Solutions to HW #5

1. The corresponding homogeneous equation is
$$xy'' - (x+1)y' + y = 0.$$

(i) Since the sum of the coefficients in this equation is zero, $y = e^x$ is an obvious solution!

(ii) The second solution may be found using the general result
$$y_2 = \text{const. } y_1 \int^x \frac{\exp \int^x -p_1(t)\,dt}{y_1^2}\,dx \qquad \left[p_1(t) = -\left(1+\frac{1}{t}\right)\right]$$
$$= \text{const. } e^x \int^x \frac{x \cdot e^x}{e^{2x}}\,dx$$
$$= \text{const. } e^x\left[-e^{-x}(x+1)\right] = \text{const. } (x+1).$$

(iii) As for $y_p(x)$, since $f(x)$ is a const. and the coefficient of y also a const., $y_p(x) = f(x) = -1$.

Hence, $$y(x) = c_1 e^x + c_2(x+1) - 1.$$

The conditions $y(1) = y'(1) = 0$ impel us to choose $c_1 = -e^{-1}$ and $c_2 = 1$, with the result that
$$y(x) = x - e^{x-1}. \checkmark$$

2. For the differential equation $y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = c e^{rx}$ [where c and all the p's are constants],

$$y_p(x) = \frac{c}{r^n + p_{n-1} r^{n-1} + \cdots + p_1 r + p_0} e^{rx}.$$

If e^{rx} were replaced by $x^m e^{rx}$, which is <u>no different</u> from $\frac{\partial^m}{\partial r^m} e^{rx}$, the particular solution will be

$$\frac{\partial^m}{\partial r^m}\left[\frac{c}{r^n + p_{n-1} r^{n-1} + \cdots + p_1 r + p_0} e^{rx}\right].$$

A little reflection will show that this solution will be of the form

$$\sum_{l=0}^{m} a_l x^l \cdot e^{rx},$$

where a_l's are fully determined by the operation above. However, in practice, it is much easier to determine them via substitution!

<u>Example:</u> $$y'' + 3y' + 2y = x e^x.$$

(i) The complementary function in this case is $y = e^{rx}$, with the requirement that $r^2 + 3r + 2 = 0$, i.e., $r_{1,2} = \underline{-1 \; \& \; -2}$.

So, this part of the solution is $c_1 e^{-x} + c_2 e^{-2x}$.

(ii) The particular solution in this case (with $r = 1$ & $m = 1$) is

$$y_p(x) = (a_0 + a_1 x) e^x .$$

Substituting this into the given diff. eqn., we get

$$(a_0 + a_1 x + 2a_1) + 3(a_0 + a_1 x + a_1) + 2(a_0 + a_1 x) = x,$$

that is, $(6a_0 + 5a_1) + 6a_1 x = x$.

It follows that $a_1 = \frac{1}{6}$ & $a_0 = -5/36$.

(iii) So, the full solution is

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{6x - 5}{36} e^x . \checkmark$$

3. The corresponding homogeneous equation is $x^2 y'' - 2y = 0$, which of the Euler type. So, the trial solution is x^r, with the requirement that $r(r-1) - 2 = 0$, i.e., $r_{1,2} = 2$ & -1.

So, the complementary solutions are x^2 & x^{-1}.

As regards $y_p(x)$, we could well follow the spirit of Problem 2 and propose that $y_p(x) = cx$, where c may be determined by substitution. We readily get:

$$x^2 \cdot 0 - 2cx = x,$$

so, $c = -1/2$ and the problem is completely solved!

HOWEVER, for practice, we may fall back on the general formula

$$y_p(x) = -y_1 \int^x \frac{y_2(t)}{W(t)} f(t)\,dt + y_2 \int^x \frac{y_1(t)}{W(t)} f(t)\,dt.$$

Now, with $y_1 = x^2$ & $y_2 = x^{-1}$, $W(x) = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix} = -3.$

So, for the diff. eqn. $y'' - \frac{2}{x^2} y = \frac{1}{x}$, we get

$$y_p(x) = -x^2 \int^x \frac{t^{-1}}{-3} \frac{1}{t}\,dt + x^{-1} \int^x \frac{t^2}{-3} \frac{1}{t}\,dt$$

$$= -\frac{1}{3}x - \frac{1}{6}x = -\frac{1}{2}x.$$

So, finally, $y(x) = c_1 x^2 + c_2 x^{-1} - \frac{1}{2}x.$

4. The Hermite eqn. is $y'' - 2xy' + 2\alpha y = 0$.

Let $y(x)$ be $\sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$ $(a_0 \neq 0)$.

Substituting this into the given eqn., we get

$$\sum_{\lambda=0}^{\infty} \Big\{ a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} - 2a_\lambda (k+\lambda) x^{k+\lambda} + 2\alpha a_\lambda \cdot x^{k+\lambda} \Big\} = 0,$$

i.e.,

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} + 2 \sum_{\lambda=0}^{\infty} a_\lambda (\alpha - k - \lambda) x^{k+\lambda} = 0.$$

The coeff. of x^{k-2} gives $a_0 k(k-1) = 0$ (the indicial eqn.).
It follows that $k = 0$ or 1. ✓

The coeff. of x^{k-1} gives $a_1(k+1)k = 0$, which means that a_1 is arbitrary if $k=0$ but $a_1 = 0$ if $k=1$. ✓

Case 1 : $k=1$ & $a_1 = 0$.

Our equation now gives

$$\sum_{\lambda=0}^{\infty} a_\lambda (\lambda+1)\lambda x^{\lambda-1} + 2\sum_{\lambda=0}^{\infty} a_\lambda (\alpha - 1 - \lambda) x^{\lambda+1} = 0.$$

The coeffs. of x^{-1} and x^0 have already been considered. So now we have to consider coeffs. of x^1 & beyond. For this, we look at the coeff. of x^{j+1} $(j = 0, 1, 2, \ldots)$.

We get: $a_{j+2}(j+3)(j+2) + 2a_j(\alpha - 1 - j) = 0.$

It follows that $a_{j+2} = \dfrac{2(j+1-\alpha)}{(j+2)(j+3)} a_j$. ✓

Since $a_1 = 0$, we also have: $a_{3,5,7,\ldots} = 0$. ✓

The desired solution then is (remember that $\underline{k=1}$)

$$y(x) = a_0 x^1 \left[1 + \frac{2(1-\alpha)}{2.3} x^2 + \frac{2(1-\alpha)}{2.3} \cdot \frac{2(3-\alpha)}{4.5} x^4 + \ldots \right]$$

$$= a_0 \left[x + \frac{2(1-\alpha)}{3!} x^3 + \frac{2^2(1-\alpha)(3-\alpha)}{5!} x^5 + \ldots \right];$$

clearly, this is $y_{\underline{odd}}(x)$.

<u>The question of convergence:</u>

The ratio $\dfrac{a_{j+2}}{a_j} x^2 \xrightarrow[(j \gg 1)]{} \dfrac{2}{j} x^2 \xrightarrow[j \to \infty]{} 0$ for <u>finite values</u> of x but NOT if x itself goes to ∞.

This behavior is very much like the Taylor expansion of the function e^{2x^2}, where the corresponding ratio is

$$\frac{1/(j+1)!}{1/j!}(2x^2) = \frac{2}{j+1}x^2 \approx \frac{2}{j}x^2.$$

It follows that, as $x \longrightarrow \infty$, our solution will diverge unless the parameter α is <u>an odd integer</u>. Imp.

Writing $\alpha = n$, our soln. takes the form

$$\underline{y_{odd}}(x) = a_0 \left[x - \frac{2(n-1)}{3!}x^3 + \frac{2^2(n-1)(n-3)}{5!}x^5 - \dots \right],$$

which terminates at x^n. This soln., apart from a constant factor, is the <u>Hermite polynomial</u> $H_n(x)$ for <u>odd</u> n.

Similarly, with $k = 0$ (and setting $a_1 = 0$) we arrive at the solution

$$\underline{y_{even}}(x) = a_0 \left\{ 1 - \frac{2n}{2!}x^2 + \frac{2^2 n(n-2)}{4!}x^4 - \dots \right],$$

which terminates at x^n if n is an <u>even</u> integer. This soln., apart from a constant factor, is the <u>Hermite polynomial</u> $H_n(x)$ for <u>even</u> n.

The general soln., of course, is

$$y(x) = c_1 y_{odd}(x) + c_2 y_{even}(x).$$

5. The given eqn. is $\rho u'' + (2-\rho) u' + (\lambda - 1) u = 0$.

Let the solution be $u = \sum_{n=0}^{\infty} a_n \rho^{k+n}$ $(a_0 \neq 0)$.

Substituting into the given eqn., we get

$$\sum_{n=0}^{\infty} \left\{ a_n (k+n)(k+n-1) \rho^{k+n-1} + (2-\rho) \cdot a_n (k+n) \rho^{k+n-1} + (\lambda-1) a_n \rho^{k+n} \right\} = 0, \text{ i.e.,}$$

$$\sum_{n=0}^{\infty} a_n (k+n)(k+n+1) \rho^{k+n-1} + \sum_{n=0}^{\infty} a_n (\lambda - 1 - k - n) \rho^{k+n} = 0.$$

The coefficient of ρ^{k-1} is $a_0 k(k+1)$, leading to the <u>indicial eqn.</u> $k(k+1) = 0$, so $k = 0$ or -1. ✓

Imp. The solution with $k = -1$ will diverge at $\rho = 0$, so we discard it.

The solution with $k = 0$ will be <u>regular</u> at $\rho = 0$; for this, we have

$$\sum_{n=0}^{\infty} a_n n(n+1) \rho^{n-1} + \sum_{n=0}^{\infty} a_n (\lambda - 1 - n) \rho^n = 0,$$

leading to the <u>recurrence relation</u>

$$a_{j+1} = \frac{j+1-\lambda}{(j+1)(j+2)} a_j \qquad (j = 0, 1, 2, \dots).$$

The question of convergence:

The relevant ratio here is

$$\frac{a_{j+1}}{a_j}\rho = \frac{j+1-\lambda}{(j+1)(j+2)}\rho \xrightarrow[j \gg 1]{} \frac{1}{j}\rho \xrightarrow[j\to\infty]{} 0$$

for all finite x. So, the series does converge (like e^{ρ}, so long as ρ is finite) but will diverge if $\rho \longrightarrow \infty$.

The series will terminate and become a polynomial in x if $\lambda \doteq 1, 2, 3, \cdots$ — a positive integer !

We then have

$$u(\rho) = a_0\left\{1 + \frac{1-\lambda}{1.2}\rho + \frac{1-\lambda}{1.2}\cdot\frac{2-\lambda}{2.3}\rho^2 + \frac{1-\lambda}{1.2}\cdot\frac{2-\lambda}{2.3}\cdot\frac{3-\lambda}{3.4}\rho^3 + \cdots\right\}$$

$$= a_0\left\{1 - \frac{\lambda-1}{1!2!}\rho + \frac{(\lambda-1)(\lambda-2)}{2!3!}\rho^2 - \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{3!4!}\rho^3 + \cdots\right\},$$

✓

terminating at $\rho^{\lambda-1}$.

Imp. {Apart from a numerical factor, this solution is an associated Laguerre polynomial $L^1_{\lambda-1}(\rho)$; see Eq. (13.72) of AW.}

6. The given differential equation is

$$x^2 y'' + 2x y' + x^2 y = 0$$

and the given function (supposedly a solution to this equation) is

$$f(x) = x^{-1} \sin x .$$

If so, we have

$$f'(x) = -x^{-2} \sin x + x^{-1} \cos x$$

& $$f''(x) = 2x^{-3} \sin x - 2x^{-2} \cos x - x^{-1} \sin x .$$

It follows that

$$\left(x^2 f'' + 2x f' + x^2 f\right)$$

$$= \left[2x^{-1} \sin x - 2\cos x - x \sin x\right] + \left[-2x^{-1} \sin x + 2\cos x\right] + x \sin x = 0.$$

Hence, $x^{-1} \sin x$ is indeed a soln. of the given eqn.

Now, to determine the second solution of this equation, we write it as

$$y'' + \frac{2}{x} y' + y = 0 ,$$

so that $p_1(x) = 2/x$. The second soln. is then given by

$$y_2(x) = \text{const. } x^{-1}\sin x \int^x \frac{\exp\left\{-\int^x \frac{2}{x}\,dx\right\}}{x^{-2}\sin^2 x}\,dx$$

$$= \text{const. } x^{-1}\sin x \int^x \frac{x^{-2}}{x^{-2}\sin^2 x}\,dx$$

$$= \text{const. } x^{-1}\sin x \,(-\cot x)$$

$$= \text{const. } x^{-1}\cos x. \checkmark$$