Physics 214 UCSD/225a UCSB

Lecture 12

- Halzen & Martin Chapter 6
  - Spin averaged e-mu scattering
    - Introduce traces, and allude to trace theorems.
  - Use crossing to derive spin averaged e^+e^- -> mu^+mu^-
  - Detailed discussion of the result.

Spinless vs Spin 1/2

$$\phi\left(t,\vec{x}\right) = Ne^{-ip_\mu x^\mu}$$

$$\psi\left(t,\vec{x}\right) = u(p)e^{-ip_\mu x^\mu}$$

$$J_\mu = -ie\left(\phi^*(\partial_\mu\phi\ ) - (\partial_\mu\phi^*)\phi\ \right)$$

$$T_{fi} = -i\int J^\mu_{fi}A_\mu d^4x + O(e^2)$$

$$J^\mu = -e\overline{\psi}\gamma^\mu\psi$$

$$T_{fi} = -i\int J^\mu_{fi}A_\mu d^4x + O(e^2)$$

We basically make a substitution:

$$\left(p_f + p_i\right)_\mu \rightarrow \overline{u}_f\gamma_\mu u_i$$

And all else in calculating |M|² remains the same.

Example: e⁻ e⁻ scattering

For Spinless (i.e. bosons) we showed:

$$M = -e^2\left(\frac{(p_A+p_C)^\mu (p_B+p_D)_\mu}{(p_A-p_C)^2} + \frac{(p_A+p_D)^\mu (p_B+p_C)_\mu}{(p_A-p_D)^2}\right)$$

For Spin 1/2 we thus get:

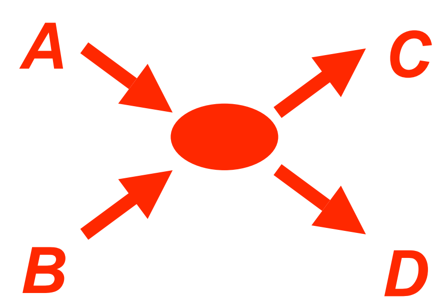
$$M = -e^2\left(\frac{(\bar{u}_c\gamma^\mu u_A)(\bar{u}_D\gamma_\mu u_B)}{(p_A-p_C)^2} - \frac{(\bar{u}_D\gamma^\mu u_A)(\bar{u}_C\gamma_\mu u_B)}{(p_A-p_D)^2}\right)$$

Minus sign comes from fermion exchange !!!

Spin Averaging

- The M from previous page includes spinors in initial and final state.
- In many experimental situations, in particular in hadron collissions, you neither fix initial nor final state spins.
- We thus need to form a spin averaged amplitude squared before we can compare with experiment:

$$\overline{|M|^2} = \frac{1}{(2s_A+1)(2s_B+1)} \sum_{spin} |M|^2 = \frac{1}{4} \sum_{spin} |M|^2$$



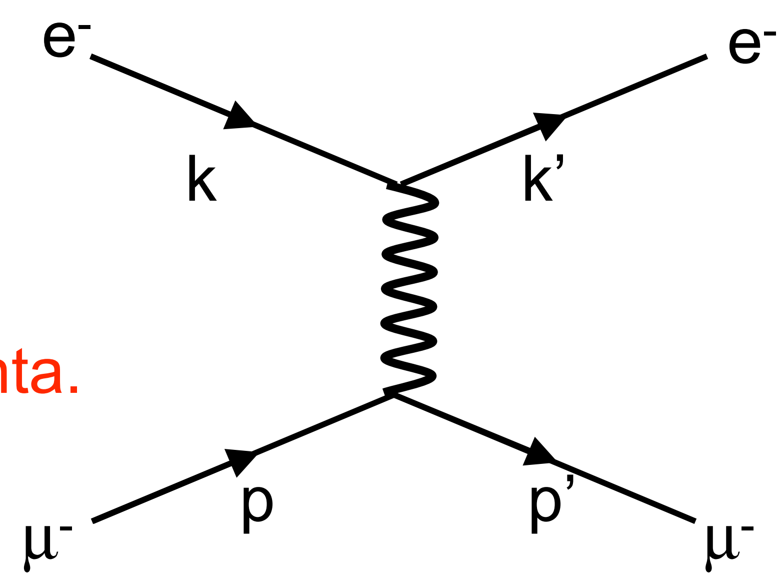
Last time we did the nonrelativistic case.
This time we do the complete derivation.

Note: This is quite possibly the most painful derivation we do this quarter.

“Easiest”: e- mu- -> e- mu-

- Easiest because it has only one diagram !!!

k for electron momenta.
p for muon momenta.
prime for outgoing momenta.



$$M = -e^2 \frac{\left[\bar{u}(k')\gamma^{\mu}u(k)\right]\left[\bar{u}(p')\gamma_{\mu}u(p)\right]}{t}$$

$t = q^2 = (k'-k)^2 = (p'-p)^2$ = scalar product of 4-momenta

Spin averaged $|M|^2$ (1)

$$\overline{|M|^2} = \frac{1}{(2s_A+1)(2s_B+1)}\sum_{spin}|M|^2 = \frac{1}{4}\sum_{spin}|M|^2$$

$$\overline{|M|^2} = \frac{e^4}{4t^2}\sum_{spin}\left(\bar{u}(k')\gamma^{\mu}u(k)\right)g_{\mu\nu}\left(\bar{u}(p')\gamma^{\nu}u(p)\right)\left[\left(\bar{u}(k')\gamma^{\rho}u(k)\right)g_{\rho\sigma}\left(\bar{u}(p')\gamma^{\sigma}u(p)\right)\right]^*$$

This is a Scalar product of 4-vectors,
multiplied by its complex conjugate,
to get a positive definite number.

Unfortunately, the scalar product mixes e and mu currents.

To form the spin average, we separate e and mu,
to execute the spin average on e and mu independently

Spin averaged $|M|^2$ (2)

$$\overline{|M|^2} = \frac{e^4}{4t^2}\sum_{spin}\left(\bar{u}(k')\gamma^{\mu}u(k)\right)g_{\mu\nu}\left(\bar{u}(p')\gamma^{\nu}u(p)\right)\left[\left(\bar{u}(k')\gamma^{\rho}u(k)\right)g_{\rho\sigma}\left(\bar{u}(p')\gamma^{\sigma}u(p)\right)\right]^*$$

$$\overline{|M|^2} = \frac{e^4}{t^2} L^{\mu\rho}_{electron} L^{\nu\sigma}_{muon} g_{\mu\nu} g_{\rho\sigma}$$

Where we defined:

$$L^{\mu\rho}_{electron} = \frac{1}{2}\sum_{spin}\left(\bar{u}(k')\gamma^{\mu}u(k)\right)\left(\bar{u}(k')\gamma^{\rho}u(k)\right)^*$$

We can now focus on doing the sum over spins for this tensor!

Summary of where we are

$$M = -e^2 \frac{\left[\bar{u}(k')\gamma^{\mu}u(k)\right]\left[\bar{u}(p')\gamma_{\mu}u(p)\right]}{t}$$

$$\overline{|M|^2} = \frac{e^4}{t^2} L_{electron}{}^{\mu\nu} L^{muon}{}_{\mu\nu}$$

$$L_{electron}{}^{\mu\nu} = \frac{1}{2}\sum_{e-spins}\left[\bar{u}(k')\gamma^{\mu}u(k)\right]\left[\bar{u}(k')\gamma^{\nu}u(k)\right]^*$$

$$L_{muon}{}^{\mu\nu} = \frac{1}{2}\sum_{muon-spins}\left[\bar{u}(p')\gamma^{\mu}u(p)\right]\left[\bar{u}(p')\gamma^{\nu}u(p)\right]^*$$

All that's left to do is the sum over spins.

Note the structure of this expression:

$$\left[\bar{u}(p')\gamma^{\nu}u(p)\right]^{*}=\left[\left(\qquad\qquad\right)(4x4)\begin{pmatrix}\ \\ \ \\ \ \end{pmatrix}\right]^{\nu}=(1x1)^{\nu}$$

The complex conjugate of this object is thus identical to the hermitian conjugate of this !!!

We can use the latter in order to rearrange terms, while ignoring the lorentz index for the moment.

$$\left[\bar{u}(k')\gamma^{\nu}u(k)\right]^{T*} = \left[u^{T*}(k')\gamma^{0}\gamma^{\nu}u(k)\right]^{T*} = \left[u^{T*}(k)\gamma^{\nu T*}\gamma^{0}u(k')\right] = \left[\bar{u}(k)\gamma^{\nu}u(k')\right]$$

Where in the last step , we used the commutation properties.
To be explicit:

$$\left[\gamma^{0}\gamma^{\nu}\right]^{T*} = \gamma^{\nu T*}\gamma^{0} = -\gamma^{\nu}\gamma^{0} = \gamma^{0}\gamma^{\nu}$$

$$\left[\gamma^{0}\right]^{T*} = \gamma^{0}$$

for ν=1,2,3

$$\left[\gamma^{\nu}\right]^{T*} = -\gamma^{\nu}$$

At this point we get the electron tensor:

$$L_{electron}{}^{\mu\nu} = \frac{1}{2}\sum_{s,s'}\left[\bar{u}(k')\gamma^{\mu}u(k)\right]\left[\bar{u}(k)\gamma^{\nu}u(k')\right]$$

Next, we are going to make all summations explicit, by writing out the gamma-matrices and spinor-vectors as components.

$$L_{electron}{}^{\mu\nu} = \frac{1}{2}\sum_{s,s'}\left[\overline{u}(k')\gamma^{\mu}u(k)\right]\left[\overline{u}(k)\gamma^{\nu}u(k')\right]$$

$$= \frac{1}{2}\sum_{s,s'}\sum_{ijlm}\left[\overline{u}_i^{s'}(k')\gamma_{ij}^{\mu}u_j^{s}(k)\right]\left[\overline{u}_l^{s}(k)\gamma_{lm}^{\nu}u_m^{s'}(k')\right]$$

$$= \frac{1}{2}\sum_{ijlm}\gamma_{ij}^{\mu}\gamma_{lm}^{\nu}\sum_{s'}\left[\overline{u}_i^{s'}(k')u_m^{s'}(k')\right]\sum_{s}\left[\overline{u}_l^{s}(k)u_j^{s}(k)\right]$$

Here we now apply the completeness relations:

$$\sum_{s} u^{s}(p)\overline{u}^{s}(p) = p_{\mu}\gamma^{\mu} + m = (4x4)$$

See H&M exercise 5.9 for more detail on completeness relation.

$$L_{electron}{}^{\mu\nu} = \frac{1}{2}\sum_{ijlm}\gamma^{\mu}_{ij}\gamma^{\nu}_{lm}\sum_{s'}\left[\bar{u}^{s'}_{i}(k')u^{s'}_{m}(k')\right]\sum_{s}\left[\bar{u}^{s}_{l}(k)u^{s}_{j}(k)\right]$$

Now apply to this the relationship:

$$\sum_{s} u^{s}(p)\bar{u}^{s}(p) = p_{\mu}\gamma^{\mu} + m = (4x4)$$

And you get as a result:

$$L_{electron}{}^{\mu\nu} = \frac{1}{2}\sum_{ijlm}\left[k'_{\alpha}\gamma^{\alpha} + m\right]_{mi}\gamma^{\mu}_{ij}\left[k_{\alpha}\gamma^{\alpha} + m\right]_{jl}\gamma^{\nu}_{lm}$$

Mathematical aside:

Let A,B,C,D be 4 matrices.

$$\sum_{ijlm} A_{ij} B_{jl} C_{lm} D_{mi} \equiv Tr\left(A \cdot B \cdot C \cdot D\right)$$

This weird sum is thus nothing more than the trace of the product of matrices !!!

I won't prove this here, but please feel free to convince yourself.

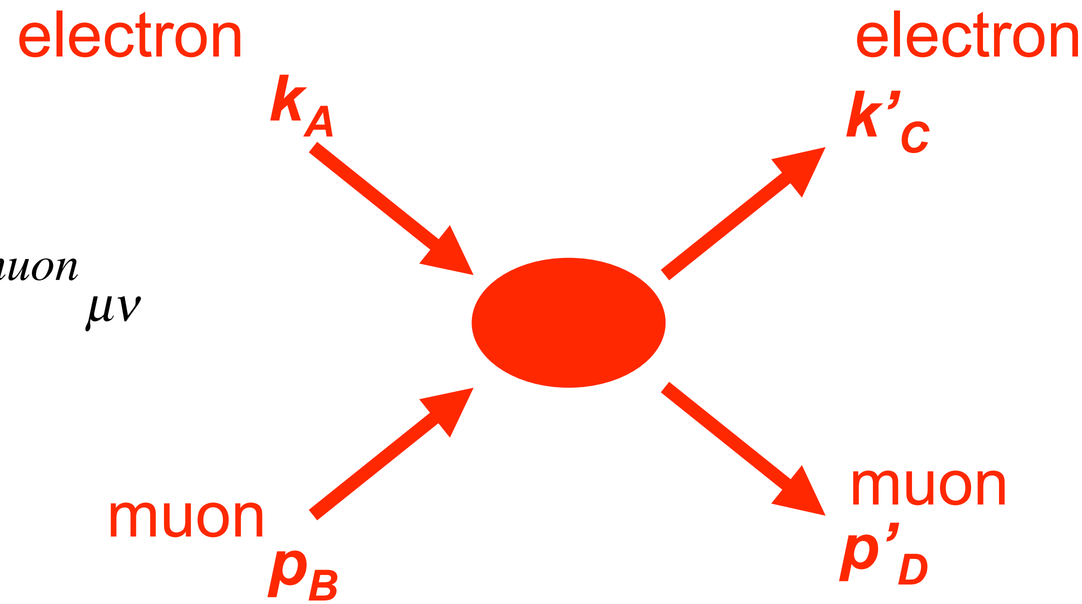
$$L_{electron}{}^{\mu\nu} = \frac{1}{2} \sum_{ijlm} \left[k'_\alpha \gamma^\alpha + m\right]_{mi} \gamma^\mu_{ij} \left[k_\alpha \gamma^\alpha + m\right]_{jl} \gamma^\nu_{lm}$$

$$L_{electron}{}^{\mu\nu} = \frac{1}{2} Tr\left[\left(k'_\alpha \gamma^\alpha + m\right) \gamma^\mu \left(k_\beta \gamma^\beta + m\right) \gamma^\nu\right]$$

$$L_{electron}{}^{\mu\nu} = \frac{1}{2} Tr\left[\left(k'_\alpha \gamma^\alpha + m \right) \gamma^\mu \left(k_\beta \gamma^\beta + m \right) \gamma^\nu \right]$$

$$L_{muon}{}^{\mu\nu} = \frac{1}{2} Tr\left[\left(p'_\alpha \gamma^\alpha + m \right) \gamma^\mu \left(p_\beta \gamma^\beta + m \right) \gamma^\nu \right]$$

$$\overline{\left| M \right|^2} = \frac{e^4}{t^2} L_{electron}{}^{\mu\nu} L^{muon}{}_{\mu\nu}$$



"All" that's left to do is apply trace theorems.
(There's a whole bunch of them in H&M p.123)

$$L_{electron}{}^{\mu\nu} = \frac{1}{2} Tr\left[\left(k'_\alpha \gamma^\alpha + m\right)\gamma^\mu\left(k_\beta \gamma^\beta + m\right)\gamma^\nu\right]$$

$$= \frac{1}{2} Tr\left[k'_\alpha \gamma^\alpha \gamma^\mu k_\beta \gamma^\beta \gamma^\nu + k'_\alpha \gamma^\alpha \gamma^\mu \gamma^\nu m + m\gamma^\mu k_\beta \gamma^\beta \gamma^\nu + m^2 \gamma^\mu \gamma^\nu\right]$$

Trace of product of any 3 gamma matrices is zero!

$$L_{electron}{}^{\mu\nu} = \frac{1}{2} Tr\left[k'_\alpha \gamma^\alpha \gamma^\mu k_\beta \gamma^\beta \gamma^\nu\right] + \frac{m^2}{2} Tr\left[\gamma^\mu \gamma^\nu\right]$$

Next, define unit vectors a,b for μ and ν coordinate:

$$a_\iota \gamma^\iota \equiv \gamma^\mu$$

$$b_\kappa \gamma^\kappa \equiv \gamma^\nu$$

This will allow us to use trace theorems to evaluate the remaining traces.

Aside

- What does this mean?

$$a_\iota \gamma^\iota \equiv \gamma^\mu$$

$$b_\kappa \gamma^\kappa \equiv \gamma^\nu$$

$a_\iota \gamma^\iota$

Is a scalar product of 4-vectors.
As a_l is a unit vector, it projects out a component of γ^l .
The components of γ^l are themselves 4x4 matrices.

We introduce this to be able to use this trace theorem:

$$Tr\left[\not{a}\not{b}\not{c}\not{d}\right] = 4\left[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)\right]$$

2 Trace Theorems to use:

(1) (2) (3) (4)

$$Tr\left[(k'_\alpha\gamma^\alpha)(a_\iota\gamma^\iota)(k_\beta\gamma^\beta)(b_\kappa\gamma^\kappa)\right]=$$

(1x2) (3x4) - (1x3) (2x4) + (1x4) (2x3)

$$=4\left[(k'\cdot a)(k\cdot b)-(k'\cdot k)(a\cdot b)+(k'\cdot b)(k\cdot a)\right]$$

$$=4\left[k'^\mu k^\nu-(k'\cdot k)g^{\mu\nu}+k^\mu k'^\nu\right]$$

$$Tr\left[(a_\iota\gamma^\iota)(b_\kappa\gamma^\kappa)\right]=4a\cdot b=4g^{\mu\nu}$$

Now put it all together …

Electron - Muon scattering

$$L^{\mu\nu}_{electron} = 2\left[k'^{\mu}k^{\nu} + (m^2 - k'\cdot k)g^{\mu\nu} + k^{\mu}k'^{\nu}\right]$$

$$L^{muon}_{\mu\nu} = 2\left[p'_{\mu}p_{\nu} + (M^2 - p'\cdot p)g_{\mu\nu} + p_{\mu}p'_{\nu}\right]$$

$$\overline{|M|^2} = \frac{e^4}{t^2} L_{electron}{}^{\mu\nu} L^{muon}{}_{\mu\nu}$$

$$= \frac{8e^4}{t^2}\left[(k'p')(kp) + (k'p)(kp') - m^2 pp' - M^2 kk' + 2M^2m^2\right]$$

This is the "exact" form. Next look at relativistic approx.

Aside: $g_{\mu\nu}g^{\mu\nu} = 4$

Relativistic approx. for e-mu scattering:

$$\overline{|M|^2} = \frac{8e^4}{t^2}\left[(k'p')(kp) + (k'p)(kp')\right]$$

Let's use the invariant variables:

s = (k+p)^2 ~ 2kp ~ 2k'p'

u = (k - p')^2 ~ -2kp' ~ -2k'p

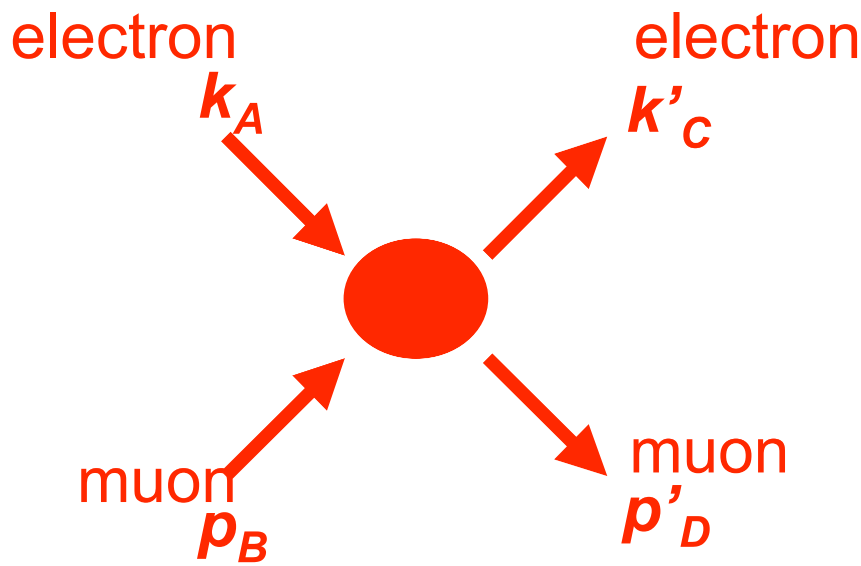
$$\boxed{\overline{|M|^2} = 2e^4 \frac{\left(s^2 + u^2\right)}{t^2}}$$

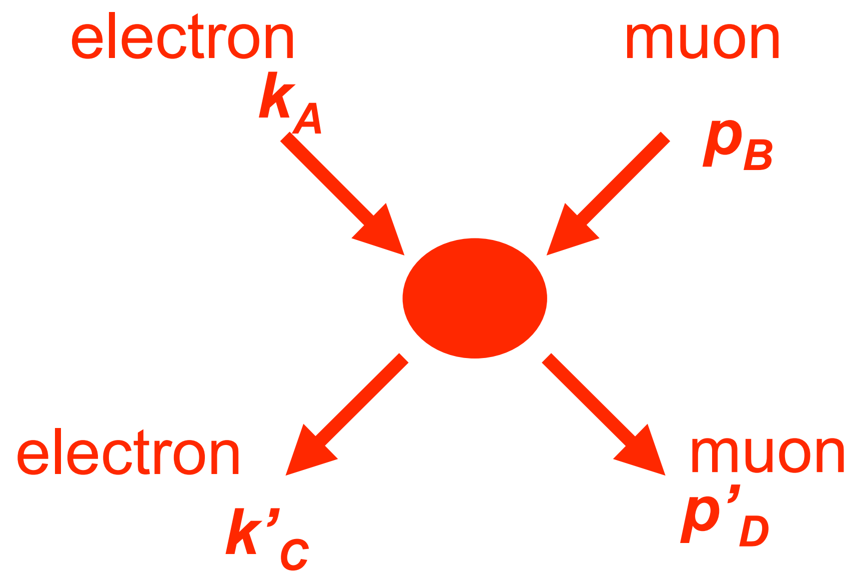
Next look at ee -> mumu, and get it via crossing.

emu -> emu => ee -> mumu

via crossing

$$k'_C \longleftrightarrow -p_B$$





$s = (k+p)^2$
$t = (k' - k)^2$

$$s \longleftrightarrow t$$

"s" $= (k' - k)^2 = t$
"t" $= (k+p)^2 = s$

e-mu- -> e-mu- => e+e- -> mu+mu-

$$\overline{|M|^2} = 2e^4 \frac{\left(s^2 + u^2\right)}{t^2} \qquad \overline{|M|^2} = 2e^4 \frac{\left(t^2 + u^2\right)}{s^2}$$

Recall exercise 4.2 from H&M: $$\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \frac{|M|^2}{64\pi^2 s} \frac{p_f}{p_i}$$

We will now use this for relativistic e+e- -> mu+mu- scattering.

$p_f \sim p_i$ **=>** $$\left.\frac{d\sigma}{d\Omega}\right|_{cm} \approx \frac{|M|^2}{64\pi^2 s}$$

(because we can neglect masses)

(see next slide)

$t = -2\,k^2\,(1 - \cos\theta)$

$u = -2\,k^2\,(1 + \cos\theta)$ **=>** $$\overline{|M|^2} = e^4\left(1 + \cos^2\theta\right)$$

$s \sim 4k^2$

$\alpha = e^2/4\pi$

$$\boxed{\begin{gathered}\left.\frac{d\sigma}{d\Omega}\right|_{cm} \approx \frac{\alpha^2}{4s}\left(1 + \cos^2\theta\right) \\ \sigma(e^+e^- \to \mu^+\mu^-) \approx \frac{4\pi\alpha^2}{3s}\end{gathered}}$$

Relativistic limit

Aside on Algebra

$t = -2\,k^2\,(1 - \cos\theta)$

$u = -2\,k^2\,(1 + \cos\theta)$

$s \sim 4k^2$

$$\overline{|M|^2} = 2e^4 \frac{\left(t^2 + u^2\right)}{s^2}$$

$$\frac{\left(t^2 + u^2\right)}{s^2} = \frac{4k^4\left[1 - 2\cos\theta + \cos^2\theta + 1 + 2\cos\theta + \cos^2\theta\right]}{16k^4}$$

$$\frac{\left(t^2 + u^2\right)}{s^2} = \frac{\left(1 + \cos^2\theta\right)}{2}$$

$$\boxed{\overline{|M|^2} = e^4\left(1 + \cos^2\theta\right)}$$

Result worthy of discussion

1. $\sigma \propto 1/s$ must be so on dimensional grounds
2. $\sigma \propto \alpha^2$ two vertices!
3. At higher energies, Z-propagator also contributes:

More discussion

4. Calculation of e+e- -> q qbar is identical as long as sqrt(s) >> Mass of quark.

$$\sigma(e^+e^- \to q\bar{q}) = \underset{\text{\# of flavors}}{3} \cdot \underbrace{\sum_{q-flavor}}_{\text{Sum over quark flavors}} \underbrace{e_q^2}_{\text{Charge of quark}} \sigma(e^+e^- \to \mu^+\mu^-)$$

Measurement of this cross section was very important !!!

Measurement of R

$$R = \frac{\sigma(e^+e^- \rightarrow q\overline{q})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = 3 \cdot \sum_{q-flavor} e_q^2$$

Below charm threshold: $R = 3\,[\,(2/3)^2 + (1/3)^2 + (1/3)^2\,] = 2$

Between charm and bottom: $R = 2 + 3(4/9) = 10/3$

Above bottom: $R = 10/3 + 3(1/9) = 11/3$

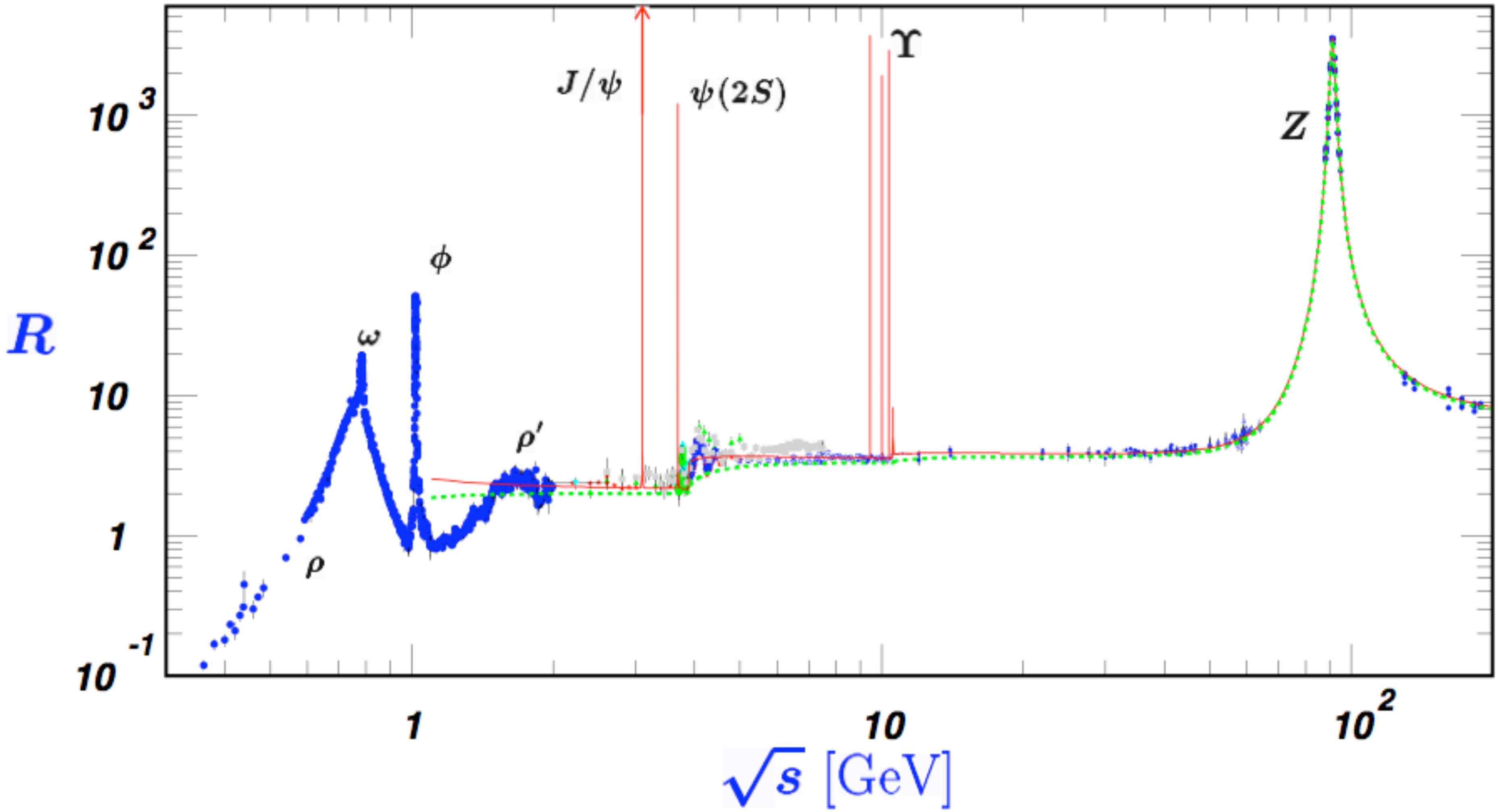
Measurement of R was crucial for:

a. Confirm that quarks have 3 colors

b. Search for additional quarks

c. Search for additional leptons

Experimental Result



Ever more discussion

5. $d\sigma/d\Omega \propto (1 + \cos^2\theta)$

5.1 θ is defined as the angle between e+ and mu+ in com. $\cos^2\theta$ means that the outgoing muons have no memory of the direction of incoming particle vs antiparticle.

Probably as expected as the e+e- annihilate before the mu+mu- is created.

5.2 Recall, phase space is flat in $\cos\theta$. $\cos^2\theta$ dependence thus implies that the initial state axis matters to the outgoing particles. Why?

Helicity Conservation in relativistic limit

- You showed as homework that u_L and u_R are helicity eigenstates in the relativistic limit, and thus:

$$\bar{u}\gamma^{\mu}u = (\bar{u}_L + \bar{u}_R)\gamma^{\mu}(u_L + u_R)$$

- We'll now show that the cross terms are zero, and helicity is thus conserved at each vertex.
- We then show how angular momentum conservation leads to the cross section we calculated.

Let' do one cross product explicitly:

$$\bar{u}\gamma^{\mu}u = (\bar{u}_L + \bar{u}_R)\gamma^{\mu}(u_L + u_R)$$

$$\bar{u}_L = u_L^{T*}\gamma^0 = u^{T*}\frac{1}{2}(1-\gamma^5)\gamma^0 = \bar{u}\frac{1}{2}(1+\gamma^5)$$

$$u_R = \frac{1}{2}(1+\gamma^5)u$$

$$\bar{u}_L\gamma^{\mu}u_R = \bar{u}\frac{1}{4}(1+\gamma^5)\gamma^{\mu}(1+\gamma^5)u = \bar{u}\gamma^{\mu}\frac{1}{4}(1-\gamma^5)(1+\gamma^5)u = 0$$

Here we have used:

$$\gamma^5\gamma^{\mu} = -\gamma^{\mu}\gamma^5$$

$$\gamma^5 = \gamma^{5T*}$$

$$\gamma^5\gamma^5 = 1$$

Helicity conservation holds for all vector and axialvector currents as E>>m.

- $e_L^- e_R^+ \rightarrow mu_L^- mu_R^+$ $\quad J_z$ +1 -> +1
- $e_L^- e_R^+ \rightarrow mu_R^- mu_L^+$ $\quad J_z$ +1 -> -1
- $e_R^- e_L^+ \rightarrow mu_L^- mu_R^+$ $\quad J_z$ -1 -> +1
- $e_R^- e_L^+ \rightarrow mu_R^- mu_L^+$ $\quad J_z$ -1 -> -1
- *Next look at the rotation matrices:*

$$d^1_{11}(\theta) = \frac{1}{2}\left(1+\cos\theta\right) \approx \frac{-u}{s}$$

$$d^1_{-1-1}(\theta) = \frac{1}{2}\left(1+\cos\theta\right) \approx \frac{-u}{s}$$

$$d^1_{-11}(\theta) = \frac{1}{2}\left(1-\cos\theta\right) \approx \frac{-t}{s}$$

$$d^1_{1-1}(\theta) = \frac{1}{2}\left(1-\cos\theta\right) \approx \frac{-t}{s}$$

Cross products cancel in Spin average:

$$\overline{|M|^2} \propto \left(1+\cos^2\theta\right)$$

d^J_{1-1}

Initial J_z final J_z

Conclusion on relativistic limit

- Dependence on scattering angle is given entirely by angular momentum conservation !!!
- This is a generic feature for any vector or axialvector current.
- We will thus see the exact same thing also for V-A coupling of Electroweak interactions.