Phys. 201

Solutions to HW #1

1. $\sum_{n=0}^{N-1} \cos nx + i \sum_{n=0}^{N-1} \sin nx = \sum_{n=0}^{N-1} e^{inx}$, a geometric series,

$$= \frac{e^{iNx} - 1}{e^{ix} - 1} = \frac{e^{i\frac{1}{2}Nx}\left(e^{i\frac{1}{2}Nx} - e^{-i\frac{1}{2}Nx}\right)}{e^{i\frac{1}{2}x}\left(e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}\right)}$$

$$= e^{i\frac{1}{2}(N-1)x} \frac{\cancel{2i} \sin\left(\frac{1}{2}Nx\right)}{\cancel{2i} \sin\left(\frac{1}{2}x\right)}$$

Now resolve into real & imaginary parts, to get the desired results.

2. $\sum_{n=0}^{\infty} p^n \cos nx + i \sum_{n=0}^{\infty} p^n \sin nx = \sum_{n=0}^{\infty} \left(pe^{ix}\right)^n$

$$= \frac{1}{1 - pe^{ix}},$$ provided that $|pe^{ix}| < 1$, i.e. $|p| < 1$.

Rationalizing this expression, we get

$$\frac{1}{1-pe^{ix}} \cdot \frac{1-pe^{-ix}}{1-pe^{-ix}} = \frac{1-pe^{-ix}}{1-p\cdot 2\cos x + p^2}.$$

Now resolve into real & imaginary parts, to get the desired results.

Continued →

For $p \to 1$, $\displaystyle\sum_{n=0}^{\infty} \cos nx \longrightarrow \frac{1-\cos x}{2(1-\cos x)} = \frac{1}{2}$ &

$$\sum_{n=0}^{\infty} \sin nx \longrightarrow \frac{\sin x}{2(1-\cos x)} = \frac{1}{2}\cot\frac{x}{2},$$

provided that $x \neq 0, 2\pi, 4\pi, \ldots$ <u>Why?</u>

— · — · —

Going back to Q. 1, we may write

$$\sum_{n=0}^{N-1} \cos nx = \frac{\sin(Nx/2)\cos[(N-1)x/2]}{\sin(x/2)} = \frac{\sin(N-\frac{1}{2})x + \sin(\frac{1}{2}x)}{2\sin(x/2)}$$

&

$$\sum_{n=0}^{N-1} \sin nx = \frac{\sin(Nx/2)\sin[(N-1)x/2]}{\sin(x/2)} = \frac{\cos(\frac{1}{2}x) - \cos(N-\frac{1}{2})x}{2\sin(x/2)}$$

If we now let $N \to \infty$ and replace the "rapidly oscillating" functions $\sin(N-\frac{1}{2})x$ & $\cos(N-\frac{1}{2})x$ by their "mean values," namely zero, we do get the desired results! Clearly, this <u>cannot</u> be done if x were $0, 2\pi, 4\pi, \ldots$.

Imp. Strictly speaking, these series are not truly convergent in the limit $N \to \infty$, because $|e^{ix}| = 1$, not < 1 — as required for absolute convergence.

3. $v(x,y) = e^{-y} \sin x$ gives

$$\left.\begin{aligned} &\frac{\partial v}{\partial x} = e^{-y}\cos x = -\frac{\partial u}{\partial y} \implies u = e^{-y}\cos x + a(x) + b, \\ \& \quad &\frac{\partial v}{\partial y} = -e^{-y}\sin x = \frac{\partial u}{\partial x} \implies u = e^{-y}\cos x + c(y) + d. \end{aligned}\right\}$$

For consistency, we must have: $u(x,y) = \underbrace{e^{-y}\cos x + \text{a const.}}$

$$\therefore \quad f(z) = \left(e^{-y}\cos x + \text{a const.}\right) + i e^{-y}\sin x$$

$$= e^{iz} + \text{a real const.} \checkmark$$

4. $$\left|\Gamma\left(\tfrac{1}{2}+iy\right)\right|^2 = \Gamma\left(\tfrac{1}{2}+iy\right)\left[\Gamma\left(\tfrac{1}{2}+iy\right)\right]^*$$
$$= \Gamma\left(\tfrac{1}{2}+iy\right)\Gamma\left(\tfrac{1}{2}+iy\right)^* = \Gamma\left(\tfrac{1}{2}+iy\right)\Gamma\left(\tfrac{1}{2}-iy\right).$$

Since $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, it follows that
$$\left|\Gamma\left(\tfrac{1}{2}+iy\right)\right|^2 = \frac{\pi}{\sin\left[\pi\left(\tfrac{1}{2}+iy\right)\right]} = \frac{\pi}{\cos(i\pi y)} = \frac{\pi}{\cosh(\pi y)}. \checkmark$$

In the limit $y \to 0$, we get: $\Gamma(1/2) = \sqrt{\pi}$. ✓

Similarly, $$\left|\Gamma(iy)\right|^2 = \Gamma(iy)\Gamma(-iy) = \frac{\Gamma(1+iy)}{iy}\Gamma(-iy)$$
$$= \frac{1}{iy}\,\frac{\pi}{\sin[\pi(1+iy)]}$$
$$= \frac{1}{iy}\,\frac{\pi}{-\sin(\pi i y)} = \frac{\pi}{y\sinh(\pi y)}. \checkmark$$

In the limit $y \to 0$, we get: $\left|\Gamma(iy)\right|^2 \approx \frac{1}{y^2}$,
consistent with the known result: $\lim_{\epsilon\to 0}\Gamma(-n+\epsilon) \approx \frac{(-1)^n}{n!\,\epsilon}$,

which gives $$\lim_{y\to 0}\Gamma(iy) \approx \frac{1}{iy}.$$

5. The function $f(z) = \dfrac{1}{z(1-z)(2+z)}$ has simple poles at $z = 0, 1$ & -2. So we write it in the form

$$f(z) = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+2},$$

so that A, B & C are the respective residues.

In the limit $z \to 0$, the given function $\approx \dfrac{1}{2z}$; so $A = 1/2$.

In the limit $z \to 1$, " " " $\approx \dfrac{1}{3\underline{(1-z)}}$; so $B = \dfrac{-1}{3}$.

note

In the limit $z \to -2$, " " " $\approx \dfrac{-1}{6(2+z)}$; so $C = \dfrac{-1}{6}$.

Hence

$$f(z) = \frac{1}{2z} - \frac{1}{3(z-1)} - \frac{1}{6(z+2)}.$$

Continued →

Laurent expansions of $f(z)$ around $z=0$.

(i) $|z|<1$. Write

$$f(z) = \frac{1}{2z} + \frac{1}{3}(1-z)^{-1} - \frac{1}{12}\left(1+\tfrac{1}{2}z\right)^{-1}$$

$$= \frac{1}{2z} + \frac{1}{3}\left(1+z+z^2+\dots\right) - \frac{1}{12}\left(1-\tfrac{1}{2}z+\tfrac{1}{4}z^2-\dots\right)$$

$$= \frac{1}{2z} + \frac{1}{4} + \frac{3}{8}z + \frac{5}{16}z^2 + \dots\dots,$$

with $a_{-1} = \frac{1}{2} = A$. ✓

(ii) $1<|z|<2$. Now write

$$f(z) = \frac{1}{2z} \underbrace{- \frac{1}{3z}\left(1-\frac{1}{z}\right)^{-1}}_{\text{note!}} - \frac{1}{12}\left(1+\tfrac{1}{2}z\right)^{-1}$$

$$= \frac{1}{2z} - \frac{1}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right) - \frac{1}{12}\left(1-\tfrac{1}{2}z+\tfrac{1}{4}z^2-\dots\right)$$

$$= \dots\dots - \frac{1}{3z^3} - \frac{1}{3z^2} + \frac{1}{6z} - \frac{1}{12} + \frac{1}{24}z - \frac{1}{48}z^2 + \dots\dots,$$

with $a_{-1} = \frac{1}{6} = A+B$. ✓

(iii) $|z|>2$. Now write

$$f(z) = \frac{1}{2z} - \frac{1}{3z}\left(1-\frac{1}{z}\right)^{-1} \underbrace{- \frac{1}{6z}\left(1+\frac{2}{z}\right)^{-1}}_{\text{note!}}$$

$$= \frac{1}{2z} - \frac{1}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right) - \frac{1}{6z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\dots\right).$$

$$= \dots\dots - \frac{1}{z^3} + 0\,\frac{1}{z^2} + 0\,\frac{1}{z},$$

with $a_{-1} = 0 = A+B+C$. ✓

Since both a_{-1} & a_{-2} are zero here, we should include a few more terms of the original expansions. Doing so, we get in this case

$$f(z) = \cdots\cdots - \frac{3}{z^5} + \frac{1}{z^4} - \frac{1}{z^3}. \checkmark \quad \underline{\text{Check!}}$$

Note that for $|z| \gg 1$, the given function $\approx -1/z^3$. So, we couldn't have here any terms beyond $1/z^3$.

6. $f(z) = \cot z = \dfrac{\cos z}{\sin z}$ has simple poles wherever $\sin z$ has (simple) zeros, viz. at $z = z_n$ where

$$z_n = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Residue at z_n is $\dfrac{\cos z_n}{\left[d/dz \sin z\right]_{z=z_n}} = 1$, for all n.

Near $z = 0$, $\cot z \approx 1/z$.

Laurent expansion of $\cot z$ about $z = 0$ should be of the form $\sum_{n=-1}^{\infty} a_n z^n$, with $a_{-1} = 1$.

Since $\cot z$ is an odd fn. of z, we expect that

$$a_0 = a_2 = a_4 = \ldots = 0.$$

Our aim now is to determine a_1 & a_3.

The simplest way is to use the series expansions of $\cos z$ and $\sin z$. We get

Continued →

$$\cot z = \left(1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots\right)\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \cdots\right)^{-1}$$

$$= \left(1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots\right) \cdot z^{-1}\left[1 - \left(\frac{1}{6}z^2 - \frac{1}{120}z^4 + \cdots\right)\right]^{-1}.$$

The last factor $= 1 + \left(\frac{1}{6}z^2 - \frac{1}{120}z^4 + \cdots\right) + \left(\frac{1}{6}z^2 - \frac{1}{120}z^4 + \cdots\right)^2 + \cdots$

$$= 1 + \frac{1}{6}z^2 \underbrace{- \frac{1}{120}z^4 + \frac{1}{36}z^4}_{\text{Same order!}} + \cdots$$

$$= 1 + \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots$$

Hence

$$\cot z = z^{-1}\left(1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots\right)\left(1 + \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots\right)$$

$$= z^{-1}\left[1 \underbrace{- \frac{1}{2}z^2 + \frac{1}{6}z^2} + \underbrace{\frac{1}{24}z^4 + \frac{7}{360}z^4 - \frac{1}{12}z^4} + \cdots\right]$$

$$= \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \cdots \checkmark$$

{ Note that we obtain the same result if we carry out a "Taylor expansion" of the function $\left[\cot z - \frac{1}{z}\right]$ about $z = 0$.

Range of validity: Since the pole next to the one at $z = 0$ lies at $z = +\pi$ or $-\pi$, the above expansion is valid only for $|z| < \pi$.