Physics 214 UCSD/225a UCSB

Lecture 10

- Halzen & Martin Chapter 4
  - Electron-muon scattering
  - Cross section definition
  - Decay rate definition
  - treatment of identical particles => symmetrizing
  - crossing

Electrodynamics of Spinless particles

- We replace p^{μ} with $p^{\mu} + eA^{\mu}$ in classical EM for a particle of charge -e moving in an EM potential A^{μ}
- In QM, this translates into: $i\partial^{\mu} \rightarrow i\partial^{\mu} + eA^{\mu}$
- And thus to the modified Klein Gordon Equation:

$$\left(\partial^{\mu}\partial_{\mu} + m^{2}\right)\phi = -V\phi$$

$$V = -ie(\partial^{\mu}A_{\mu} + A^{\mu}\partial_{\mu}) - e^{2}A^{2}$$

V here is the potential energy of the perturbation.

Two-by-two process Overview

- Start with general discussion of how to relate number of scatters in AB -> CD scattering to “beam & target independent” cross section in terms of W_{fi} .
- Calculate W_{fi} for electron-muon scattering.
- Calculate cross section from that
- Show relationship between cross section and “invariant amplitude” (or “Matrix Element”).

$$j^{\mu} = \left(\rho, \vec{j}\right)$$

Reminder from last lecture

$$\partial^{\mu} j_{\mu} = 0$$

$$\rho = i\left(\phi^{*}\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^{*}}{\partial t}\right)$$

$$J = -i\left(\phi^{*}\nabla\phi - \phi\nabla\phi^{*}\right)$$

Plane wave solutions are:

$$\boxed{\phi\left(t, \vec{x}\right) = N e^{-ip_{\mu}x^{\mu}}}$$

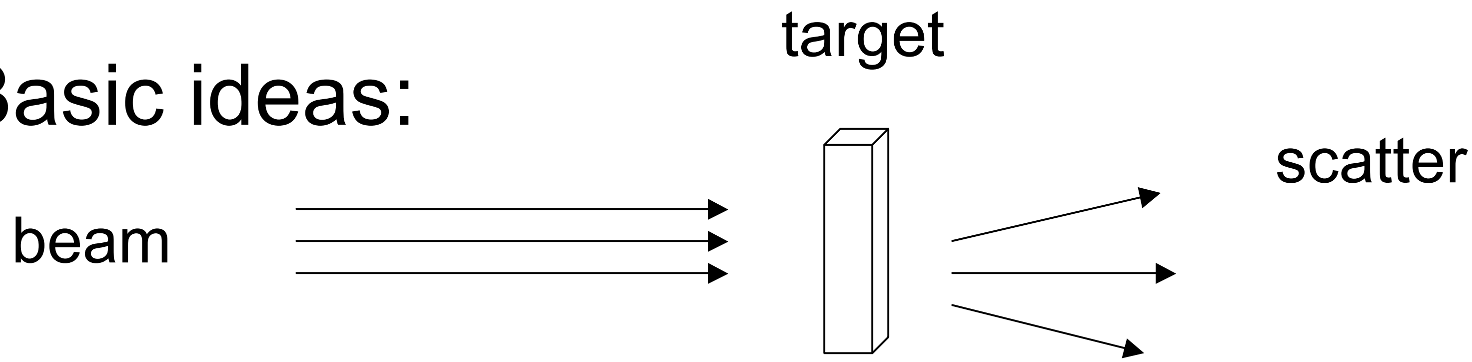
4-vector current for the plane wave solutions we find:

$$\left.\begin{aligned} \rho &= 2E\,|N|^{2} \\ \vec{j} &= 2\vec{p}\,|N|^{2} \end{aligned}\right\} J^{\mu} = 2p^{\mu}\,|N|^{2}$$

The $2|N|^2$ is an arbitrary normalization

Cross Section for AB -> CD

- Basic ideas:



# of scatters = (flux of beam) x (# of particles in target) x σ

$$\text{Cross section} = \sigma = \frac{W_{fi}}{\text{(initial flux)}} \text{ (number of final states)}$$

W_{fi} = rate per unit time and volume

"Cross section" is independent of characteristics of beam and target !!!

Aside on wave function Normalization

$$\text{Cross section} = \sigma = \frac{W_{fi}}{\text{(initial flux)}} \text{ (number of final states)}$$

$W_{fi} \propto N^4$

Number of final states/initial flux $\propto N^{-4}$

Cross section is thus independent of choice of wave function normalization

(as it should, of course!)

We will see this explicitly as we walk through this now.

Two-Two process AB -> CD

- Normalize plane wave in constant volume
  - This is obviously not covariant, so the volume normalization better cancel out before we're done!

$$\int_V \rho dV = 2E \Rightarrow N = \frac{1}{\sqrt{V}}$$

- # of particles per volume = 2E/V = n
- # of particles A crossing area per time = $v_A\, n_A$
- Flux(AB) = $v_A\, n_A\, (2E_B/V) = v_A\, (2E_A/V)\, (2E_B/V)$

Aside on covariant flux

- Flux = $v_A\,(2E_A/V)\,(2E_B/V)$
- Now let target (i.e. B) move collinear with beam (i.e. A): Flux = $(\mathbf{v_A} - \mathbf{v_B})\,(2E_A/V)\,(2E_B/V)$
- Now take v=p/E: Flux = $(E_B\,p_A + E_A\,p_B)\,4/V^2$
- Now a little relativistic algebra:

$$\left(p_A^\mu p_\mu^B\right)^2 - m_A^2 m_B^2 = \left(E_A E_B - \overrightarrow{p_A}\,\overrightarrow{p_B}\right)^2 - m_A^2 m_B^2$$

$$(E_A E_B)^2 = \left(p^2 + m^2\right)_A \left(p^2 + m^2\right)_B$$

$$\overrightarrow{p_A} = -\overrightarrow{p_B}$$

Putting the pieces together and adding some algebra:

$$\left(p_A^\mu p_\mu^B\right)^2 - m_A^2 m_B^2 = \left(p_A E_B + p_B E_A\right)^2$$

$$\boxed{Flux = \frac{4}{V^2}\sqrt{\left(p_A^\mu p_\mu^B\right)^2 - m_A^2 m_B^2}}$$

Obviously covariant!

(up to $1/V^2$ normalization factor that is arbitrary, and will cancel)

Number of final states/particle

- QM restricts the number of final states that a single particle in a box of volume V can have:

$$\frac{\text{Number of final states}}{\text{2E particles}} = \frac{Vdp^3}{(2\pi)^3 2E}$$

This follows from Exercise 4.1 in H&M
that you will do as homework exercise.

Putting the pieces together

$$\text{Cross section} = \sigma = \frac{W_{fi}}{\text{(initial flux)}} \text{ (number of final states)}$$

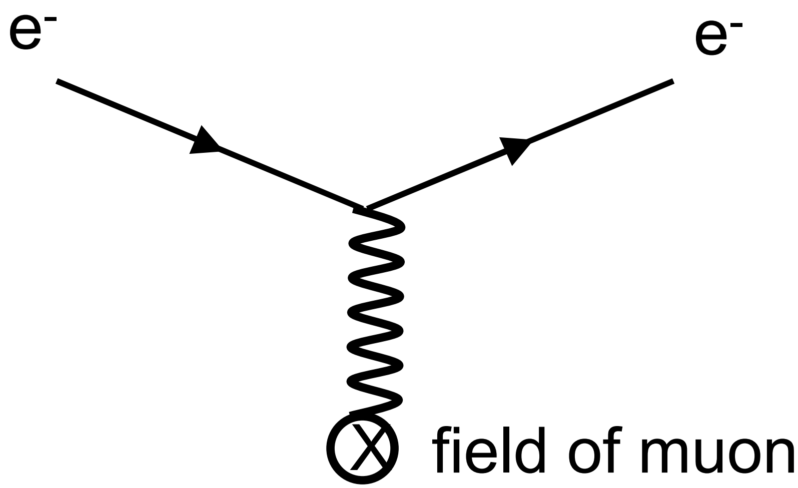
$$\sigma = \frac{W_{fi}}{v_A\,(2E_A/V)\,(2E_B/V)} \frac{V dp_C^3}{(2\pi)^3 2E_C} \frac{V dp_D^3}{(2\pi)^3 2E_D}$$

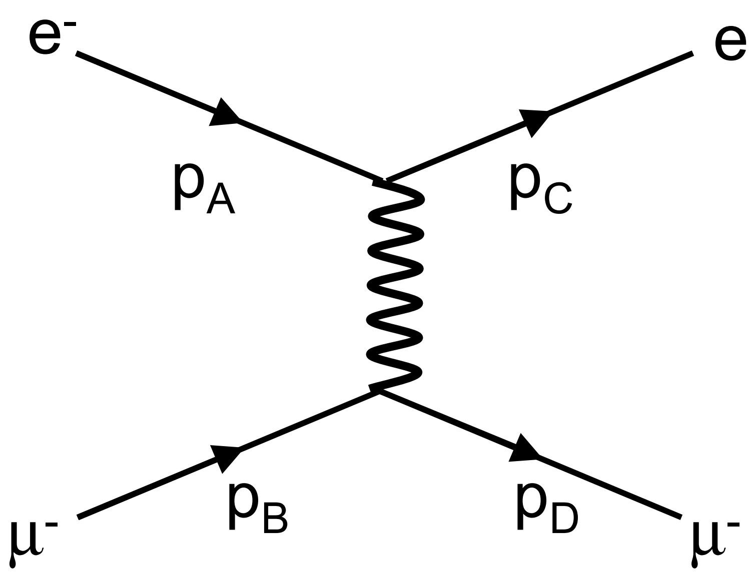
Next we calculate W_{fi}

Electron Muon Scattering

- Use what we did last lecture
  - Electron scattering in EM field
- With the field being the one generated by the muon as source.
  - Use covariant form of maxwell's equation in Lorentz Gauge to get V, the perturbation potential.
- Plug it into T_{fi}

In form of diagrams





Electron-muon scattering

Electron Muon scattering

$\Box^2 A^\mu = J^\mu_{(2)}$ Maxwell Equation

Note: $\Box^2 e^{iqx} = -q^2 e^{iqx}$

$$J^\mu_{(2)} = -eN_B N_D (p_D + p_B)^\mu e^{i\boxed{(p_D - p_B)}x}$$

= q

$$A^\mu = -\frac{1}{q^2} J^\mu_{(2)}$$

$$T_{fi} = -i \int J^{(1)}_\mu \frac{-1}{q^2} J^\mu_{(2)} d^4x$$

Note the symmetry: (1) <-> (2)

$$T_{fi} = -iN_A N_B N_C N_D (2\pi)^4 \delta^{(4)}(p_D + p_C - p_A - p_B) M$$

$$-iM = (ie(p_A + p_C)^\mu) \frac{-ig_{\mu\nu}}{q^2} (ie(p_D + p_B)^\nu)$$

Note the structure: Vertex x propagator x Vertex

Reminder

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu}$$

$$A^{\mu} B_{\mu} = g_{\mu\nu} A^{\mu} B^{\nu}$$

Reminder: T_{fi} -> W_{fi}

$$W_{fi} = \lim_{t \to \infty} \frac{\left|T_{fi}\right|^2}{t} \Rightarrow \frac{\left|T_{fi}\right|^2}{tV}$$

Last time we didn't work in a covariant fashion. This time around, we want to do our integrations across both time and space, i.e. W is a rate per unit time and volume.

$$T_{fi} = -iN_A N_B N_C N_D (2\pi)^4 \delta^{(4)}(p_D + p_C - p_A - p_B) M$$

As last time, we argue that one δ-function remains after $||^2$ while the other gives us a tV to cancel the tV in the denominator.

Putting it all together for W_{fi}

$$N = \frac{1}{\sqrt{V}}$$

$$W_{fi} = \frac{|T_{fi}|^2}{tV}$$

$$T_{fi} = \frac{-i(2\pi)^4}{V^2}\delta^{(4)}(p_D + p_C - p_A - p_B)M$$

$$\boxed{W_{fi} = (2\pi)^4 \frac{\delta^{(4)}(p_D + p_C - p_A - p_B)}{V^4}|M|^2}$$

Putting it all together for σ

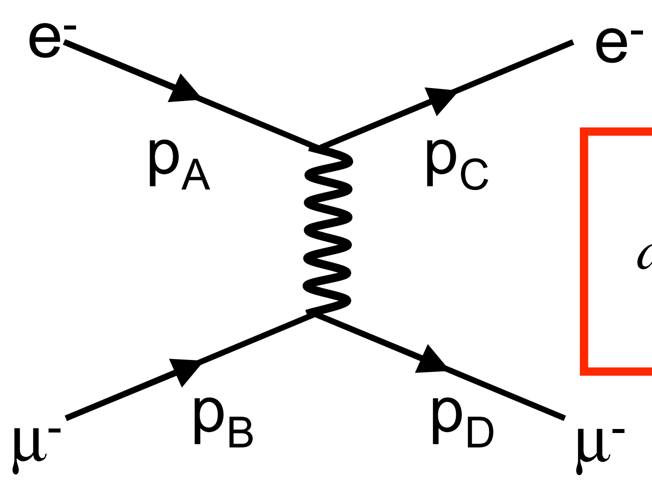
$$\sigma = \frac{W_{fi}}{v_A\,(2E_A/V)\,(2E_B/V)}\,\frac{V dp_C^3}{(2\pi)^3 2E_C}\,\frac{V dp_D^3}{(2\pi)^3 2E_D}$$

$$d\sigma = \frac{V^2}{4v_A E_A E_B}(2\pi)^4\,\frac{\delta^{(4)}(p_D + p_C - p_A - p_B)}{V^4}\left|M\right|^2\frac{V^2 dp_c^3 dp_D^3}{(2\pi)^6 4E_c E_D}$$

$$\boxed{d\sigma = \frac{1}{64\pi^2}\,\frac{\delta^{(4)}(p_D + p_C - p_A - p_B)}{v_A E_A E_B}\left|M\right|^2\frac{dp_c^3 dp_D^3}{E_c E_D}}$$

Aside on outgoing states

- While the incoming states have definite momentum, the outgoing states can have many momenta.
- The cross section is thus a differential cross section in the outgoing momenta.



$$\boxed{d\sigma = \frac{|M|^2}{4v_A E_A E_B}\frac{1}{16\pi^2}\delta^{(4)}(p_D + p_C - p_A - p_B)\frac{dp_c^3 dp_D^3}{E_c E_D}}$$

(incoming flux is still not covariant)

It is customery to re-express

$$d\sigma = \frac{|M|^2}{4v_A E_A E_B}\frac{1}{16\pi^2}\delta^{(4)}(p_D + p_C - p_A - p_B)\frac{dp_c^3 dp_D^3}{E_c E_D}$$

As: $$d\sigma = \frac{|M|^2}{F}dQ$$

F = flux factor: $$F = 4\sqrt{\left(p_A^\mu p_\mu^B\right)^2 - m_A^2 m_B^2} = 4v_A E_A E_B$$

dQ = Lorentz invariant phase space:

$$dQ = \frac{1}{16\pi^2}\delta^{(4)}(p_D + p_C - p_A - p_B)\frac{dp_c^3 dp_D^3}{E_c E_D}$$

In the center-of-mass frame:

$$F = 4p_i\sqrt{(E_A + E_B)^2} = 4p_i\sqrt{s}$$

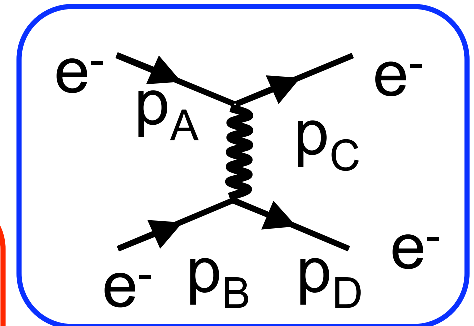
$$dQ = \frac{1}{4\pi^2}\frac{p_f}{4\sqrt{s}}d\Omega$$

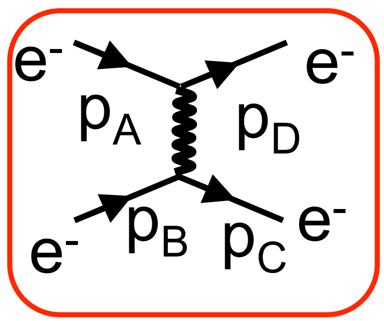
$$\boxed{\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \frac{|M|^2}{64\pi^2 s}\frac{p_f}{p_i}}$$

You get to show this as homework !

Electron-electron scattering

- With identical particles in the final state, we obviously need to allow for two contributions to M.
  - Option 1:
    - C attaches at vertex with A
    - D attaches at vertex with B
  - Option 2:
    - C attaches at vertex with B
    - D attaches at vertex with A

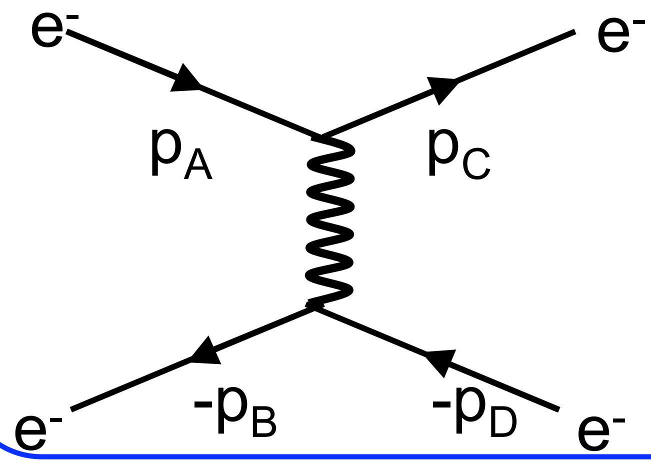




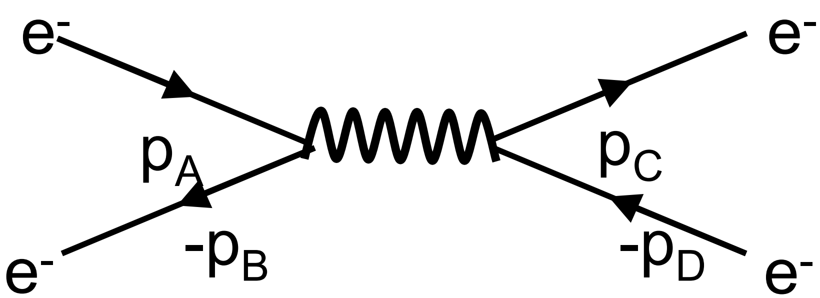
- As we can't distinguish C and D,the amplitudes add before M is squared.

$$M = -e^2\left(\boxed{\frac{(p_A+p_C)^\mu(p_B+p_D)_\mu}{(p_D-p_B)^2}}+\boxed{\frac{(p_A+p_D)^\mu(p_B+p_C)_\mu}{(p_C-p_B)^2}}\right)$$

Electron-positron and crossing



$$\frac{(p_A + p_C)^\mu(-p_B - p_D)_\mu}{\left((-p_D) - (-p_B)\right)^2}$$



$$\frac{(p_A - p_B)^\mu(-p_D + p_C)_\mu}{\left(p_C - (-p_D)\right)^2}$$

$$M = -e^2\left(\boxed{\frac{(p_A + p_C)^\mu(-p_B - p_D)_\mu}{\left((-p_D) - (-p_B)\right)^2}} + \boxed{\frac{(p_A - p_B)^\mu(-p_D + p_C)_\mu}{\left(p_C - (-p_D)\right)^2}}\right)$$

Electron-positron and crossing

Electron - electron

$$M = -e^2 \left(\frac{(p_A + p_C)^\mu (p_B + p_D)_\mu}{(p_D - p_B)^2} + \frac{(p_A + p_D)^\mu (p_B + p_C)_\mu}{(p_C - p_B)^2} \right)$$

Electron - positron

$$M = -e^2 \left(\frac{(p_A + p_C)^\mu (-p_B - p_D)_\mu}{((-p_D) - (-p_B))^2} + \frac{(p_A - p_B)^\mu (-p_D + p_C)_\mu}{(p_C - (-p_D))^2} \right)$$

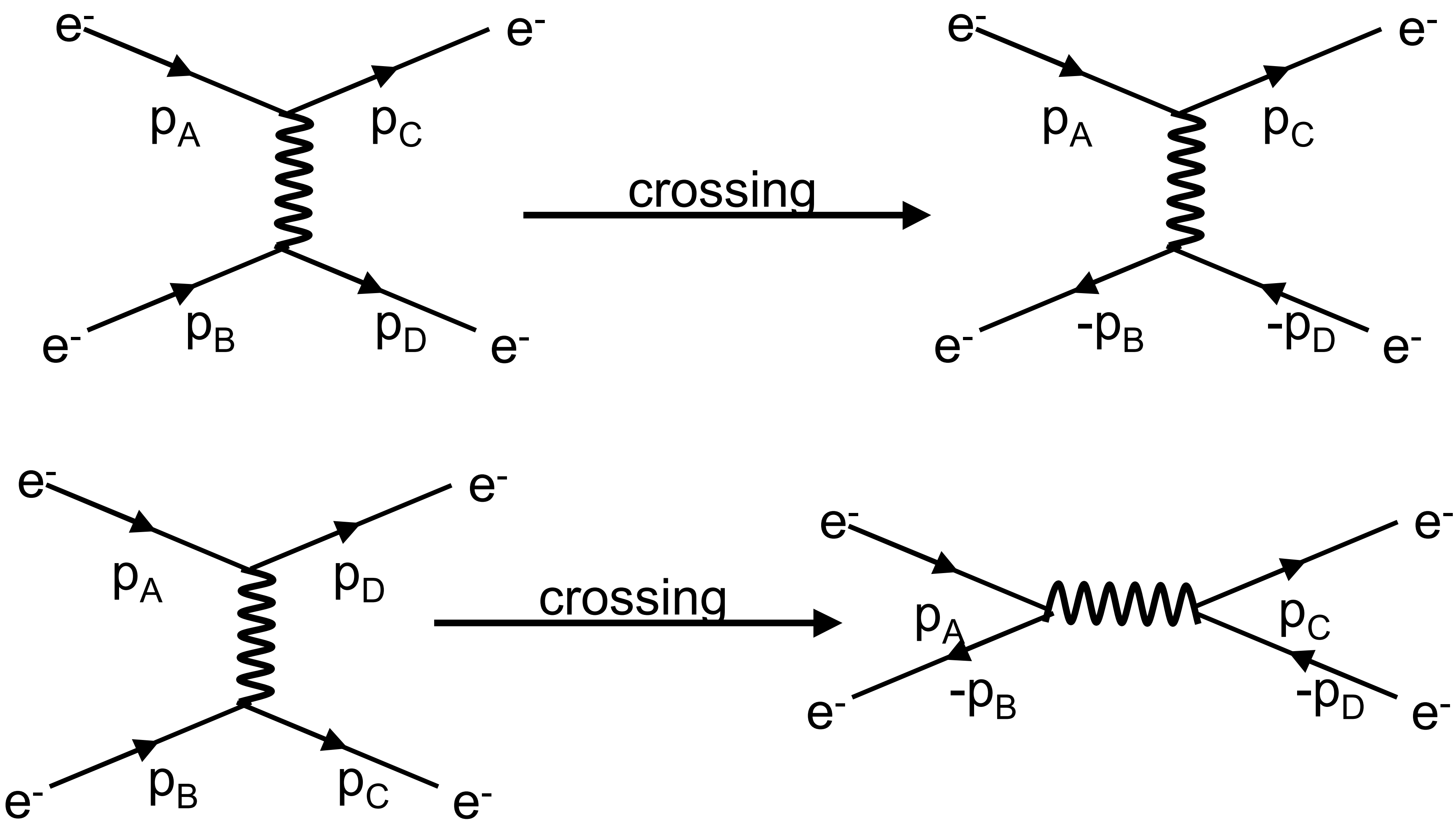
Only difference is:

$p_D \rightarrow -p_B$

$p_B \rightarrow -p_D$

Electron-Electron scattering

Electron-Positron scattering



E-mu vs e-e vs e-ebar scattering

Electron - muon

$$M = -e^2 \left(\frac{(p_A + p_C)^\mu (p_B + p_D)_\mu}{(p_D - p_B)^2} \right)$$

Electron - electron

$$M = -e^2 \left(\frac{(p_A + p_C)^\mu (p_B + p_D)_\mu}{(p_D - p_B)^2} + \frac{(p_A + p_D)^\mu (p_B + p_C)_\mu}{(p_C - p_B)^2} \right)$$

Electron - positron

$$M = -e^2 \left(\frac{(p_A + p_C)^\mu (-p_B - p_D)_\mu}{((-p_D) - (-p_B))^2} + \frac{(p_A - p_B)^\mu (-p_D + p_C)_\mu}{(p_C - (-p_D))^2} \right)$$