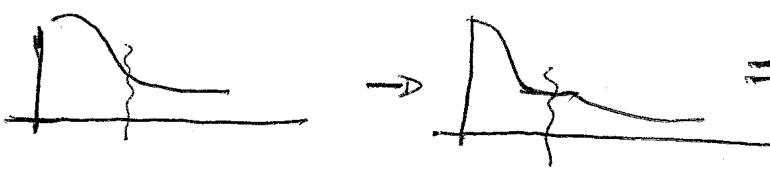
- This assumes small fluctuation levels, so stochastic deflection is 'weak'

i.e. $\underline{x}(t) = \underline{x}(0) + \underline{v}t + \underbrace{\delta(\underline{x}(t))}_{\text{deflection}}$

- How weak? ↔ take care $\tau_{ac} < \tau_T$ condition

but - $\langle \phi^2 \rangle_k \sim (\ )\, \dfrac{F(\omega_{pe}/k)}{|F'(\omega_{pe}/k)|}$

∴ Fluctuations diverge as $F' \to 0$, from below

i.e.  ⇒ "Critical coalescence"

- Note $F' > 0$ not necessary ↔ theory fails for stable plasma ....., approaching marginality.

→ As fluctuations grow, linearizations fail

∴ must renormalize!

→ particle propagator

$$i/\omega - kv \rightarrow i/\omega - kv + \underbrace{\Sigma}_{\text{self energy, decorrelation rate}}$$

→ mode propagator / response

$$1/\epsilon \rightarrow 1\Big/\left[\underbrace{(\omega-(\omega_k+\delta\omega_k))}_{\substack{\text{nonlinear}\\ \text{frequency}\\ \text{shift}}}\frac{\partial\epsilon}{\partial\omega} + i\underbrace{(\epsilon^{L}_{Im}+\epsilon^{NL}_{Im})}_{\text{nonlinear dissipation}}\right]$$

(w-w interaction)
(w-p interaction)
⇒ γ^{NL}

(recall NL oscillator, driven)

Calculating all this is aim of plasma turbulence theory .....

→ Relaxation and Transport

- till now: focus on thermal equilibrium fluctuation spectrum
  - → FDT
  - → collective modes
  - → kinetic equations (BBGKY)
  - → Vlasov modes / Landay damping
  - → test particle model

- now: focus on resistivity and effective Ohm's Law for plasma i.e. J vs E relation

$$\begin{cases} C(f) = \frac{e}{m} E \cdot \frac{\partial \langle f \rangle}{\partial v} \\ -\nu \delta f = \frac{e}{m} E \cdot \frac{\partial \langle f \rangle}{\partial v} \\ \delta J = \int dv\, v\, \delta f \end{cases}$$

elements:

(1.) → collisional resistivity ↔ how compute collisional transport coefficients?

⇒ transport coefficient as linear response coefficients

⇒ Boltzmann - Landau - Rosenbluth Eqn.

via:

- Lenard-Balescu Eqn. (from TPM)
- Fokker-Planck Eqn

⇒ Equilibrium

⇒ perturbations ⇒ relaxation

⇒ transport coefficients, incl. $\sigma_{Spitzer}$

(2.) → beyond collisional resistivity

⇒ why? low collisionality ⇒ current

⇒ basis of waves instabilities
⇒ wave-particle momentum transfer
⇒ quasilinear theory for mean $\langle f \rangle$ evolution → relaxation, momentum budget
⊛ ⇒ turbulent resistivity

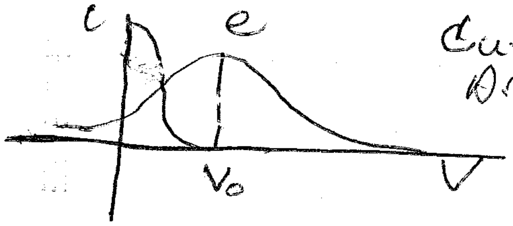
∴ nonlinear $J(E)$ relation

n.b. : Supplementary reading/study of Fokker-Planck theory, Boltzmann Equation, H-theorem is strongly suggested.

iii) Near-Equilibrium Relaxation : Lenard--Balescu Equation → where does C(f) come from

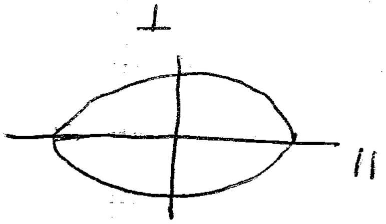
→ here, seek understand evolution of $\langle f \rangle$ (i.e. transport, relaxation) in near-equilibrium plasma

i.e.



Current Driven Ion Acoustic → resistivity

$V_0 < V_{0crit}$



$T_\parallel > T_\perp$ → isotropization/thermalization

→ systematic procedure for evolution equation of $\langle f \rangle$ → i.e. beyond Landau Collision Integral ? or where from Landau Collision Integral.

Now, $C(f) \to$ # ans.

velocity space flux → drives relaxation

$$\frac{\partial \langle f \rangle}{\partial t} = -\frac{q}{m}\frac{\partial}{\partial v}\langle E\, \delta f \rangle = -\frac{\partial}{\partial v}\Gamma_v$$

$q = -|e| \Leftrightarrow$ electrons, FYI

i.e. "transport"
→ how does mean $\langle f \rangle$ evolve in response to small

$$\delta f = f^c + \tilde{f}$$

f^c: Q.L. response (coherent)

$\tilde{f}$: discreteness fluctuation

(every particle both a (pea / element of pea soup))

simple mean field theory with total δf

$\therefore$

$$\frac{\partial \langle f \rangle}{\partial t} = -\frac{\partial}{\partial v}\left\langle \frac{q}{m}\hat{E} f^c \right\rangle - \frac{\partial}{\partial v}\left\langle \frac{q}{m}\hat{E}\tilde{f} \right\rangle$$

$$f^c = \frac{-\frac{q}{m}\hat{E}_{k,\omega}\,\frac{\partial\langle f\rangle}{\partial v}}{-i(\omega - kv)}$$

[but fluctuations due other particles!]

so

$$\Gamma_v = -D(v)\frac{\partial\langle f\rangle}{\partial v} + F\langle f\rangle$$

$D(v)$: [quasi-linear] diffusion, with T.P.M. spectrum (i.e. random kicks from thermal fluctuations)

$F\langle f\rangle$: drag due discreteness (i.e. a/a wake, wave emission exerts force on emitter)

$$D = \sum_{k,\omega} \frac{q^2}{m^2} k^2 \langle \phi^2 \rangle_{k,\omega}\, \pi\delta(\omega - kv)$$

{i.e. note ω-summation

$$F = -\frac{q}{m}\sum_{k,\omega} ik \frac{\langle \phi\tilde{f} \rangle_{k,\omega}}{\langle f \rangle}$$

$$\langle \tilde{\phi}^2 \rangle_{k,\omega} = \underbrace{\left(\frac{4\pi n_0 q}{k^2}\right)^2}_{\text{Coulomb spectrum}} \frac{\overbrace{\langle \tilde{n}^2 \rangle_{k,\omega}}^{\text{test particle correlator}}}{\underbrace{|\epsilon(k,\omega)|^2}_{\text{screening}}}$$

$$\langle \tilde{n}^2 \rangle_{k,\omega} = \int d\underline{v} \; \frac{2\pi}{n} \delta(\omega - \underline{k}\cdot\underline{v}) \langle f(\underline{v}) \rangle \qquad (3D)$$

$$= \int dv \; \frac{2\pi}{n} \delta(\omega - kv) \langle f(v) \rangle \qquad (1D)$$

so

$$\boxed{\langle \tilde{\phi}^2 \rangle_{k,\omega} = \left(\frac{4\pi n_0 q}{k^2}\right)^2 \left(2\pi / |k| v_{Te} n_0\right) \frac{\overbrace{\langle \bar{f}(\omega/k v_{Te}) \rangle}^{\text{dim-less.}}}{|\epsilon(k,\omega)|^2}}$$

$$\langle \tilde{\phi} \tilde{f} \rangle_{k,\omega} = \left(\frac{4\pi n_0 q}{k^2}\right) \int dv \; \frac{\langle \tilde{f}\tilde{f} \rangle_{k,\omega}}{\epsilon^*(k,\omega)}$$

→ determines diffusion D

$$= \left(\frac{4\pi n_0 q}{k^2}\right) \frac{1}{\epsilon^*(k,\omega)} \int dv \; \frac{\delta(\omega - kv)}{n_0} \langle f \rangle \delta(v_1 - v_2)$$

$$= \frac{4\pi n_0 q}{k^2} \frac{1}{\epsilon^*(k,\omega)} \; 2\pi \delta(\omega - kv) \frac{\langle f \rangle}{n_0}$$

→ determines drag F

$\rightarrow$ velocity space current

So $$\frac{\partial \langle f \rangle}{\partial t} = -\frac{\partial}{\partial v} J(v)$$

$$J(v) = -D \frac{\partial \langle f \rangle}{\partial v} + F\langle f \rangle$$

$$= \sum_{k,\omega} \left(\frac{4\pi n_0 q}{k^2}\right) \frac{q}{m} \left(\frac{2\pi\,\delta(\omega - kv)}{n_0 |\epsilon(k,\omega)|^2}\right) * k \left\{ \left(\frac{4\pi n_0 q}{k^2}\right) \frac{\pi k}{|k| v_{Te}} \left(\frac{q}{m}\right) * \langle \bar{f}(\omega/k) \rangle \frac{\partial \langle f \rangle}{\partial v} + \epsilon_{IM}(k,\omega) \langle f \rangle \right\}$$

but

$$\epsilon_{IM} = -\frac{\pi \omega_p^2}{|k|^2} \frac{k}{|k| v_{Te}} \frac{\partial \langle f \rangle}{\partial v}\Big|_{\omega/k} + \epsilon_{IM}^{ion}$$

$\omega)$ w/o ϵ_{IM}^{ion}

So, in 1D:

(vanishes — Maxwellian)

$$J(v) = \sum_{k,\omega} \left(\frac{\omega_p^2}{k^2}\right)^2 \left(\frac{2\pi^2 k}{n_0 k^2 v_T}\right) \frac{\delta(\omega - kv)}{|\epsilon(k,\omega)|^2} \left\{ \langle \bar{f}(\omega/k v_T) \rangle \frac{\partial \langle f \rangle}{\partial v} - \langle f(v) \rangle \frac{\partial \langle \bar{f}(\omega/k v_T) \rangle}{\partial v}\Big|_{\omega/k} + \epsilon_{IM}^{ion} \langle f(v) \rangle \right\}$$

$$= D_{e,e} \frac{\partial \langle f_e \rangle}{\partial v} + F_{e,e} \langle f_e(v) \rangle + F_{e,i} \langle f_e(v) \rangle$$

$$= 0$$