Solutions to HW #4

1. $F(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{(x^2+a^2)^2}\,dx$ VIA $I_C = \int_C \frac{e^{ikz}}{(z^2+a^2)^2}\,dz$

Assume $k>0$ and treat this integral as of category I_2. Then,

$F(k)$ = $2\pi i$. the residue of the function $\frac{e^{ikz}}{(z+ia)^2(z-ia)^2}$

at the <u>double pole</u> $z=ia$

$$= 2\pi i \cdot \frac{d}{dz}\left\{\frac{e^{ikz}}{(z+ia)^2}\right\}\Bigg|_{z=ia}$$

$$= 2\pi i \cdot \left\{\frac{ike^{ikz}}{(z+ia)^2} - \frac{2e^{ikz}}{(z+ia)^3}\right\}_{z=ia}$$

$$= 2\pi i \cdot \left\{\frac{ike^{-ka}}{4i^2a^2} - \frac{2e^{-ka}}{8i^3a^3}\right\}$$

$$= \frac{\pi}{2a^3}e^{-ka}(ka+1).$$ ✓

2. For $x \to 0$, only small values of t will really matter.

$$\therefore K_\nu(x) \simeq \frac{1}{2}\left(\frac{x}{2}\right)^\nu \int_0^\infty e^{-x^2/4t}\, t^{-\nu-1}\, dt.$$

Let $t = \frac{x^2}{4} \cdot \frac{1}{u}$, with the result

$$K_\nu(x) \simeq \frac{1}{2}\left(\frac{x}{2}\right)^\nu \int_\infty^0 e^{-u} \left(\frac{x^2}{4}\right)^{-\nu-1} u^{\nu+1} \cdot \frac{x^2}{4} \frac{-1}{u^2}\, du$$

$$= \frac{1}{2}\left(\frac{x}{2}\right)^\nu \cdot \left(\frac{x^2}{4}\right)^{-\nu} \int_0^\infty e^{-u} u^{\nu-1}\, du$$

$$= \frac{1}{2} \frac{\Gamma(\nu)}{(x/2)^\nu} \qquad (\nu > 0). \checkmark$$

3. With $\nu \to \infty$, we better write the given integral as

$$K_\nu(x) = \int_0^\infty e^{-x\cosh t + \ln\cosh(\nu t)}\,dt.$$

We may write

$$f(t) = -x\cosh t + \ln\cosh(\nu t), \text{ so that}$$

$$f'(t) = -x\sinh t + \nu\tanh(\nu t).$$

$\therefore$ t_0 is determined by the condition $\nu\tanh(\nu t_0) = x\sinh t_0$.

For $\nu \gg 1$, $t_0 \approx \sinh^{-1}\dfrac{\nu}{x} \approx \ln\dfrac{2\nu}{x}$.

Now, $f''(t_0) = -x\cosh t_0 + \underbrace{\nu^2\operatorname{sech}^2(\nu t_0)}_{\text{negligible}} \approx -x.\frac{1}{2}e^{t_0} = -\nu$,

which is negative and large in magnitude, hence a sharp maximum.

$$\therefore \quad K_\nu(x) \approx e^{f(t_0)}\sqrt{\frac{2\pi}{\nu}},$$

while

$$f(t_0) = -x\cosh t_0 + \ln\cosh(\nu t_0)$$

$$\approx -\nu + \ln\left\{\frac{1}{2}\left(\frac{2\nu}{x}\right)^\nu\right\}.$$

$$\therefore K_\nu(x) \approx e^{-\nu}.\frac{1}{2}\left(\frac{2\nu}{x}\right)^\nu.\sqrt{\frac{2\pi}{\nu}} \checkmark$$

In view of that, for $\nu \gg 1$, $\Gamma(\nu) \approx \sqrt{\dfrac{2\pi}{\nu}}\left(\dfrac{\nu}{e}\right)^2$, $\checkmark$

this result is asymptotically the same as $\dfrac{1}{2}\dfrac{\Gamma(\nu)}{(x/2)^\nu}$.

4. $I(x) = \int_0^\infty e^{-t + x.t^\alpha}\, t^{\beta-1}\, dt \qquad (\beta > 0,\ 0 < \alpha < 1).$

Here,

$f(t) = -t + x.t^\alpha$, so

$f'(t) = -1 + x.\alpha\, t^{\alpha-1}$; $t_0 = (\alpha x)^{\frac{1}{1-\alpha}}$.

$f''(t) = x\alpha(\alpha-1)\, t^{\alpha-2}$; $f''(t_0) = -(1-\alpha)(\alpha x)^{\frac{-1}{1-\alpha}}$

$< 0.$

At the same time,

$$f(t_0) = -(\alpha x)^{\frac{1}{1-\alpha}} + x.(\alpha x)^{\frac{\alpha}{1-\alpha}}$$

$$= (-\alpha+1)\,(\alpha^\alpha . x)^{\frac{1}{1-\alpha}}.$$ Check!

Hence

$$I(x) \approx e^{(1-\alpha)(\alpha^\alpha . x)^{\frac{1}{1-\alpha}}} . (\alpha x)^{\frac{\beta-1}{1-\alpha}} \sqrt{\frac{2\pi}{(1-\alpha)(\alpha x)^{-\frac{1}{1-\alpha}}}}$$

$$= \sqrt{\frac{2\pi}{1-\alpha}}\, (\alpha x)^{\frac{2\beta-1}{2(1-\alpha)}} . e^{(1-\alpha)(\alpha^\alpha . x)^{\frac{1}{1-\alpha}}}$$ ✓

Continued

By the way, the special case $\alpha = \beta = \frac{1}{2}$ gives

$$Fi(\tfrac{1}{2}, \tfrac{1}{2}; x) \approx \sqrt{4\pi} \cdot 1 \cdot e^{\frac{1}{2}\left(\frac{1}{\sqrt{2}}x\right)^2}$$

$$= 2\sqrt{\pi}\, e^{\frac{1}{4}x^2}.$$

The <u>exact</u> answer in this case is $\sqrt{\pi}\, e^{\frac{1}{4}x^2}\left[1 + \operatorname{erf}\left(\tfrac{1}{2}x\right)\right]$

which, for $x >> 1$, is indeed $\approx 2\sqrt{\pi}\, e^{\frac{1}{4}x^2}$.

In contrast,

$$\int_0^\infty e^{-t - x t^\alpha}\, t^{\beta-1}\, dt = \int_0^\infty e^{-x t^\alpha}\left(1 - t + t^2 \ldots\right) t^{\beta-1}\, dt \qquad \underbrace{t_{eff} = O\left(\frac{1}{x^{1/\alpha}}\right)}$$

$$\approx \int_0^\infty e^{-x t^\alpha} \cdot 1 \cdot t^{\beta-1}\, dt$$

$$= \int_0^\infty e^{-xu} \cdot u^{\frac{\beta-1}{\alpha}} \cdot \frac{1}{\alpha} u^{\frac{1}{\alpha}-1}\, du$$

$$= \frac{1}{\alpha}\,\frac{\Gamma(\beta/\alpha)}{x^{\beta/\alpha}} \qquad (\beta > 0) \checkmark$$

5. Asymptotic Behaviour of $J_\nu(\nu) = \dfrac{(\frac{1}{2}\nu)^\nu}{2\pi i}\displaystyle\int_C e^{t-\frac{\nu^2}{4t}}\, t^{-\nu-1}\,dt$ (for $\nu \gg 1$).

Write your integrand as $e^{f(t)}\cdot g(t)$, with $f(t) = t - \frac{\nu^2}{4t} - \nu \ln t$ & $g(t) = t^{-1}$.

Then
$$f'(t) = 1 + \frac{\nu^2}{4t^2} - \frac{\nu}{t} = \left(1 - \frac{\nu}{2t}\right)^2 = 0 \longrightarrow t_0 = \tfrac{1}{2}\nu, \checkmark$$

$$f''(t) = -\frac{\nu^2}{2t^3} + \frac{\nu}{t^2} \longrightarrow f''(t_0) = -\frac{\nu^2}{2\cdot\frac{1}{8}\nu^3} + \frac{\nu}{\frac{1}{4}\nu^2} = 0. \text{ Oops!}$$

So, we have to go to
$$f'''(t) = \frac{3\nu^2}{2t^4} - \frac{2\nu}{t^3} \longrightarrow f'''(t_0) = \frac{3\nu^2}{2\cdot\frac{1}{16}\nu^4} - \frac{2\nu}{\frac{1}{8}\nu^3} = \frac{8}{\nu^2}. \checkmark$$

$$\therefore\ J_\nu(\nu) \approx \frac{\cancel{(\frac{1}{2}\nu)^\nu}}{2\pi i}\int_C e^{\left[\cancel{\frac{1}{2}\nu} - \cancel{\frac{\nu^2}{2\nu}} - \cancel{\nu\ln(\frac{1}{2}\nu)}\right] + \frac{8}{\nu^2}(t-\frac{1}{2}\nu)^3/3!}\, t_0^{-1}\,dt$$

$\{$Put $t = \frac{1}{2}\nu(1+u)\}$

$$= \frac{1}{2\pi i}\int_C e^{\nu u^3/6}\, \cancel{(\tfrac{1}{2}\nu)}^{-1}\left(\cancel{\tfrac{1}{2}\nu}\,du\right) \longrightarrow |u_{\text{eff}}| = O\left(\frac{1}{\nu^{1/3}}\right)$$

$\{$Put $\frac{\nu u^3}{6} = v$, i.e. $u = \left(\frac{6}{\nu}\right)^{1/3} v^{1/3}\}$

$$= \frac{1}{2\pi i}\int_C e^{v}\left(\frac{6}{\nu}\right)^{1/3}\tfrac{1}{3}\,v^{-2/3}\,dv$$

$$= \frac{1}{3}\left(\frac{6}{\nu}\right)^{1/3}\cdot\frac{1}{\Gamma(2/3)} \quad \left[\text{by Hankel's defn. of } \frac{1}{\Gamma(z)}\right]$$

$$= \frac{2^{1/3}}{3^{2/3}\,\Gamma(2/3)}\;\frac{1}{\nu^{1/3}}. \checkmark$$

6. Estimate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(st)}{(1+t^2)^s}\,dt$ for $s \gg 1$.

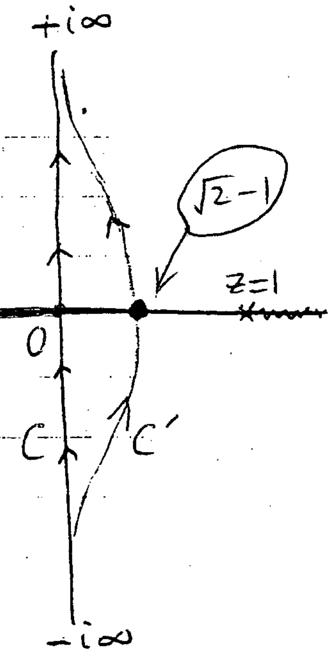
$$\text{The given integral} = \int_{-\infty}^{\infty} e^{ist - s\ln(1+t^2)}\,dt \qquad \text{Put } t = iz, \text{ so that } z = -it.$$

$$= \int_{+i\infty}^{-i\infty} e^{s\{-z-\ln(1-z^2)\}}\, i\,dz$$

$$= -i\int_{-i\infty}^{+i\infty} e^{sf(z)}\,dz,$$

where $f(z) = -z - \ln(1-z^2)$,

$$f'(z) = -1 - \frac{-2z}{1-z^2} \Rightarrow z_0 = -1 \pm \sqrt{2}.$$



The relevant z_0 here is $\sqrt{2}-1$; let's call it c. Then

$$f''(z) = \frac{2}{1-z^2} + \frac{2z(-1)(-2z)}{(1-z^2)^2} = 2\,\frac{1+z^2}{(1-z^2)^2}, \text{ so that}$$

$$f''(z_0) = 2\,\frac{1+(2+1-2\sqrt{2})}{(2c)^2} = 2\,\frac{2\sqrt{2}\,c}{4c^2} = \frac{\sqrt{2}}{c},$$

which is real & positive;

so we want $(z-z_0)^2$ to be real and negative, i.e. $\alpha = \pi/2$.

Hence, the given integral

$$\approx -i \cdot \sqrt{\frac{2\pi}{s(\sqrt{2}/c)}} \cdot e^{s\{-c-\ln(2c)\} + i\pi/2}$$

$$= \sqrt{\frac{\pi\sqrt{2}\,c}{s}}\;\frac{e^{-cs}}{(2c)^s}. \checkmark \qquad \text{QED}$$