SOLUTIONS TO HW # 6

1. For the given functions <u>NOT</u> to be linearly independent, their Wronskian must vanish, i.e.

$$\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3''' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix} = 0.$$

A "short cut" is provided by the observation that the given y's are linear combinations of the functions e^x, e^{-x}, xe^x & xe^{-x} which are (obviously) linearly independent:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = M \begin{pmatrix} e^x \\ e^{-x} \\ xe^x \\ xe^{-x} \end{pmatrix}, \text{ where } M = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \alpha & 0 & 1 & 0 \\ 0 & \beta & 0 & 1 \end{pmatrix}.$$

If y's are linearly related, $\det M$ will vanish and that will give us the desired values of α and β.

A little algebra shows that $\det M = \frac{1}{2}(1-\alpha)(1+\beta)$, which will vanish if $\alpha = 1$ and/or $\beta = -1$. ✓

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Solution BY INSPECTION:

The given functions are

$$y_1 = \frac{1}{2}(x+1)e^{x} + \frac{1}{2}(x-1)e^{-x},$$

$$y_2 = \frac{1}{2}(x+1)e^{x} - \frac{1}{2}(x-1)e^{-x},$$

$$y_3 = (x+\alpha)e^{x}, \quad \text{and}$$

$$y_4 = (x+\beta)e^{-x}.$$

We observe that

(i) y_1 and y_2 are (obviously) LIN. IND., and so are y_3 and y_4. Next,

(ii) y_1, y_2 and y_3 are LIN. IND., except when $\alpha = 1$ — for then we'll have $y_1 + y_2 - y_3 = 0$. Similarly,

(iii) y_1, y_2 and y_4 are LIN. IND., except when $\beta = -1$ — for then we'll have $y_1 - y_2 - y_4 = 0$.

That solves the problem.

2. Substituting $y = fz$ into the equation $y'' + Py' + Qy = 0$, we get

$$(f''z + 2f'z' + fz'') + P(f'z + fz') + Qfz = 0, \text{ i.e.}$$

$$fz'' + \underbrace{(2f' + Pf)}\, z' + (f'' + Pf' + Qf)\, z = 0.$$

For $(2f' + Pf)$ to be zero, we require that $f' = -\frac{1}{2}Pf$, so that

$$f = \text{const.} \exp\left[-\tfrac{1}{2}\int^{x} P\,dx\right],$$

while

$$f'' + Pf' + Qf = (-\tfrac{1}{2}P'f - \underbrace{\tfrac{1}{2}Pf'}) + \underbrace{Pf'} + Qf$$

$$= -\tfrac{1}{2}P'f - \tfrac{1}{4}P^2 f + Qf.$$

So, finally, we get

$$z'' + q(x)\, z = 0, \quad \text{with } q = Q - \tfrac{1}{2}P' - \tfrac{1}{4}P^2. \checkmark$$

3. Here, $p_1(x) = -\dfrac{2x}{1-x^2}$, so $\exp\int^x -p_1(x)\,dx = \exp[-\ln(1-x^2)]$

$$= \frac{1}{1-x^2}.$$

Thus, apart from an arbitrary constant,

$y_2(x)$ for case (i) is $1.\int^x \dfrac{1}{1-x^2}\,dx = \int^x \frac{1}{2}\left(\dfrac{1}{1+x} + \dfrac{1}{1-x}\right)dx$

$$= \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \tanh^{-1} x, \checkmark$$

which diverges as $x \to \pm 1$.

$y_2(x)$ for case (ii) is $x.\int^x \dfrac{1}{x^2(1-x^2)}\,dx$

$$= x\int^x \left(\frac{1}{x^2} + \frac{1}{1-x^2}\right)dx$$

$$= x\left[-\frac{1}{x} + \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)\right]$$

$$= -\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \cdots\right)$$

$$= -1 + x\tanh^{-1} x, \checkmark$$

which too diverges as $x \to \pm 1$.

$\left\{\right.$ Note that our $y_1(x)$ for various values of l are $P_l(x)$, Legendre fns. of the 1st kind and the corresponding $y_2(x)$ are $Q_l(x)$, Legendre fns. of the 2nd kind.

4. The given equation is $x^2y'' - xy' + y = x$ and, obviously, the corresponding homog. eqn. is in the Euler form. So, setting $y = x^r$, we get (in the homog. case)

$$r(r-1) - r + 1 = 0, \text{ i.e. } r^2 - 2r + 1 = 0, \text{ so } \underbrace{r_{1,2} = 1, 1}.$$

So, following the approach of $\underbrace{\text{a limiting process}}$, we get:

$$y(x)\Big|_{homog.} = c_1 x + c_2 x \ln x,$$ which suggests that we better transform the given equation using the $\underbrace{\text{substitution } x = e^t}$.

Doing so, we have

$$y' = \frac{dy}{d(e^t)} = \frac{1}{e^t}\frac{dy}{dt} \;\&\; y'' = \frac{d}{d(e^t)}\left[\frac{1}{e^t}\frac{dy}{dt}\right] = \frac{1}{e^t}\frac{d}{dt}\left[\frac{1}{e^t}\frac{dy}{dt}\right]$$

$$= \frac{-1}{e^{2t}}\frac{dy}{dt} + \frac{1}{e^{2t}}\frac{d^2y}{dt^2},$$

with the result that our original equation reduces to the form

$$\left(-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right) - \frac{dy}{dt} + y = e^t, \text{ i.e.}$$

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^t. \checkmark$$

Clearly, we are in a better shape now!

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The solution of the corresponding homog. eqn. can now be obtained by setting $y = e^{rt}$, with the result

$$r^2 - 2r + 1 = 0, \text{ so } r_{1,2} = 1, 1 \text{ and hence}$$

$$y_{compl.}(t) = (c_1 + c_2 t)\, e^t .$$

As for the particular solution, we note that [in $f(t) = e^{rt}$], r is equal to <u>both</u> r_1 & r_2; so, our $y_p(t)$ will be of the form $(c' + c'' t + c''' t^2)\, e^t$.

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Since the terms with c' and c'' are going to merge into the complementary solution, we may consider only $c'''t^2e^t$.

Substituting this into the relevant diff. eqn., we get

$$c'''\left[(2+4t+t^2)-2(2t+t^2)+t^2\right]e^t = e^t,$$

so $c''' = \frac{1}{2}$.

Hence, our full soln. is

$$y(t) = \left(c_1 + c_2 t + \frac{1}{2}t^2\right)e^t.$$

Let's now impose the required conditions on $y(t)$, viz.

$y_{t=0} = 1$ & $y_{t=1} = e$, with the result that

$c_1 = 1$ & $c_1 + c_2 + \frac{1}{2} = 1$, so $c_1 = 1$ & $c_2 = -\frac{1}{2}$.

So, we finally get

$$y(t) = \left(1 - \frac{1}{2}t + \frac{1}{2}t^2\right)e^t,$$

that is,

$$y(x) = \left[1 - \frac{1}{2}\ln x + \frac{1}{2}(\ln x)^2\right]x.$$

5. Writing $\frac{1}{y}\frac{dy}{x} = u$, we have a <u>1st order</u> diff. eqn. in u:

$$u' + \frac{2a}{\tanh(2ax)}\, u = 2a^2.$$

The solution of the corresponding homog. eqn. is

$$u = c \exp\left[-\int^x \frac{2a}{\tanh(2ax)}\, dx\right] = \frac{c}{\sinh(2ax)}.$$

The particular solution of the equation, with $f(x) = 2a^2$, is

$$\frac{c}{\sinh(2ax)}\left[\int_0^x 2a^2 \cdot \frac{\sinh(2ax)}{c}\, dx + c'\right].$$

Thus,

$$u(x) = \frac{1}{y}\frac{dy}{dx} = \frac{\text{const}}{\sinh(2ax)} + a\coth(2ax).$$

Integrating once again, we get

$$\ln y = \text{const} \ln|\tanh(ax)| + \frac{1}{2}\ln|\sinh(2ax)| + \text{const}.$$

$$\therefore\ y(x) = c_1 |\sinh(2ax)|^{1/2}\, |\tanh(ax)|^{c_2}.$$

For small x, $y(x) \sim |x|^{\frac{1}{2}+c_2}$; for this to be linear in x, we must have: $c_2 = \frac{1}{2}$, so finally

$$y(x) = \text{const}\, |\sinh x|.$$

6. Substituting $y_p = y_1 v$ into the eqn. $y'' + p_1 y' + p_0 y = f$, we get

$$(\underline{y_1'' v} + 2y_1' v' + y_1 v'') + p_1(\underline{y_1' v} + y_1 v') + p_0 \underline{y_1 v} = f \text{, i.e.}$$

$$v'' + \left(2\frac{y_1'}{y_1} + p_1\right) v' = \frac{f}{y_1}, \text{ which is}$$

a <u>1st-order</u> diff. eqn. in v'.

The solution of the corresponding homog. eqn. is

$$\text{const.} \exp \int^x -\left(2\frac{y_1'}{y_1} + p_1\right) dx, \text{ i.e. } \frac{cW}{y_1^2}.$$

The particular solution then is

$$\frac{cW}{y_1^2}\left[\int^x \frac{(f/y_1)}{cW/y_1^2}\, dx + c'\right] = \frac{W}{y_1^2}\int^x \frac{f y_1}{W}\, dx + \frac{cc'W}{y_1^2}.$$

Adding the two results, we have

$$v' = \frac{\text{const.} W}{y_1^2} + \underline{\frac{W}{y_1^2}} \cdot \underline{\int^x \frac{f y_1}{W}\, dx}.$$

Integrating again, we get $\left[\text{remembering that } \frac{W}{y_1^2} = \frac{d}{dx}\left(\frac{y_2}{y_1}\right)\right]$

$$v = \frac{\text{const.} y_2}{y_1} + \left[\frac{y_2}{y_1} \cdot \int^x \frac{f y_1}{W}\, dx - \int^x \frac{y_2}{y_1} \cdot \frac{f y_1}{W}\, dx\right] + \text{const}.$$

It follows that

$$y = y_1 v = \text{const.} y_2 + y_p + \text{const.} y_1, \text{ where}$$

$$y_p = y_2 \int^x \frac{f y_1}{W}\, dx - y_1 \int^x \frac{f y_2}{W}\, dx, \text{ a result}$$

that is <u>symmetric in y_1 & y_2</u> and carries <u>no arbitrary constants.</u>