HOMEWORK ASSIGNMENT #3

Oct. 15, 2009

1. Using contour integration, show that

$$\int_0^{\pi} \frac{d\theta}{a + b\sin^2\theta} = \frac{\pi}{\sqrt{a(a+b)}}.$$

What restrictions must a and b satisfy in order that the integral exists?

2. Using a suitable contour, evaluate the integral

$$\oint_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\, dz,$$

to show that

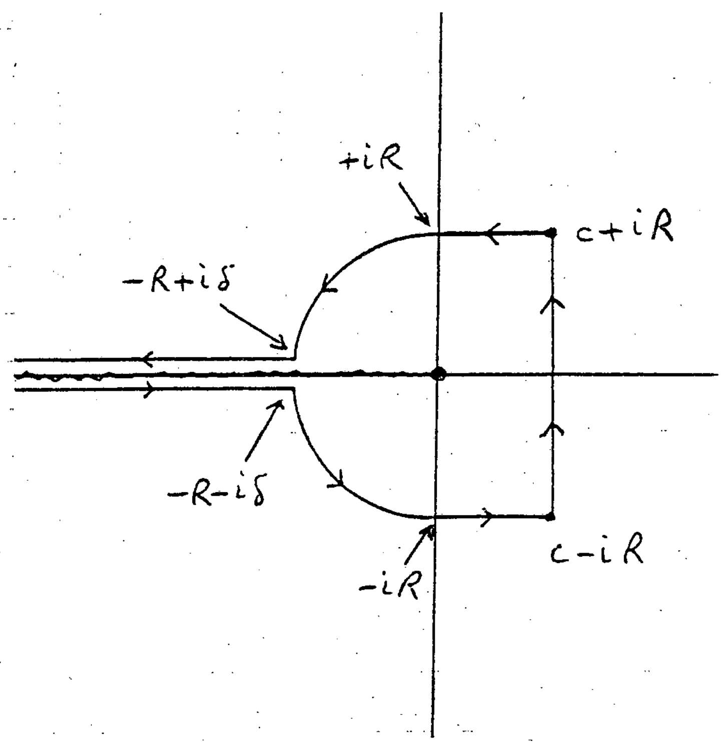
$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx = \frac{5\pi}{12}.$$

3. Using contour integration, evaluate the integral

$$\int_0^{\infty} \frac{\ln x}{x^{\beta}(1+x)}\, dx \qquad (0<\beta<1).$$

4. (a) Modifying Hankel's loop as shown below (with c a real and positive constant, $\delta \to 0$ and $R \to \infty$), deduce that for $\operatorname{Re} z > 0$

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{t}\, t^{-z}\, dt \qquad \text{(Laplace)}.$$



(b) Show that, with a suitable substitution, the above result can be expressed in the form

$$\frac{1}{\Gamma(z)} = \frac{e^{c}\, c^{1-z}}{\pi} \int_0^{\pi/2} \cos(c\tan\theta - z\theta)\, \cos^{z-2}\theta \, d\theta.$$

5. Show that

$$\int_{-\infty}^{\infty} \frac{\cos(bx)}{x^2+a^2}\,dx = \frac{\pi}{a} e^{-a|b|}$$

and

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{\cos(bx)}{x^2-a^2}\,dx = -\frac{\pi}{a}\sin(a|b|).$$

Using either of these results, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(bx)-\cos(cx)}{x^2}\,dx.$$

6. Verify that the function $f(z) = e^{iz}$ satisfies conditions conforming to dispersion relations.

Next, check by direct integration that the real part, $\cos x$, and the imaginary part, $\sin x$, of the function $f(x) = e^{ix}$ indeed satisfy dispersion relations.