Problem 1:

HW3 p.1

(a) Define $A_{\alpha\beta} = \text{Amp}(\nu_\alpha \to \nu_\beta)$

$$A_{\alpha\beta} = \sum_j U^*_{\alpha j} U_{\beta j} \exp\left(-i\, m_j^2 \frac{L}{2E}\right)$$

$$|A_{\alpha\beta}|^2 = \sum_{jk} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} \exp\left[-i \frac{L}{2E}\left(m_j^2 - m_k^2\right)\right]$$

define $\Delta_{jk} = \left(m_j^2 - m_k^2\right)\frac{L}{2E}$

Rewrite, isolating ~~sum~~ $j=k$ from $j \neq k$:

$$|A_{\alpha\beta}|^2 = \underbrace{\sum_j U^*_{\alpha j} U_{\beta j} U_{\alpha j} U^*_{\beta j}}_{\text{term } ①} + \underbrace{2\sum_{j>k} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} e^{-i\Delta_{jk}}}_{\text{term } ②}$$

Use unitarity to reexpress ① as follows:

$$\sum_i U^*_{\alpha i} U_{\beta i} = \delta_{\alpha\beta} \Rightarrow \sum_{ij} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} = \delta_{\alpha\beta}$$

$$\rightarrow \quad ① = \delta_{\alpha\beta} - 2\sum_{j>k} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k}$$

Sub ① into eq. for $|A_{\alpha\beta}|^2$ and ~~[illegible]~~ combine the sums.

Problem 1 continued:

the same old

$$|A_{\alpha\beta}|^2 = \delta_{\alpha\beta} - 2\sum_{j>k} (UUUU)\left(1 - e^{-i\Delta_{jk}}\right) \quad (4)$$

$$1 - e^{-i\Delta_{jk}} = 1 - \cos\Delta_{jk} + i\sin\Delta_{jk}$$

$$(3) \quad = 2\sin^2\frac{\Delta_{jk}}{2} + i\sin\Delta_{jk}$$

$$\sum_{j>k} \text{"}UUUU\text{"} = \mathrm{Re}\sum_{j>k} UUUU + i\,\mathrm{Im}\sum UUUU$$

$$= \sum_{j>k} \mathrm{Re}(UUUU) + i\sum_{j>k} \mathrm{Im}\, UUUU$$

as $|A_{\alpha\beta}|^2 = $ Real the real and imaginary parts of (3) and (4) must combine such that only real parts survive !!!

$$\boxed{\begin{aligned} |A_{\alpha\beta}|^2 = \delta_{\alpha\beta} &- 4\sum_{i<j} \mathrm{Re}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\right) \sin^2\frac{\Delta_{ij}}{2} \\ &+ 2\sum_{i<j} \mathrm{Im}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\right) \sin\Delta_{ij} \end{aligned}}$$

Problem 1(b) | HW3 p.3

(b) The majorana phases affect U as follows:

$$U \rightarrow U \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

~~only~~ all $U_{\alpha 1}$ and $U_{\alpha 2}$ are affected equally among $U_{\alpha\beta}$

~~However, in $|A_{\alpha\beta}|^2$~~

e.g.
$$\left.\begin{aligned} U_{11} &\rightarrow U_{11}\, e^{i\alpha_1/2} \\ U_{21} &\rightarrow U_{21}\, e^{i\alpha_1/2} \\ U_{31} &\rightarrow U_{31}\, e^{i\alpha_1/2} \end{aligned}\right\} \quad \begin{aligned} &\left(U_{11}\, e^{i\alpha_1/2}\right)^* \cdot U_{12}\, e^{i\alpha_1/2} \\ &= U_{11}^* \cdot U_{12} \end{aligned}$$

and thus remains unchanged.

In $|A_{\alpha\beta}|^2$ only such combinations show up

$$U_{\alpha 1}^* U_{\beta 1} \rightarrow U_{\alpha 1}^* U_{\beta 1} \cdot e^{-i\alpha_1/2} \cdot e^{+i\alpha_1/2}$$
$$= U_{\alpha 1}^* U_{\beta 1} \quad \text{etc}$$

The majorana phases are thus irrelevant for neutrino mixing.

Problem 1c:

(i) $\operatorname{Im}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\right) =$ same for all i,j up to a sign and as long as $i \neq j$, $\alpha \neq \beta$

This can be shown by unitarity of U only as follows:

$$\sum_{j=1,2,3} U^*_{\alpha j} U_{\beta j} = 0 \quad \Big| \text{ multiply by } U_{\alpha 2} U^*_{\beta 2}$$

$$U_{\alpha 2} U^*_{\beta 2}\left(U^*_{\alpha 1} U_{\beta 1} + U^*_{\alpha 3} U_{\beta 3}\right) + |U_{\alpha 2} U_{\beta 2}|^2 = 0$$

$$\Rightarrow -\operatorname{Im} U_{\alpha 2} U^*_{\beta 2} U^*_{\alpha 1} U_{\beta 1} = \operatorname{Im} U_{\alpha 2} U^*_{\beta 2} U^*_{\alpha 3} U_{\beta 3}$$

you can repeat this for any two indices i,j from equation (i). qed

For the standard representation (PDG 2006 p.138) we get:

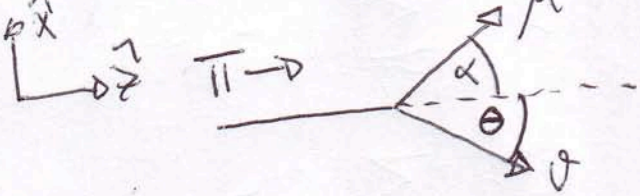
$$\operatorname{Im}\left(U^*_{11} U_{21} U_{12} U^*_{22}\right) = c_{12} c_{13} s_{12} c_{13} \operatorname{Im}\left(U_{21} U^*_{22}\right)$$

$$= -c_{12} c^2_{13} c_{23} s_{12} s_{13} s_{23} \sin\delta$$

This shows that all 4 angles $\theta_{12}, \theta_{13}, \theta_{23}, \delta$ have to be different from 0 and $\theta_{12}, \theta_{13}, \theta_{23}$ can't be $\pi/2$ either, in order for $\cancel{CP} \neq 0$!!!

Problem 2: Exploring Off-axis ν

(a) Assume π decays in x-z plane for concreteness.



π: $(E_\pi, 0, 0, P_\pi)$

μ: $(E_\mu, P_\mu \sin\alpha, 0, P_\mu \cos\alpha)$

ν: $(E_\nu, -E_\nu \sin\theta, 0, E_\nu \cos\theta)$

4-vector conservation:

(i) $E_\pi = E_\mu + E_\nu$

(ii) $P_\mu \sin\alpha = E_\nu \sin\theta$

(iii) $P_\mu \cos\alpha = P_\pi - E_\nu \cos\theta$

(ii)^2+(iii)^2: $\underbrace{P_\mu^2 = E_\nu^2 + P_\pi^2 - 2 P_\pi E_\nu \cos\theta}_{\text{call this (iv)}}$

now use (i) to rearrange and express as $E_\mu^2 = P_\mu^2 + m_\mu^2$

(v) $P_\mu^2 + m_\mu^2 = E_\pi^2 + E_\nu^2 - 2 E_\pi E_\nu$

Now plug (iv) into (v) to eliminate P_μ

$$2 E_\nu \left(E_\pi - P_\pi \cos\theta\right) = \underbrace{E_\pi^2 - P_\pi^2}_{m_\pi^2} - m_\mu^2 = m_\pi^2 - m_\mu^2$$

$$\boxed{E_\nu = \frac{1}{2} \frac{m_\pi^2 - m_\mu^2}{\left(E_\pi - P_\pi \cos\theta\right)}}$$

Next page shows how even a small angle makes the E_ν roughly independent of E_π !!!

Problem 3: $\sin\theta_{13}$ reactor ν disappearance experiments

Start with eq. (15) of hep-ex 0506165

$$P(\bar{\nu}_e \to \bar{\nu}_\beta) = -4\sum_{i>j} \mathrm{Real}\left(U^*_{ei} U_{\beta i} U_{ej} U^*_{\beta j}\right) \sin^2\Delta_{ij}$$
$$-2\sum_{i>j} \mathrm{Im}\left(\quad '' \quad '' \quad \right) \sin 2\Delta_{ij}$$

where $\Delta_{ij} \equiv \Delta m^2_{ij} \cdot \frac{L}{4E}$; $\Delta m^2_{ij} = m_i^2 - m_j^2$

and $\beta = \mu$ or τ

The probability for disappearance is then:

$$P = P(\bar{\nu}_e \to \bar{\nu}_\mu) + P(\bar{\nu}_e \to \bar{\nu}_\tau)$$

(A)
$$P = -4\sum_{i>j} \mathrm{Real}\left(U^*_{ei} U_{ej}\left[U_{\mu i} U^*_{\mu j} + U_{\tau i} U^*_{\tau j}\right]\right) \sin^2\Delta_{ij}$$
$$-2\sum_{i>j} \mathrm{Im}\underbrace{\left(\quad '' \quad '' \quad '' \quad \right)}_{= -|U_{ei} U^*_{ej}|^2} \sin 2\Delta_{ij}$$

for $i \neq j$ because of Unitarity as follows:

$\cancel{U_{ei} U^*_{ej} [U_{ei} U^*_{ej}]}$

$$U^*_{ei} U_{ej} \cdot \overbrace{\left[U_{ei} U^*_{ej} + U_{\mu i} U^*_{\mu j} + U_{\tau i} U^*_{\tau j}\right]}^{\delta_{ij}} = 0 \quad \text{for } i \neq j$$

Problem 3 continued:

This then leads to rewriting (A) as:

$$P = -4 \sum_{i>j} \text{Real}\left[-|U_{ei} U_{ej}^*|^2\right] \sin^2 \Delta_{ij}$$
$$-2 \sum_{i>j} \text{Im}\left[\quad '' \quad\right] \sin 2\Delta_{ij}$$

The Im[ ] is thus identically $\varnothing$

Note: We have thus shown that disappearance experiments are fundamentally insensitive to $\sin 2\Delta_{ij}$, but only sensitive to $\sin^2$. As such, they are insensitive to the hierarchy of masses.

Now use $U_{e1} = c_{12} c_{13}$ $\quad U_{e2} = s_{12} c_{13}$ $\quad U_{e3} = s_{13} e^{-i\delta}$

$$P = 4\left[s_{12}^2 c_{13}^2 c_{12}^2 c_{13}^2 \sin^2 \Delta_{21} + s_{13}^2 c_{12}^2 c_{13}^2 \sin^2 \Delta_{31} + s_{12}^2 c_{13}^2 s_{13}^2 \sin^2 \Delta_{32}\right]$$

now combine last 2 terms because $\Delta_{31} \approx \Delta_{32}$

$$P = 4\left[s_{12}^2 c_{12}^2 c_{13}^4 \sin^2 \Delta_{21} + s_{13}^2 c_{13}^2 \sin^2 \Delta_{31}\right]$$

now use double angle theorems: $\sin x \cdot \cos x = \frac{1}{2} \sin 2x$

$$\boxed{P = \sin^2[2\theta_{13}] \sin^2 \Delta_{31} + \cos^4 \theta_{13} \sin^2[2\theta_{12}] \sin^2 \Delta_{21}} \quad \boxed{\text{qed}}$$

Pluging in the numbers: $\Delta_{21} = 1.27 * 8\,10^{-5} * \frac{1.5}{3\,10^{-3}} = 0.051$

$$\Downarrow$$

$\Delta_{31} = 1.27 * 2.4\,10^{-3} * \frac{1.5}{3\,10^{-3}} = 1.52 \simeq \frac{\pi}{2}$ $\qquad \sin^2 \Delta_{21} = 0.0026$

We thus get: $P = \sin^2[2\theta_{13}] \cdot \sin^2 \Delta_{31} + 2.6\,10^{-3} \sin^2[2\theta_{12}]$