# Physics 214 UCSD/225a UCSB

## Lecture 11

- Finish Halzen & Martin Chapter 4
  - origin of the propagator
- Halzen & Martin Chapter 5
  - Continue Review of Dirac Equation
- Halzen & Martin Chapter 6
  - start with it if time permits

# Origin of propagator

- When we discussed perturbation theory a few lectures ago, we did what some call “old fashioned perturbation theory”.
  - It was not covariant
  - We required momentum conservation at vertex but not Energy conservation
  - At second order, we need to consider time ordered products.
- When you do this “more modern”
  - Fully covariant
  - 4-momentum is conserved at each vertex
  - However, “propagating particles” are off-shell
  - This is what you’ll learn in QFT!

# Spinless massive propagator

H&M has a more detailed discussion for how to go from a time ordered 2nd order perturbation theory to get the propagator. Here, we simply state the result:

$$\frac{1}{(p_A + p_B)^2 - m^2} = \frac{1}{p^2 - m^2}$$

For more details see Halzen & Martin

# H&M Chapter 5
# Review of Dirac Equation

- Dirac's Quandery
- Notation Reminder
- Dirac Equation for free particle
  - Mostly an exercise in notation
- Define currents
  - Make a complete list of all possible currents
- Aside on Helicity Operator
  - Solutions to free particle Dirac equation are eigenstates of Helicity Operator
- Aside on "handedness"

# Dirac's Quandery

- Can there be a formalism that allows wave functions that satisfy the linear and quadratic equations simultaneously:

$$H\psi = (\vec{\alpha}\vec{p} + \beta m)\psi$$

$$H^2\psi = (P^2 + m^2)\psi$$

- If such a thing existed then the linear equation would provide us with energy eigenvalues that automatically satisfy the relativistic energy momentum relationship

# Dirac's Quandery (2)

- Such a thing does indeed exist:
  - Wave function is a 4 component object

$$\alpha_i = \begin{pmatrix} & \sigma_i \\ \sigma_i & \end{pmatrix}$$

$$\beta = \begin{pmatrix} I & \\ & -I \end{pmatrix}$$

$\alpha$ is thus a 3-vector of 4x4 matrices with special commutator relationships like the Pauli matrices.
While $\beta$ is a diagonal 4x4 matrix as shown.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

# The rest is history

In the following I provide a very limited reminder of notation and a few facts.

If these things don't sound familiar, then I encourage you to work through ch. 5 carefully.

# Notation Reminder (1)

- Sigma Matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Gamma Matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix};\ \gamma^{j=1,2,3} = \begin{pmatrix} 0 & \sigma_{j=1,2,3} \\ -\sigma_{j=1,2,3} & 0 \end{pmatrix};\ \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

- State Vectors:

$$\psi = \begin{pmatrix} . \\ . \\ . \\ . \end{pmatrix};\ \overline{\psi} \equiv \psi^{T*}\gamma^0 = \begin{pmatrix} . & . & . & . \end{pmatrix}$$

# Notation Reminder (2)

- Obvious statements about gamma matrices

$$\gamma^{(j=0,2,5)T} = \gamma^{(j=0,2,5)}; \quad \gamma^{(j=1,3)T} = -\gamma^{(j=1,3)}$$

$$\gamma^{(j=0,5)T*} = \gamma^{(j=0,5)}; \quad \gamma^{(j=1,2,3)T*} = -\gamma^{(j=1,2,3)}$$

- Probability density

$$\overline{\psi} \equiv \psi^{T*}\gamma^0$$

$$\overline{\psi}\gamma^0\psi = \psi^{T*}\psi = \# \geq 0$$

- Scalar product of gamma matrix and 4-vector

$$\not{A} \equiv \gamma^\mu A_\mu = \gamma^0 A_0 - \gamma^1 A_1 - \gamma^2 A_2 - \gamma^3 A_3$$

Is again a 4-vector

# Dirac Equation of free particle

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi=0$$

$$i\partial_{\mu}\overline{\psi}\gamma^{\mu}+m\overline{\psi}=0$$

Ansatz:

$$\psi=e^{-ipx}u(p)$$

Explore this in restframe of particle:

$$\psi_{+1/2}=\sqrt{2m}e^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\quad \psi_{-1/2}=\sqrt{2m}e^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix};$$

Normalization chosen to describe 2E particles, as usual.

# Particle vs antiparticle in restframe

$$\psi_{+1/2} = \sqrt{2m}e^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\quad \psi_{-1/2} = \sqrt{2m}e^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix};$$

particle

Recall, particle -> antiparticle means E,p -> -E,-p

Let's take a look at Energy Eigenvalues:

$$Hu = (\vec{\alpha}\vec{p} + \beta m)u = Eu$$

$$\begin{pmatrix} mI & 0 \\ 0 & -mI \end{pmatrix} u = Eu \quad \longleftarrow$$

for p=0 we get this equation to satisfy by the energy eigenvectors

*It is thus obvious that 2 of the solutions have E < 0, and are the lower two components of the 4-component object u.*

# Particle & Anti-particle

$$\psi_{+1/2} = \sqrt{2m}e^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}; \quad \psi_{-1/2} = \sqrt{2m}e^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix};$$

Positive energy solution

$$\psi_{+1/2} = \sqrt{2m}e^{-imt}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}; \quad \psi_{-1/2} = \sqrt{2m}e^{-imt}\begin{pmatrix}0\\0\\0\\1\end{pmatrix};$$

Negative energy solution

# Not in restframe this becomes:

$$Hu = (\vec{\alpha}\vec{p} + \beta m)u = Eu$$

$$Hu = \begin{pmatrix} m & \vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & -m \end{pmatrix} u = Eu$$

The lower 2 components are thus coupled to the upper 2 via this matrix equation, leading to free particle and antiparticle solutions as follows.

# (Anti-)Particle not in restframe

$$\psi_{+1/2} = Ne^{-ipx}\begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma}\vec{p}}{E+m}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix}; \quad \psi_{-1/2} = Ne^{-ipx}\begin{pmatrix} 0 \\ 1 \\ \frac{\vec{\sigma}\vec{p}}{E+m}\begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix}$$

$$\psi_{+1/2} = Ne^{+ipx}\begin{pmatrix} \frac{-\vec{\sigma}\vec{p}}{|E|+m}\begin{pmatrix}1\\0\end{pmatrix} \\ 1 \\ 0 \end{pmatrix}; \quad \psi_{-1/2} = Ne^{+ipx}\begin{pmatrix} \frac{-\vec{\sigma}\vec{p}}{|E|+m}\begin{pmatrix}0\\1\end{pmatrix} \\ 0 \\ 1 \end{pmatrix}$$

***I suggest you read up on this in H&M chapter 5 if you're not completely comfortable with it.***

# What we learned so far:

- Dirac Equation has 4 solutions for the same p:
  - Two with E>0
  - Two with E<0
- The E<0 solutions describe anti-particles.
- The additional 2-fold ambiguity describes spin +-1/2.
  - You will show this explicitly in Exercise H&M 5.4, which is part of HW next week.
- We thus have a formalism to describe all the fundamental spin 1/2 particles in nature.

# Helicity Operator

- The helicity operator commutes with both H and P.
- Helicity is thus conserved for the free spin 1/2 particle.

$$\frac{1}{2}\begin{pmatrix} \vec{\sigma}\,\hat{p} & 0 \\ 0 & \vec{\sigma}\,\hat{p} \end{pmatrix}$$

The unit vector here is the axis with regard to which we define the helicity. For (0,0,1), i.e. the Z-axis, we get the desired +-1/2 eigenvalues.

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

# Dirac Equation for particle and anti-particle spinors

It's sometimes notationally convenient to write the antiparticle spinor solution (i.e. the -p,-E) as an explicit Antiparticle spinor that satisfies a modified dirac equation:

$$\left(\gamma^{\mu} p_{\mu} - m\right) u = 0$$ particle

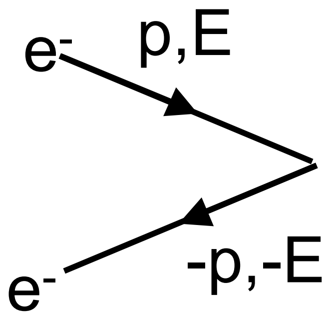
$$\left(\gamma^{\mu} p_{\mu} + m\right) v = 0$$ antiparticle

The v-spinor then has positive energy.
We won't be using v-spinors in this course.

H&M Equation (5.33) and (5.34)

# Antiparticles

- We will stick to the antiparticle description we introduced in chapter 4:




Initial state $e^+e^-$ is an initial state $e^-e^-$ with the positron being an electron going in the "wrong direction", i.e. "backwards in time".

# Some more reflections on $\gamma^\mu$

- There are exactly 5 distinct $\gamma$ matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix};\ \gamma^{j=1,2,3} = \begin{pmatrix} 0 & \sigma_{j=1,2,3} \\ -\sigma_{j=1,2,3} & 0 \end{pmatrix};\ \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

- Where $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$
- Every one of them multiplied with itself gives the unit matrix.
- As a result, any product of 5 of them can be expressed as a product of 3 of them.

# Currents

- Any bi-linear quantity can be a current as long as it has the most general form:

$$\overline{\psi}(4\,x\,4)\psi$$

- By finding all possible forms of this type, using the gamma-matrices as a guide, we can form all possible currents that can be within this formalism.

# The possible currents

$\bar{\psi}\psi$ scalar

$\bar{\psi}\gamma^5\psi$ pseudo-scalar

$\bar{\psi}\gamma^\mu\psi$ vector

$\bar{\psi}\gamma^5\gamma^\mu\psi$ pseudo-vector or "axial-vector"

$\bar{\psi}\frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)\psi$ tensor

# Chapter 6

## Electrodynamics of Spin-1/2 particles.

# Spinless vs Spin 1/2

$$\phi\left(t,\vec{x}\right) = Ne^{-ip_\mu x^\mu}$$

$$\psi\left(t,\vec{x}\right) = u(p)e^{-ip_\mu x^\mu}$$

$$J_\mu = -ie\left(\phi^*(\partial_\mu\phi) - (\partial_\mu\phi^*)\phi\right)$$

$$T_{fi} = -i\int J^\mu_{fi}A_\mu d^4x + O(e^2)$$

$$J^\mu = -e\overline{\psi}\gamma^\mu\psi$$

$$T_{fi} = -i\int J^\mu_{fi}A_\mu d^4x + O(e^2)$$

***We basically make a substitution of the vertex factor:***

$$\left(p_f + p_i\right)_\mu \rightarrow \overline{u}_f\gamma_\mu u_i$$

***And all else in calculating $|M|^2$ remains the same.***

# Example: e⁻ e⁻ scattering

For Spinless (i.e. bosons) we showed:

$$M = -e^2\left(\frac{(p_A + p_C)^\mu (p_B + p_D)_\mu}{(p_A - p_C)^2} + \frac{(p_A + p_D)^\mu (p_B + p_C)_\mu}{(p_A - p_D)^2}\right)$$

For Spin 1/2 we thus get:

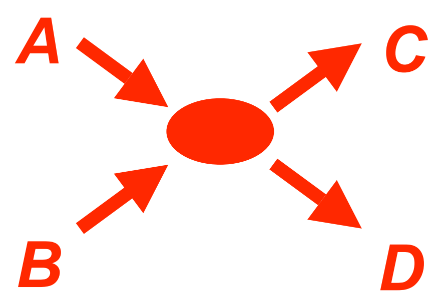
$$M = -e^2\left(\frac{(\bar{u}_c \gamma^\mu u_A)(\bar{u}_D \gamma_\mu u_B)}{(p_A - p_C)^2} - \frac{(\bar{u}_D \gamma^\mu u_A)(\bar{u}_C \gamma_\mu u_B)}{(p_A - p_D)^2}\right)$$

***Minus sign comes from fermion exchange !!!***

# Spin Averaging

- The M from previous page includes spinors in initial and final state.
- In many experimental situations, in particular in hadron collissions, you neither fix initial nor final state spins.
- We thus need to form a spin averaged amplitude squared before we can compare with experiment:

$$\overline{|M|^2} = \frac{1}{(2s_A+1)(2s_B+1)} \sum_{spin} |M|^2 = \frac{1}{4} \sum_{spin} |M|^2$$



# Spin Averaging in non-relativistic limit

- Incoming e⁻ :

$$u^{(s=+1/2)} = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

- Outgoing e⁻ :

$$\overline{u}^{(s=-1/2)} = \sqrt{2m}\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}$$

Reminder:

$$\gamma^0 = \begin{pmatrix}I & 0\\0 & -I\end{pmatrix}$$

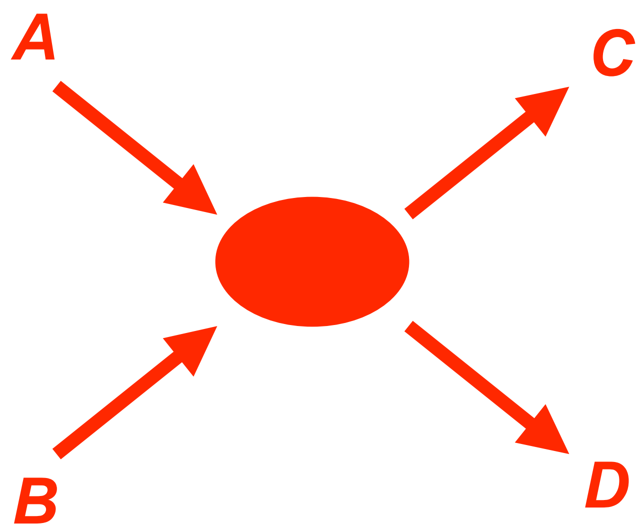
$$\gamma^{j=1,2,3} = \begin{pmatrix}0 & \sigma_{j=1,2,3}\\-\sigma_{j=1,2,3} & 0\end{pmatrix}$$

$$\overline{u}_f\gamma_\mu u_i = \begin{cases}2m \quad if \quad (\mu=0)\wedge s_i = s_f\\ 0 \quad otherwise\end{cases}$$

# Invariant variables s,t,u

Example: e- e- -> e- e-

- $s = ( p_A + p_B )^2$
- $\quad = 4 (k^2 + m^2)$



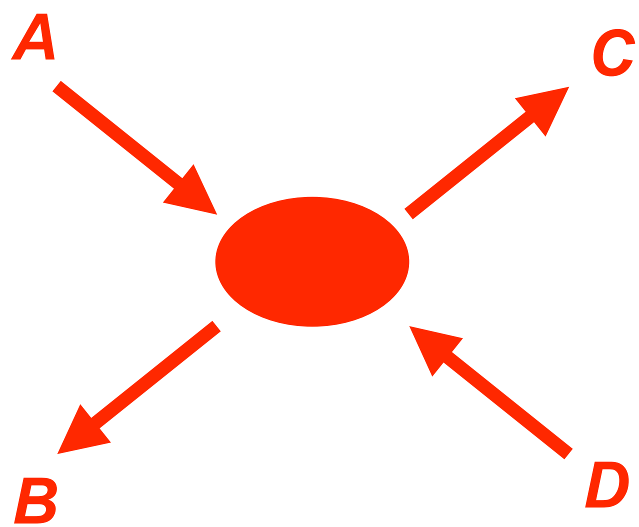

- $t = ( p_A - p_C )^2$
- $\quad = -2 k^2 (1 - \cos\theta)$

- $u = ( p_A - p_D )^2$
- $\quad = -2 k^2 (1 + \cos\theta)$

$k = |k_i| = |k_f|$ $\qquad m = m_e$ $\qquad \theta$ = scattering angle, all in com frame.

# Invariant variables s,t,u

- $s = ( p_A + p_B )^2$
- $= 4 (k^2 + m^2)$




B,D are antiparticles!
$p_B$ thus "negative", leading to the + in $(p_A+p_B)$.

$k = |k_i|= |k_f|$ $m = m_e$ $\theta$ = scattering angle, all in com frame.

# M for Different spin combos

$$M = -e^2 \left( \frac{(\bar{u}_c \gamma^\mu u_A)(\bar{u}_D \gamma_\mu u_B)}{t} - \frac{(\bar{u}_D \gamma^\mu u_A)(\bar{u}_C \gamma_\mu u_B)}{u} \right)$$

$$\left[ (\bar{u}_c \gamma^\mu u_A)(\bar{u}_D \gamma_\mu u_B) \right]_{\downarrow\uparrow\to\downarrow\uparrow} = 4m^2$$

$$\left[ (\bar{u}_D \gamma^\mu u_A)(\bar{u}_C \gamma_\mu u_B) \right]_{\uparrow\downarrow\to\downarrow\uparrow} = 4m^2$$

$$\left[ (\bar{u}_c \gamma^\mu u_A)(\bar{u}_D \gamma_\mu u_B) \right]_{\uparrow\downarrow\to\downarrow\uparrow} = 0$$

$etc.$

$$\overline{|M|^2} = \frac{1}{4} \left( 4m^2 e^2 \right)^2 2 \left[ \left( \frac{1}{t} - \frac{1}{u} \right)^2 + \frac{1}{t^2} + \frac{1}{u^2} \right]$$

# M for Different spin combos

$$M = -e^2\left(\frac{(\bar{u}_c\gamma^\mu u_A)(\bar{u}_D\gamma_\mu u_B)}{t} - \frac{(\bar{u}_D\gamma^\mu u_A)(\bar{u}_C\gamma_\mu u_B)}{u}\right)$$

A B C D

↓ ↑ ↓ ↑ $\frac{1}{t^2}$

A B C D

↓ ↑ ↑ ↓ $\frac{1}{u^2}$

A B C D

↓ ↓ ↓ ↓ $\left(\frac{1}{t} - \frac{1}{u}\right)^2$

And alike for the other permutations.