Antiparticles, Spin, and Statistics

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Outline

$SO(3)$ vs. $SU(2)$

Antiparticles

Fermions

Summary

$SO(3)$

$SO(3)$ is the group of 3-dimensional spatial rotations. Its elements are 3×3 real orthogonal matrices.

A 3×3 matrix A has in general 9 independent parameters, but the orthogonality condition $AA^T = 1$ brings 6 extra conditions since the matrix AA^T is symmetric and therefore depends on 6 independent numbers. So we are left with 3 independent parameters to characterize any rotation.

Any rotation can be characterized by an axis (2 parameters) and an angle of right-handed rotation around that axis (1 parameter).

$SU(2)$

$SU(2)$ is the group of 2×2 complex unitary matrices U with determinant 1. If we introduce 3 Pauli matrices by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which satisfy

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \quad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

then it can be shown that any element of $SU(2)$ can be written as

$$B = \exp\left(\frac{-i\vec{\sigma}\hat{n}\phi}{2}\right)$$

where $\hat{n}$ is a unit vector in 3 dimensions and ϕ is an angle.

$SO(3) vs. SU(2)$

Elements of $SU(2)$ are

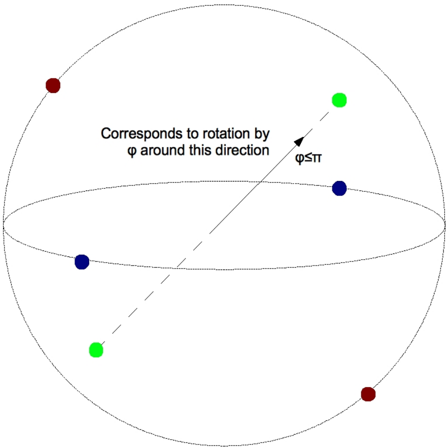
$$B = \exp\left(\frac{-i\vec{\sigma}\hat{n}\phi}{2}\right)$$

To each element of $SU(2)$ corresponds an element of $SO(3)$ which is the right-handed rotation by angle ϕ around the axis $\hat{n}$. This correspondence is not one-to-one since a rotation by 2π is equivalent to no rotation in $SO(3)$, i.e. the identity element of $SO(3)$, however this does not correspond to the identity element in $SU(2)$. The smallest nonzero value of ϕ that gives the identity element of $SU(2)$ is 4π. This means the correspondence is two-to-one.
However, to each *infinitesimal* element of $SO(3)$ corresponds a **unique** *infinitesimal* element of $SU(2)$, so locally the two groups are the same.

Topology of $SO(3)$

Consider the ball of radius π in 3 dimensional space. To each point of the ball corresponds an element of $SO(3)$ in the following way. To the center of the ball corresponds the identity element of $SO(3)$, i.e. no rotation. Every other point can be uniquely characterized by the direction $\hat{n}$ to that point from the center and the distance ϕ from the center, so we put it into correspondence with the right-handed rotation around $\hat{n}$ by angle ϕ. Thus we get all possible rotations, and each rotation only once, except the rotations by angle π since a right-handed rotation by π around $\hat{n}$ is identical with the right-handed rotation by π around $-\hat{n}$. This means that the antipodal points on the boundary of the ball correspond to identical rotations.
We may picture the topology of $SO(3)$ by considering the ball with the antipodal point on boundary glued together.

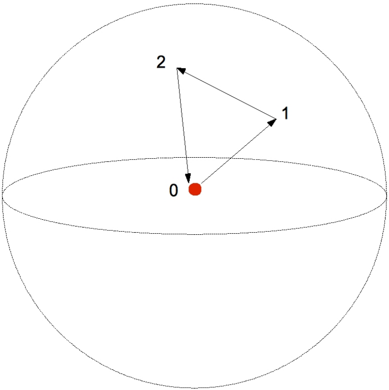
Topology of $SO(3)$



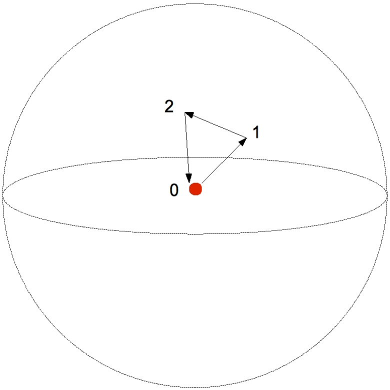
The colored points are on the boundary, and the same colored points are identified (glued together), which applies to **all of** the antipodal points of the boundary.

Topology of $SO(3)$, two types of loops

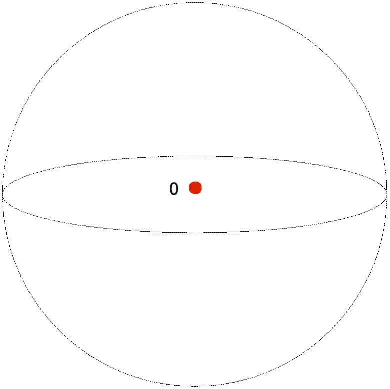
Now consider *loops* in $SO(3)$ starting at identity, i.e. we start with no rotation, then continuously do some rotations and come back to no rotation. Let us first consider the loop $0 \to 1 \to 2 \to 0$ in the picture, where we do not hit the boundary. Clearly this loop can be continuously deformed to doing nothing, i.e. this loop is *contractible*.



Topology of $SO(3)$, two types of loops

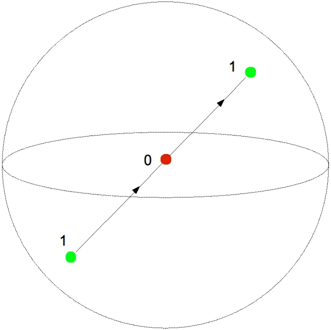


Topology of $SO(3)$, two types of loops



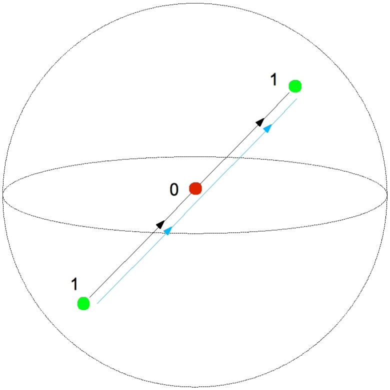
Topology of $SO(3)$, two types of loops

Now consider the loop $0 \to 1 \to 0$ below (recall that the green points are identical) which is equivalent to doing a 360° rotation around that axis. Although we again come back to no rotation, there is no way we can continuously deform this loop to doing nothing, we cannot *unwind* the loop, so this loop is *non-contractible*. Although after a 360° rotation we come back to no rotation, the *history* of doing that rotation cannot be erased!

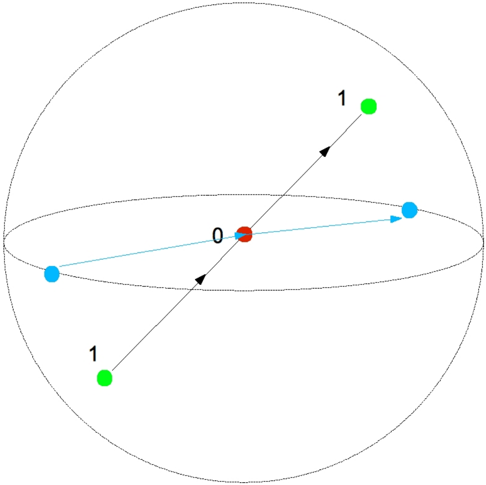


Topology of $SO(3)$, two types of loops

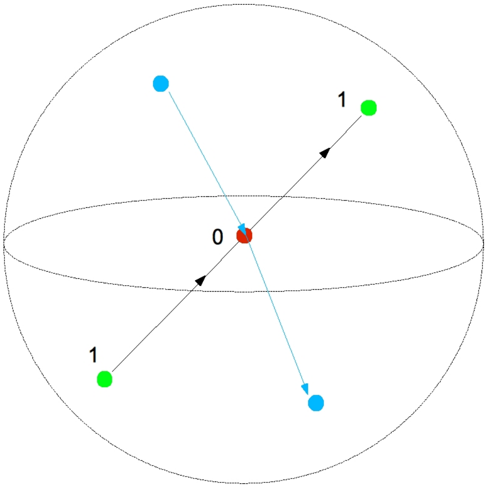
Imagine now doing a 360° rotation twice, i.e. doing the loop $0 \to 1 \to 0 \to 1 \to 0$ below. This one in fact *can be unwound*, i.e. continuously transformed to doing nothing, in the following way. Continuously rotate the second (blue) loop to become the inverse of the first loop.



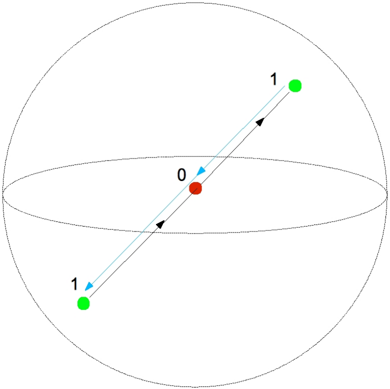
Topology of $SO(3)$, two types of loops



Topology of $SO(3)$, two types of loops

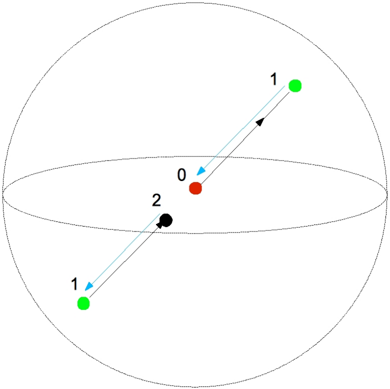


Topology of $SO(3)$, two types of loops



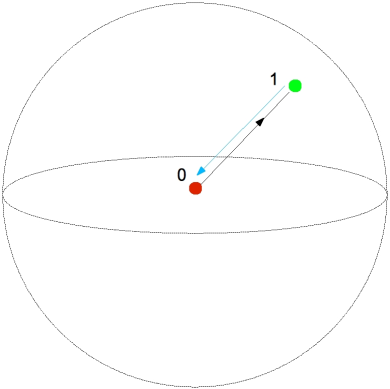
Now we can continuously cancel the two loops against each other.

Topology of $SO(3)$, two types of loops



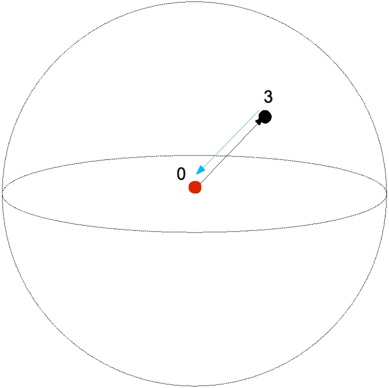
The loop is continuously transformed from $0 \to 1 \to 0 \to 1 \to 0$ to $0 \to 1 \to 2 \to 1 \to 0$.

Topology of $SO(3)$, two types of loops



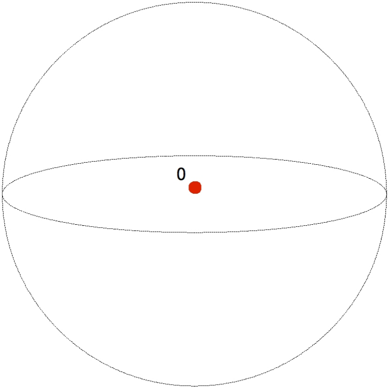
Then to $0 \rightarrow 1 \rightarrow 0$.

Topology of $SO(3)$, two types of loops



Then to $0 \to 3 \to 0$.

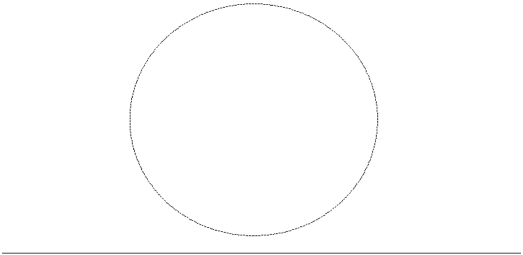
Topology of $SO(3)$, two types of loops



And finally to doing nothing!

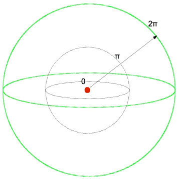
Simply connected spaces and covers

A space that does not have non-contractible loops is called *simply connected*. Under some assumptions, non-simply connected spaces can be *covered* by bigger spaces that are *locally the same* as the original space. A cover that is simply connected is called *the universal cover* in the sense that it cannot be covered by anything bigger.
An example of non-simply connected space is the circle, the universal cover of which is the infinite line.



Covering $SO(3)$

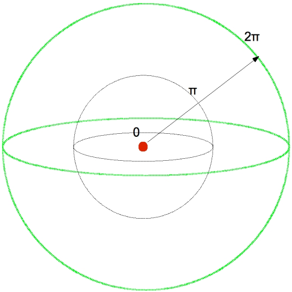
$SO(3)$ is non-simply connected because of the existence of non-contractible loops. So we may cover it by a bigger space, while still keeping the local structure. Now instead of identifying the antipodal points of the boundary, we consider them as different points and allow going further by another π.



Consider a line from the center to the boundary. Since this is a cover of $SO(3)$ we may project this line down to $SO(3)$, where it becomes a 2π rotation around the same axis. But in $SO(3)$ this corresponds to the identity element, so the endpoint of our line must correspond to the same point in $SO(3)$ independent of the direction. So we must identify all of the points of the boundary of our new space together!

Topology of $SU(2)$

The new space that we obtained is in fact $SU(2)$ which is a double cover of $SO(3)$. There are no non-contractible loops in this space, so it may not be covered by anything bigger. A full 2π rotation in $SO(3)$, which corresponds to the identity element of $SO(3)$, does not correspond to the identity element in $SU(2)$, it corresponds to the green boundary. However, two such rotations do bring us back to the identity element in $SU(2)$.



Representations of $SU(2)$

A representation of the group is a correspondence of the group elements to some linear operators. In quantum mechanics, these operators act on the states and give the change of the state under the symmetry operation of the group. To the identity element of the group must correspond the identity operator!

Let us consider possible representations of $SU(2)$. The 4π rotation in $SU(2)$ is the identity element of $SU(2)$, so its representing operator must be the identity. 2π rotations do not satisfy this requirement, however, after rotating by 2π we must clearly come back to the same *physical state*, i.e. measurements should not be affected, so it may at most correspond to a multiplication by a phase factor $e^{i\phi}$. However, doing this twice gives a 4π rotation which **must** correspond to the identity, so $(e^{i\phi})^2 = 1$ which implies $e^{i\phi} = \pm 1$. The representations for which $e^{i\phi} = 1$ are integer spin states, those with $e^{i\phi} = -1$ are half-integer spin states.

Representations of $SU(2)$

Note that if we restricted ourselves to $SO(3)$, we would get only integer spin states, such as orbital angular momentum. The possibility of extending it to a bigger group allowed for the possibility of half-integer states.

Since $SU(2)$ is the biggest cover, we are not able to go any further and get states like $1/4$ integer spin states and so on!

Reason for antiparticles (following Feynman's Dirac Memorial Lecture in 1986)

Let us consider spin-zero particles for the moment (which are not affected by any rotation). Consider a particle which is initially in state $|\phi_0>$. Consider a disturbing potential U_1 at time $t = 0$ which acts for a very short time. Then the amplitude to end up in another state $|\phi>$ is given by

$$A(\phi_0 \to \phi) = -i < \phi|U_1|\phi_0 >$$

Reason for antiparticles

Now suppose that after some time t has passed another disturbing potential U_2 acts. We are interested in the amplitude to end up in the initial state $|\phi_0>$. We calculate the amplitude by perturbation theory and always keep terms up to second order in both disturbances. Assuming for simplicity that each one of the disturbances does not take the particle back to $|\phi_0>$ we get two possibilities up to second order, either the particle is not affected by either of the disturbances or it is taken to some intermediate state $|p>$ by the first one, then back to the original state by the second one. We consider the complete set of free eigenstates with momentum p and energy E to account for the free motion of the particle between the two disturbances.

Reason for antiparticles

The total amplitude is then given by

$$A(\phi_0 \to \phi_0) =< \phi_0|\phi_0 >$$

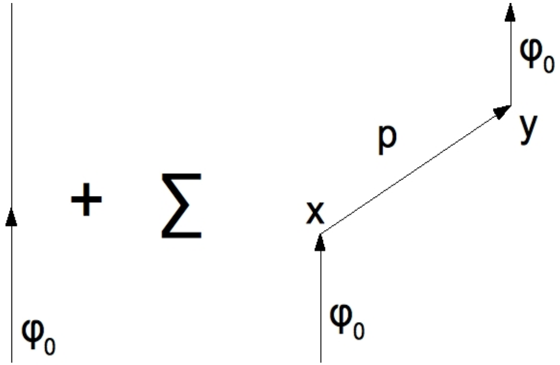
$$+\int d^3p(-i < \phi_0|U_2 \exp(-i\hat{H}t)|p > (-i < p|U_1|\phi_0 >)$$

$$= 1 - \int d^3pe^{-iEt} < \phi_0|U_2|p >< p|U_1|\phi_0 >$$

$$= 1 - \int d^3x\, d^3y \int d^3pe^{-iEt-ip(y-x)}\phi_0^*(y)U_2(y)U_1(x)\phi_0(x)$$

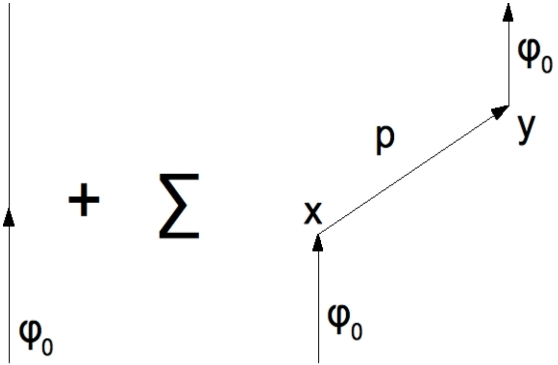
Reason for antiparticles

We can represent the two terms in the amplitude diagrammatically



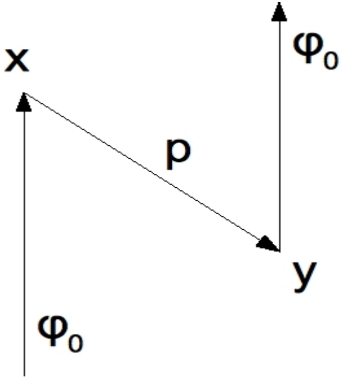
Reason for antiparticles

Now let us consider the requirement of causality. We would expect that the second term in the amplitude would give no contribution for *x* and *y* separated by a spacelike interval. However, this turns out not to be the case, even if we use the relativistic expression for the energy (or even any other reasonable function of momentum)!



Reason for antiparticles

If x and y are spacelike separated then by boosting to another frame we see that y happens earlier then x, and our second diagram looks like



So what we see is that at point y a pair of particles is created, one of which *travels backwards in time*, hits the initial particle at point x later on and both are destroyed!

Antiparticles, CPT

We see that for quantum mechanics and relativity to consistently work together, particles “going back in time” are required. We identify these to be the *antiparticles* of the original particles. So a particle going from x to y backwards in time is equivalent to a particle going forward in time from y to x. So if we invert the space (operation P) and the time (operation T) then the result will be to change the particles to their antiparticles, which we call charge conjugation (operation C). If we do all these operations together then we must come back to our initial state, in other words

$$CPT = 1$$

A virtual particle in one reference frame may become a virtual antiparticle in another!

Causality

If we allow the existence of antiparticles, then the causality violation problem observed above can be solved in the following way. Although a virtual particle can travel between two spacelike points, an antiparticle is also allowed to travel in the opposite direction. It can be shown that the amplitudes of these two processes exactly cancel each other, which means that no measurement can detect any causality violation!

Spin-half particles, time reversal

Let us now consider spin-half particles. We denote the spin up eingenstate in the z direction by $|\uparrow>$ and the spin down eigenstate by $|\downarrow>$. We know from quantum mechanics that the spin eingenstates in the x direction are given by

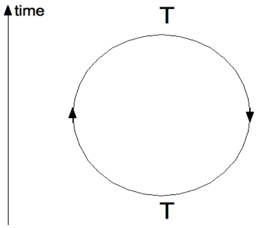
$$|x \uparrow>= \frac{1}{\sqrt{2}}(|\uparrow> + |\downarrow>)$$

$$|x \downarrow>= \frac{1}{\sqrt{2}}(|\uparrow> - |\downarrow>)$$

Time reversal flips the spins, however there may be some additional phase. By the corresponding choice of the arbitrary phase difference between $|\uparrow>$ and $|\downarrow>$ we may always assume that $T|\uparrow>=|\downarrow>$. However, in order to get a flip of the spin in the x direction as well, we must take $T|\downarrow>= -|\uparrow>$. So applying T twice on $|\uparrow>$ (and similarly on any other state) we get back the same state but with an extra minus sign!

Spin-half particles, loops, exchange

If we consider Feynman diagrams, then any loop with a fermion involves two time reversals, which means that it gains an extra minus sign!



The fact that an exchange of two identical fermions must be accompanied by a minus sign is a direct consequence of the minus sign of the loops (the proof involves some technical details)!
It follows directly that two Fermions may not be found in the same state, which is the Pauli exclusion principle!

Summary

- Half integer spin states are allowed because of the presence of non-contractible loops in $SO(3)$, and the possible extension of $SO(3)$ to $SU(2)$.

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- Since $SU(2)$ is the biggest allowed extension of $SO(3)$, states with smaller spin (such as $1/4$) are not allowed.
- Fermions get a minus sign after a 360° rotation, as well as two time reversals. As a result, fermionic loops get a minus sign.

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- Since $SU(2)$ is the biggest allowed extension of $SO(3)$, states with smaller spin (such as $1/4$) are not allowed.
- Fermions get a minus sign after a 360° rotation, as well as two time reversals. As a result, fermionic loops get a minus sign.
- The exchange of two fermions brings a minus sign. Two (or more) fermions are not allowed to be in the same state.

THANK YOU