Physics 201
Solutions to HW # 2

1. To evaluate the given integral

$$I = \int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta} \qquad (a > b > 0)$$

substitute $z = e^{i\theta}$, so that $d\theta = dz/iz$, etc. We get

$$I = \oint_C \frac{\left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^2 \cdot \frac{dz}{iz}}{a + \frac{b}{2}\left(z + \frac{1}{z}\right)}$$

$$= \frac{i}{2} \oint_C \frac{(z^2-1)^2 \, dz}{z^2 \left(b z^2 + 2az + b\right)},$$

where C is the unit circle centered at $z = 0$.

Let's write the integrand in the form

$$f(z) = \frac{(z^2-1)^2}{z^2 \, b \, (z - z_1)(z - z_2)},$$

where $z_{1,2} = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}$.

Continued →

We have <u>two simple poles</u> at $z = z_1$ & z_2 and a <u>double pole</u> at $z = 0$.

Since $b > 0$, <u>z_2 is outside C</u> while <u>z_1 is inside.</u>

X ✓

Keep in mind that the product $z_1 z_2 = 1$.

<u>Residue at $z = z_1$</u> is $\dfrac{(z_1^2-1)^2}{z_1^2 b(z_1-z_2)} = \dfrac{(z_1 - 1/z_1)^2}{b(z_1-z_2)}$

$$= \frac{(z_1-z_2)^2}{b(z_1-z_2)} \quad \left(\text{because } \frac{1}{z_1} = z_2\right)$$

$$= \frac{z_1-z_2}{b} = \frac{2}{b}\sqrt{\frac{a^2}{b^2}-1} = \frac{2}{b^2}\sqrt{a^2-b^2}. \checkmark$$

<u>Residue at $z = 0$:</u> For this, we should express $f(z)$ in the form of a Laurent expansion about $z=0$ and pick the coefficient a_{-1}, i.e.

$$f(z) = \frac{1}{z^2 b}(1-z^2)^2 \cdot \frac{-1}{z_1}\left(1-\frac{z}{z_1}\right)^{-1} \cdot \frac{-1}{z_2}\left(1-\frac{z}{z_2}\right)^{-1}$$

$$= \frac{1}{z^2 b z_1 z_2}(1-\cdots)\left(1+\frac{1}{z_1}z+\cdots\right)\left(1+\frac{1}{z_2}z+\cdots\right).$$

$$= \frac{1}{z^2 b}\left[1+\left(\frac{1}{z_1}+\frac{1}{z_2}\right)z + \cdots\cdots\right] \quad (\because z_1 z_2 = 1)$$

It follows that $a_{-1} = \dfrac{1}{b}\left(\dfrac{1}{z_1}+\dfrac{1}{z_2}\right) = \dfrac{z_1+z_2}{b z_1 z_2} = \dfrac{z_1+z_2}{b} = \dfrac{-2a}{b^2}.$ ✓

It follows that

$$I = \frac{i}{2} \cdot 2\pi i \left(\frac{2}{b^2}\sqrt{a^2-b^2} - \frac{2a}{b^2} \right)$$

$$= \frac{2\pi}{b^2}\left(a - \sqrt{a^2-b^2} \right). \checkmark$$

Note that the residue at $z=0$ could also be determined as follows:

$$(a_{-1})_{z=0} = \frac{d}{dz}\left[z^2 f(z) \right]_{z\to 0}.$$

For this purpose, $f(z) \approx \dfrac{1}{z^2(bz^2+2az+b)}$, so

$$\frac{d}{dz}\left[z^2 f(z)\right] \text{ gives } \frac{-1}{(bz^2+2az+b)^2}\cdot(2bz+2a)$$

$$\xrightarrow[z\to 0]{} \frac{-2a}{b^2}. \checkmark$$

Now, for $b \ll a$, $\sqrt{a^2-b^2} = a\left(1-\frac{b^2}{a^2}\right)^{1/2} = a\left[1 - \frac{b^2}{2a^2} + \cdots\right]$

$$\therefore \lim_{b\to 0} I = \frac{2\pi}{b^2}\left[\cancel{a} - \left(\cancel{a} - \frac{b^2}{2a}\right)\right] = \frac{\pi}{a},\ \checkmark \text{ as it should be!}$$

$\left[\text{because } \int_0^{2\pi} \sin^2\theta = \pi\right].$

2. $f(z) = \dfrac{\cosh(az)}{\cosh(\pi z)}$ has simple poles wherever $\cos(i\pi z) = 0$;

$$z_n = (n+\tfrac{1}{2})\,i \quad ; \quad n = 0, \pm 1, \pm 2, \ldots$$

Residue at $z = (n+\frac{1}{2})\,i$ is given by

$$\frac{\cosh[(n+\frac{1}{2})\,ia]}{\left[\frac{d}{dz}\cosh(\pi z)\right]_{z=(n+\frac{1}{2})i}} = \frac{\cos[(n+\frac{1}{2})a]}{\pi\sinh[\pi(n+\frac{1}{2})i]}$$

$$= \frac{\cos(n+\frac{1}{2})a}{\pi i\,(-1)^n}.$$

The desired integral then turns out to be $2\sum\limits_{n=0}^{\infty}(-1)^n\cos(n+\frac{1}{2})a$.

For the next part, we note that our result is

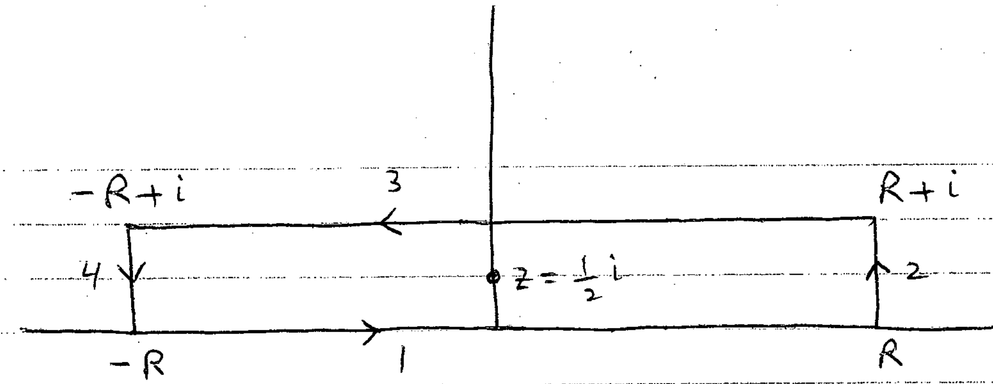
$$\mathrm{Re}\left[2\sum_{n=0}^{\infty}(-1)^n e^{i(n+\frac{1}{2})a}\right]$$

$$= \mathrm{Re}\left[2\cdot\frac{e^{i\frac{1}{2}a}}{1+e^{ia}}\right]$$

$$= \mathrm{Re}\left[\frac{1}{\cos(\frac{1}{2}a)}\right] = \sec(\tfrac{1}{2}a).$$

$\Bigg\{$ Note that the given integral <u>diverges</u> as $|a| \to \pi$.
The condition $|a| < \pi$ is also necessary for I_R to vanish, as $R \to \infty$.

3.



$$I_C = \oint_C \frac{\cosh(az)}{\cosh(\pi z)}\,dz = I_1 + I_2 + I_3 + I_4 .$$

I_2 & $I_4 \longrightarrow 0$ as $R \to \infty$ (because $|a| < \pi$).

$$I_1 \to \int_{-\infty}^{\infty} \frac{\cosh(ax)}{(\cosh \pi x)}\,dx \text{ , the desired integral.}$$

$$I_2 \longrightarrow \int_{\infty}^{-\infty} \frac{\cosh a(x+i)}{\cosh \pi(x+i)}\,dx$$

$$= \int_{\infty}^{-\infty} \frac{\cosh(ax)\cosh(ai)}{-\cosh(\pi x)}\,dx + \int_{\infty}^{-\infty} \underbrace{\frac{\sinh(ax)\sinh(ai)}{-\cosh(\pi x)}}_{\text{an } \underline{\underline{\text{odd}}} \text{ fn. of } x}\,dx$$

$$= \cos a . I_1 + \text{zero}$$

$\therefore \quad I_C \longrightarrow (1+\cos a)\, I_1 .$

The residue of $f(z)$ at $z = \frac{1}{2}i$ is $\dfrac{\cos(\frac{1}{2}a)}{\pi i}$.

$$\therefore \quad (1+\cos a)\, I_1 = 2\cancel{\pi i}\,\frac{\cos(\frac{1}{2}a)}{\cancel{\pi i}}.$$

Hence $I_1 = \dfrac{2\cos(\frac{1}{2}a)}{2\cos^2(\frac{1}{2}a)} = \sec(\frac{1}{2}a).$ ✓

4. The function e^{iz^2} is analytic for all <u>finite</u> z. As $|z| \to \infty$,

$$e^{iz^2} = e^{i(x^2-y^2+2ixy)} = e^{-2xy}\, e^{i(x^2-y^2)} \longrightarrow 0 \text{ if the}$$

product $(xy) > 0$, which is indeed the case here. ✓

$$\text{Our } I_C = I_{\text{along } \theta=0} + I_{\text{along } r=R} + I_{\text{along } \theta=\pi/4}$$

$$= 0 \text{ because } \underline{\text{no poles}} \text{ enclosed.}$$

Now, as $R \to \infty$,

$$I_{\text{along } \theta=0} \longrightarrow \int_0^\infty e^{ix^2}\,dx = \int_0^\infty \cos x^2\,dx + i\int_0^\infty \sin x^2\,dx.$$

$$I_{\text{along } r=R} \longrightarrow 0.$$

$$I_{\text{along } \theta=\pi/4} \longrightarrow \int_\infty^0 e^{i\left(re^{i\pi/4}\right)^2}\left(dr.\, e^{i\pi/4}\right) \qquad \because \left[z = re^{i\pi/4}\right]$$

$$= -\int_0^\infty e^{-r^2}\,dr.\; e^{i\pi/4}$$

$$= -\frac{\sqrt{\pi}}{2}\cdot\frac{1+i}{\sqrt{2}}.$$

It follows that

$$\int_0^\infty \cos x^2\,dx + i\int_0^\infty \sin x^2\,dx - \frac{\sqrt{\pi}}{2\sqrt{2}} - \frac{\sqrt{\pi}}{2\sqrt{2}}\,i = 0.$$

Hence the results.

5. $f(z) = \dfrac{z + i e^{iz}}{z^3}$; clearly, we have a pole at $z = 0$.

As $z \to 0$, $f(z) \longrightarrow \dfrac{i}{z^3}$; hence a pole of order 3. ✓

Since the pole lies on the contour, we must determine the principal value of the integral. However, note that, since the pole is not a simple one, we can't say that the answer is simply $\pi i a_{-1}$ (though, in this case, we do eventually get the same result)!

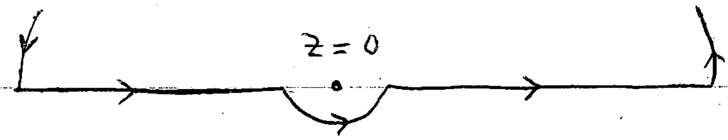
Imp.

In any case, $a_{-1} = \dfrac{1}{2!}\left(\dfrac{d^2}{dz^2}\left[z^3 f(z)\right]\right)_{z \to 0}$

$$= \frac{1}{2}\left(i^3 e^{iz}\right)_{z \to 0} = \frac{-i}{2}.$$

So, we expect that $PI_C = \pi i \cdot \dfrac{-i}{2} = \dfrac{\pi}{2}$.

The proper procedure here is to indent the contour to either enclose the pole inside C or exclude it. Adopting the former procedure, we have the following situation:



Near $z=0$, $$f(z) = \frac{z-\sin z}{z^3} + \frac{i\cos z}{z^3}$$

$$= \frac{\cancel{z} - (\cancel{z} - \frac{z^3}{6} + \cdots)}{z^3} + \frac{i(1 - \frac{z^2}{2} + \cdots)}{z^3}$$

$$= \frac{i}{z^3} + \frac{-i/2}{z} + \frac{1}{6} + \cdots\cdots$$

What about this???

This is $a_{-1} z^{-1}$.

We can readily verify that the integral of the main term over the semi-circle vanishes, and we are indeed left with $(-i/2)\cdot \pi i = \pi/2$.

The rest is straightforward. We get

$$\int_{-\infty}^{\infty} \frac{x - \sin x}{x^3} = \frac{\pi}{2} \quad \& \quad P\int_{-\infty}^{\infty} \frac{\cos x}{x^3} = 0.$$

no P here.

Note!

6. For the integral $\oint_C \frac{z^{1/2}}{(z+1)^2}\,dz$, use the same contour that we used in the class for the integral $\oint_C z^{\mu-1} f(z)\,dz$ with $0<\mu<1$. There, θ lies between 0 & 2π.

Check that, for the integral under study, $I_r \to 0$ as $r \to 0$ &

$$I_R \to 0 \text{ as } R \to \infty.$$

Then, the above integral reduces to

$$\left[1 - e^{i\,2\pi\cdot(1/2)}\right]\int_0^\infty \frac{x^{1/2}\,dx}{(x+1)^2} = 2\int_0^\infty \frac{x^{1/2}\,dx}{(x+1)^2}.$$

This should be equal to $2\pi i \times$ [the residue of the integrand at the double pole at $z=-1$]. The desired residue is

$$\frac{d}{dz}\left(z^{1/2}\right)\Big|_{z=e^{i\pi}} = \frac{1}{2}z^{-1/2}\Big|_{z=e^{i\pi}} = \frac{1}{2}(-i) = \frac{1}{2i}.$$

Hence

$$I_C = 2\int_0^\infty \frac{x^{1/2}\,dx}{(x+1)^2} = 2\pi i \times \frac{1}{2i} = \pi.$$

It follows that

$$\int_0^\infty \frac{x^{1/2}\,dx}{(x+1)^2} = \frac{\pi}{2}.$$