Chapter 11

Elastic Collisions

11.1 Center of Mass Frame

A collision or 'scattering event' is said to be *elastic* if it results in no change in the internal state of any of the particles involved. Thus, no internal energy is liberated or captured in an elastic process.

Consider the elastic scattering of two particles. Recall the relation between laboratory coordinates $\{r_1, r_2\}$ and the CM and relative coordinates $\{R, r\}$:

$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \qquad\qquad r_1 = R + \frac{m_2}{m_1 + m_2}\, r \qquad (11.1)$$

$$r = r_1 - r_2 \qquad\qquad r_2 = R - \frac{m_1}{m_1 + m_2}\, r \qquad (11.2)$$

If external forces are negligible, the CM momentum $P = M\dot{R}$ is constant, and therefore the frame of reference whose origin is tied to the CM position is an inertial frame of reference. In this frame,

$$v_1^{\mathrm{CM}} = \frac{m_2\, v}{m_1 + m_2} \qquad , \qquad v_2^{\mathrm{CM}} = -\frac{m_1\, v}{m_1 + m_2} \ , \qquad (11.3)$$

where $v = v_1 - v_2 = v_1^{\mathrm{CM}} - v_2^{\mathrm{CM}}$ is the relative velocity, which is the same in both L and CM frames. Note that the CM momenta satisfy

$$p_1^{\mathrm{CM}} = m_1 v_1^{\mathrm{CM}} = \mu v \qquad (11.4)$$

$$p_2^{\mathrm{CM}} = m_2 v_2^{\mathrm{CM}} = -\mu v \ , \qquad (11.5)$$

where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass. Thus, $p_1^{\mathrm{CM}} + p_2^{\mathrm{CM}} = 0$ and the total momentum in the CM frame is zero. We may then write

$$p_1^{\mathrm{CM}} \equiv p_0 \hat{n} \qquad , \qquad p_2^{\mathrm{CM}} \equiv -p_0 \hat{n} \qquad \Rightarrow \qquad E^{\mathrm{CM}} = \frac{p_0^2}{2m_1} + \frac{p_0^2}{2m_2} = \frac{p_0^2}{2\mu} \ . \qquad (11.6)$$
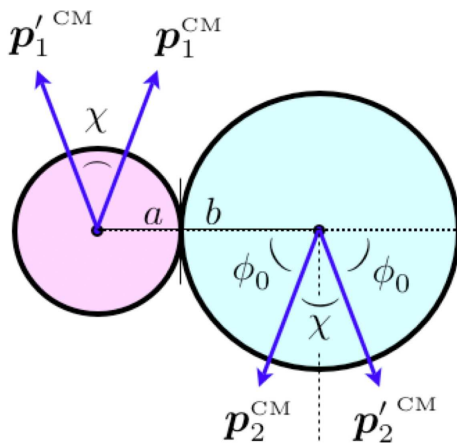
Figure 11.1: The scattering of two hard spheres of radii a and b The scattering angle is χ.

The energy is evaluated when the particles are asymptotically far from each other, in which case the potential energy is assumed to be negligible. After the collision, energy and momentum conservation require

$$\boldsymbol{p}_1'^{\,\text{CM}} \equiv p_0 \hat{\boldsymbol{n}}' \qquad , \qquad \boldsymbol{p}_2'^{\,\text{CM}} \equiv -p_0 \hat{\boldsymbol{n}}' \qquad \Rightarrow \qquad E'^{\text{CM}} = E^{\text{CM}} = \frac{p_0^2}{2\mu} \ . \tag{11.7}$$

The angle between $\boldsymbol{n}$ and $\boldsymbol{n}'$ is the *scattering angle* χ:

$$\boldsymbol{n} \cdot \boldsymbol{n}' \equiv \cos\chi \ . \tag{11.8}$$

The value of χ depends on the details of the scattering process, *i.e.* on the interaction potential $U(r)$. As an example, consider the scattering of two hard spheres, depicted in Fig. 11.1. The potential is

$$U(r) = \begin{cases} \infty & \text{if } r \leq a+b \\ 0 & \text{if } r > a+b \ . \end{cases} \tag{11.9}$$

Clearly the scattering angle is $\chi = \pi - 2\phi_0$, where ϕ_0 is the angle between the initial momentum of either sphere and a line containing their two centers at the moment of contact.

There is a simple geometric interpretation of these results, depicted in Fig. 11.2. We have

$$\boldsymbol{p}_1 = m_1 \boldsymbol{V} + p_0 \hat{\boldsymbol{n}} \qquad\qquad \boldsymbol{p}_1' = m_1 \boldsymbol{V} + p_0 \hat{\boldsymbol{n}}' \tag{11.10}$$

$$\boldsymbol{p}_2 = m_2 \boldsymbol{V} - p_0 \hat{\boldsymbol{n}} \qquad\qquad \boldsymbol{p}_2' = m_2 \boldsymbol{V} - p_0 \hat{\boldsymbol{n}}' \ . \tag{11.11}$$

So draw a circle of radius p_0 whose center is the origin. The vectors $p_0 \hat{\boldsymbol{n}}$ and $p_0 \hat{\boldsymbol{n}}'$ must both lie along this circle. We define the angle ψ between $\boldsymbol{V}$ and $\boldsymbol{n}$:

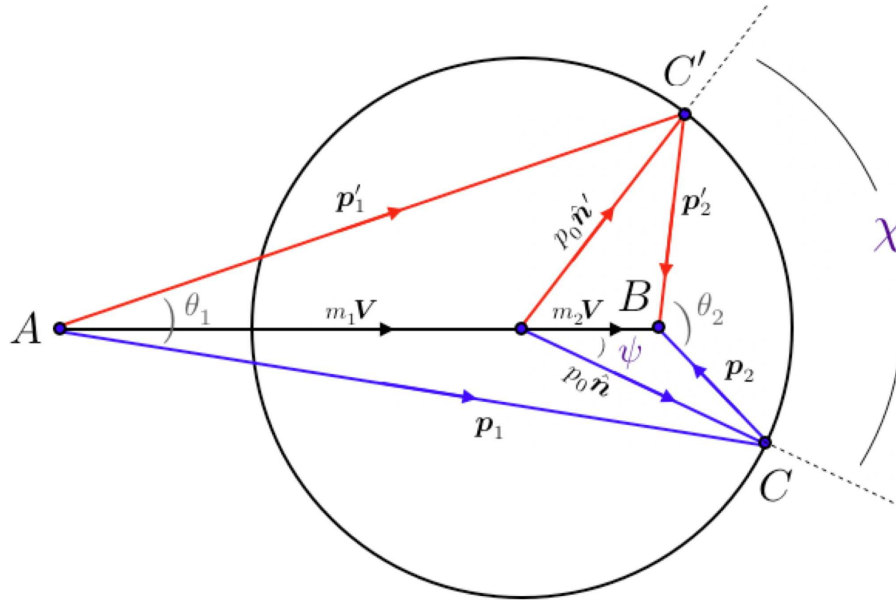$$\hat{\boldsymbol{V}} \cdot \boldsymbol{n} = \cos\psi \ . \tag{11.12}$$

Figure 11.2: Scattering of two particles of masses m_1 and m_2. The scattering angle χ is the angle between $\hat{n}$ and $\hat{n}'$.

It is now an exercise in geometry, using the law of cosines, to determine everything of interest in terms of the quantities V, v, ψ, and χ. For example, the momenta are

$$p_1 = \sqrt{m_1^2 V^2 + \mu^2 v^2 + 2m_1 \mu V v \cos \psi} \tag{11.13}$$

$$p_1' = \sqrt{m_1^2 V^2 + \mu^2 v^2 + 2m_1 \mu V v \cos(\chi - \psi)} \tag{11.14}$$

$$p_2 = \sqrt{m_2^2 V^2 + \mu^2 v^2 - 2m_2 \mu V v \cos \psi} \tag{11.15}$$

$$p_2' = \sqrt{m_2^2 V^2 + \mu^2 v^2 - 2m_2 \mu V v \cos(\chi - \psi)} \;, \tag{11.16}$$

and the scattering angles are

$$\theta_1 = \tan^{-1}\left(\frac{\mu v \sin \psi}{\mu v \cos \psi + m_1 V}\right) + \tan^{-1}\left(\frac{\mu v \sin(\chi - \psi)}{\mu v \cos(\chi - \psi) + m_1 V}\right) \tag{11.17}$$

$$\theta_2 = \tan^{-1}\left(\frac{\mu v \sin \psi}{\mu v \cos \psi - m_2 V}\right) + \tan^{-1}\left(\frac{\mu v \sin(\chi - \psi)}{\mu v \cos(\chi - \psi) - m_2 V}\right) \;. \tag{11.18}$$

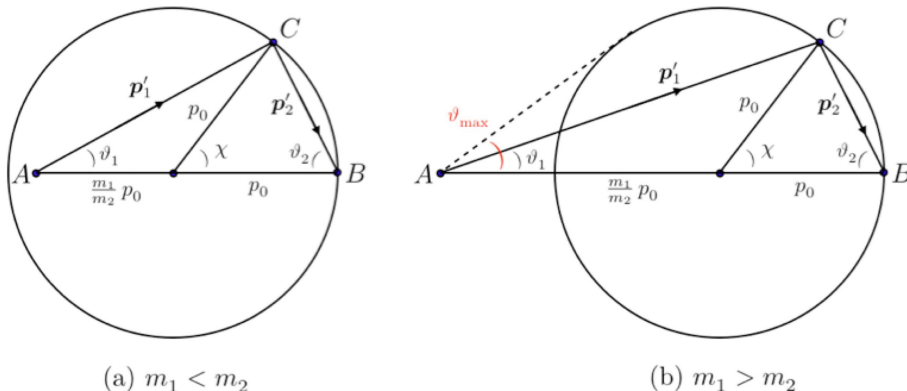(a) $m_1 < m_2$ (b) $m_1 > m_2$

Figure 11.3: Scattering when particle 2 is initially at rest.

If particle 2, say, is initially at rest, the situation is somewhat simpler. In this case, $\boldsymbol{V} = m_1\boldsymbol{V}/(m_1 + m_2)$ and $m_2\boldsymbol{V} = \mu\boldsymbol{v}$, which means the point B lies on the circle in Fig. 11.3 ($m_1 \neq m_2$) and Fig. 11.4 ($m_1 = m_2$). Let $\vartheta_{1,2}$ be the angles between the directions of motion after the collision and the direction $\boldsymbol{V}$ of impact. The scattering angle χ is the angle through which particle 1 turns in the CM frame. Clearly

$$\tan\vartheta_1 = \frac{\sin\chi}{\frac{m_1}{m_2} + \cos\chi} \qquad , \qquad \vartheta_2 = \tfrac{1}{2}(\pi - \chi) . \tag{11.19}$$

We can also find the speeds v_1' and v_2' in terms of v and χ, from

$$p_1'^{\,2} = p_0^2 + \left(\tfrac{m_1}{m_2}\,p_0\right)^2 - 2\,\tfrac{m_1}{m_2}\,p_0^2\,\cos(\pi - \chi) \tag{11.20}$$

and

$$p_2^2 = 2\,p_0^2\,(1 - \cos\chi) . \tag{11.21}$$

These equations yield

$$v_1' = \frac{\sqrt{m_1^2 + m_2^2 + 2m_1m_2\cos\chi}}{m_1 + m_2}\,v \qquad , \qquad v_2' = \frac{2m_1v}{m_1 + m_2}\,\sin(\tfrac{1}{2}\chi) . \tag{11.22}$$
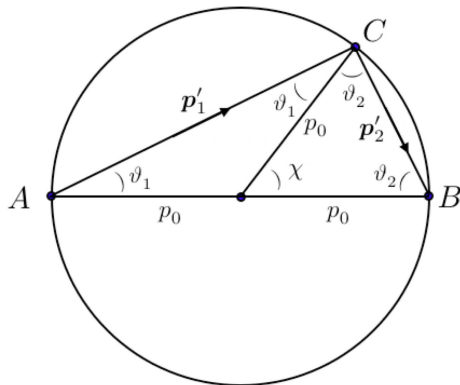

Figure 11.4: Scattering of identical mass particles when particle 2 is initially at rest.
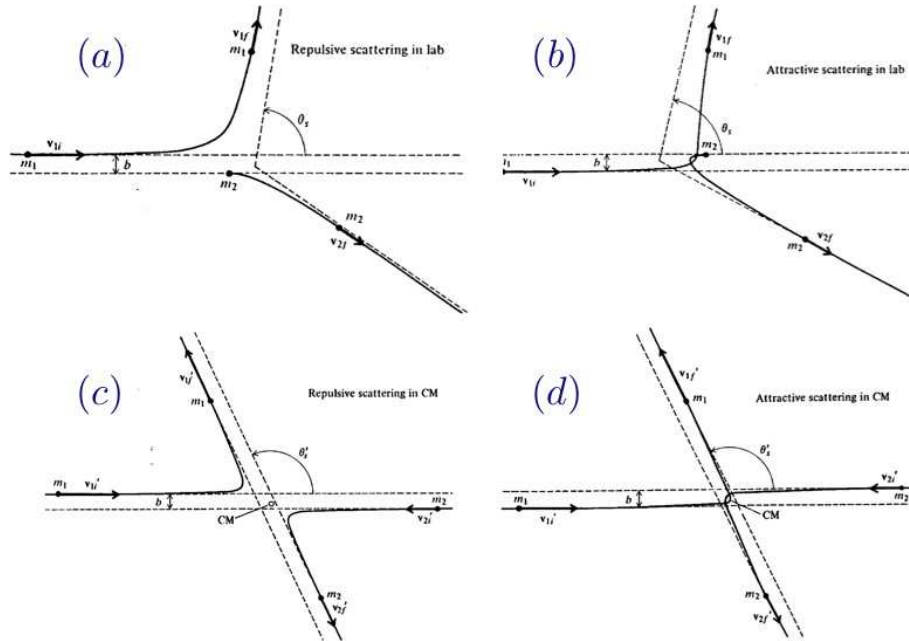
Figure 11.5: Repulsive (A,C) and attractive (B,D) scattering in the lab (A,B) and CM (C,D) frames, assuming particle 2 starts from rest in the lab frame. (From Barger and Olsson.)

The angle $\vartheta_{\max}$ from Fig. 11.3(b) is given by $\sin\vartheta_{\max} = \frac{m_2}{m_1}$. Note that when $m_1 = m_2$ we have $\vartheta_1 + \vartheta_2 = \pi$. A sketch of the orbits in the cases of both repulsive and attractive scattering, in both the laboratory and CM frames, in shown in Fig. 11.5.

11.2 Central Force Scattering

Consider a single particle of mass μ movng in a central potential $U(r)$, or a two body central force problem in which μ is the reduced mass. Recall that

$$\frac{dr}{dt} = \frac{d\phi}{dt} \cdot \frac{dr}{d\phi} = \frac{\ell}{\mu r^2} \cdot \frac{dr}{d\phi} \; , \tag{11.23}$$

and therefore

$$E = \tfrac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)$$

$$= \frac{\ell^2}{2\mu r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{\ell^2}{2\mu r^2} + U(r) \; . \tag{11.24}$$

Solving for $\frac{dr}{d\phi}$, we obtain

$$\frac{dr}{d\phi} = \pm\sqrt{\frac{2\mu r^4}{\ell^2}\big(E - U(r)\big) - r^2} \; , \tag{11.25}$$
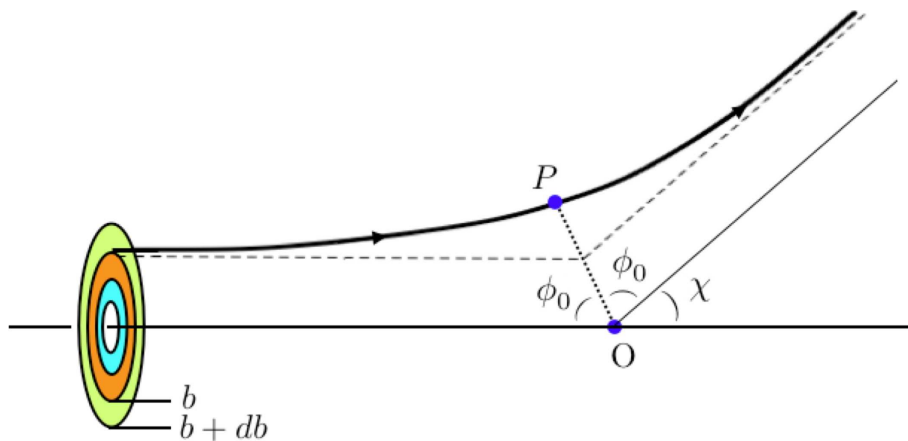
Figure 11.6: Scattering in the CM frame. O is the force center and P is the point of periapsis. The impact parameter is b, and χ is the scattering angle. ϕ_0 is the angle through which the relative coordinate moves between periapsis and infinity.

Consulting Fig. 11.6, we have that

$$\phi_0 = \frac{\ell}{\sqrt{2\mu}} \int\limits_{r_{\mathrm{p}}}^{\infty} \frac{dr}{r^2 \sqrt{E - U_{\mathrm{eff}}(r)}} \ , \tag{11.26}$$

where r_{p} is the radial distance at periapsis, and where

$$U_{\mathrm{eff}}(r) = \frac{\ell^2}{2\mu r^2} + U(r) \tag{11.27}$$

is the effective potential, as before. From Fig. 11.6, we conclude that the scattering angle is

$$\chi = \left| \pi - 2\phi_0 \right| \ . \tag{11.28}$$

It is convenient to define the *impact parameter b* as the distance of the asymptotic trajectory from a parallel line containing the force center. The geometry is shown again in Fig. 11.6. Note that the energy and angular momentum, which are conserved, can be evaluated at infinity using the impact parameter:

$$E = \tfrac{1}{2}\mu v_\infty^2 \qquad , \qquad \ell = \mu v_\infty b \ . \tag{11.29}$$

Substituting for $\ell(b)$, we have

$$\phi_0(E, b) = \int\limits_{r_{\mathrm{p}}}^{\infty} \frac{dr}{r^2} \frac{b}{\sqrt{1 - \frac{b^2}{r^2} - \frac{U(r)}{E}}} \ , \tag{11.30}$$

In physical applications, we are often interested in the deflection of a beam of incident particles by a scattering center. We define the *differential scattering cross section $d\sigma$* by

$$d\sigma = \frac{\#\ \text{of particles scattered into solid angle } d\Omega \text{ per unit time}}{\text{incident flux}} \ . \tag{11.31}$$
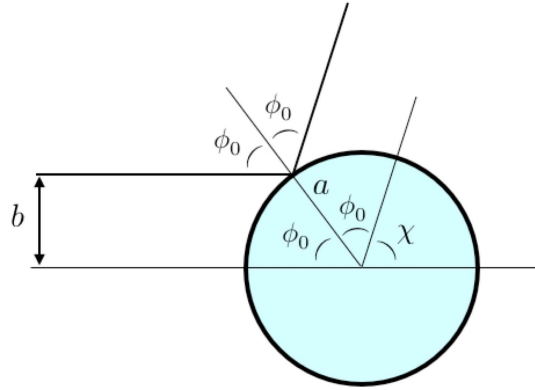
Figure 11.7: Geometry of hard sphere scattering.

Now for particles of a given energy E there is a unique relationship between the scattering angle χ and the impact parameter b, as we have just derived in eqn. 11.30. The differential solid angle is given by $d\Omega = 2\pi \sin \chi \, d\chi$, hence

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\chi} \left| \frac{db}{d\chi} \right| = \left| \frac{d\left(\frac{1}{2}b^2\right)}{d\cos\chi} \right| . \tag{11.32}$$

Note that $\frac{d\sigma}{d\Omega}$ has dimensions of area. The integral of $\frac{d\sigma}{d\Omega}$ over all solid angle is the *total scattering cross section*,

$$\sigma_{\mathrm{T}} = 2\pi \int\limits_0^\pi d\chi \, \sin\chi \, \frac{d\sigma}{d\Omega} . \tag{11.33}$$

11.2.1 Hard sphere scattering

Consider a point particle scattering off a hard sphere of radius a, or two hard spheres of radii a_1 and a_2 scattering off each other, with $a \equiv a_1 + a_2$. From the geometry of Fig. 11.7, we have $b = a \sin \phi_0$ and $\phi_0 = \frac{1}{2}(\pi - \chi)$, so

$$b^2 = a^2 \sin^2 \left(\tfrac{1}{2}\pi - \tfrac{1}{2}\chi\right) = \tfrac{1}{2}a^2 \left(1 + \cos\chi\right) . \tag{11.34}$$

We therefore have

$$\frac{d\sigma}{d\Omega} = \frac{d\left(\frac{1}{2}b^2\right)}{d\cos\chi} = \tfrac{1}{4}\,a^2 \tag{11.35}$$

and $\sigma_{\mathrm{T}} = \pi a^2$. The total scattering cross section is simply the area of a sphere of radius a projected onto a plane perpendicular to the incident flux.

11.2.2   Rutherford scattering

Consider scattering by the Kepler potential $U(r) = -\frac{k}{r}$. We assume that the orbits are unbound, *i.e.* they are Keplerian hyperbolae with $E > 0$, described by the equation

$$r(\phi) = \frac{a\,(\varepsilon^2 - 1)}{\pm 1 + \varepsilon\,\cos\phi} \quad \Rightarrow \quad \cos\phi_0 = \pm\frac{1}{\varepsilon}\ . \tag{11.36}$$

Recall that the eccentricity is given by

$$\varepsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2} = 1 + \left(\frac{\mu b v_\infty}{k}\right)^2 . \tag{11.37}$$

We then have

$$\left(\frac{\mu b v_\infty}{k}\right)^2 = \varepsilon^2 - 1$$

$$= \sec^2\phi_0 - 1 = \tan^2\phi_0 = \operatorname{ctn}^2\!\left(\tfrac{1}{2}\chi\right) . \tag{11.38}$$

Therefore

$$b(\chi) = \frac{k}{\mu v_\infty^2}\,\operatorname{ctn}\!\left(\tfrac{1}{2}\chi\right) \tag{11.39}$$

We finally obtain

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{d\left(\tfrac{1}{2}b^2\right)}{d\cos\chi} = \frac{1}{2}\left(\frac{k}{\mu v_\infty^2}\right)^2 \frac{d\operatorname{ctn}^2\!\left(\tfrac{1}{2}\chi\right)}{d\cos\chi} \\
&= \frac{1}{2}\left(\frac{k}{\mu v_\infty^2}\right)^2 \frac{d}{d\cos\chi}\left(\frac{1 + \cos\chi}{1 - \cos\chi}\right) \\
&= \left(\frac{k}{2\mu v_\infty^2}\right)^2 \operatorname{csc}^4\left(\tfrac{1}{2}\chi\right) ,
\end{aligned} \tag{11.40}$$

which is the same as

$$\frac{d\sigma}{d\Omega} = \left(\frac{k}{4E}\right)^2 \operatorname{csc}^4\left(\tfrac{1}{2}\chi\right) . \tag{11.41}$$

Since $\frac{d\sigma}{d\Omega} \propto \chi^{-4}$ as $\chi \to 0$, the total cross section σ_{T} diverges! This is a consequence of the long-ranged nature of the Kepler/Coulomb potential. In electron-atom scattering, the Coulomb potential of the nucleus is *screened* by the electrons of the atom, and the $1/r$ behavior is cut off at large distances.

11.2.3   Transformation to laboratory coordinates

We previously derived the relation

$$\tan\vartheta = \frac{\sin\chi}{\gamma + \cos\chi}\ , \tag{11.42}$$

where $\vartheta \equiv \vartheta_1$ is the scattering angle for particle 1 in the laboratory frame, and $\gamma = \frac{m_1}{m_2}$ is the ratio of the masses. We now derive the differential scattering cross section in the laboratory frame. To do so, we note that particle conservation requires

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{L}} \cdot 2\pi \sin\vartheta\, d\vartheta = \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} \cdot 2\pi \sin\chi\, d\chi\ , \tag{11.43}$$

which says

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{L}} = \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} \cdot \frac{d\cos\chi}{d\cos\vartheta}\ . \tag{11.44}$$

From

$$\cos\vartheta = \frac{1}{\sqrt{1 + \tan^2\vartheta}}$$

$$= \frac{\gamma + \cos\chi}{\sqrt{1 + \gamma^2 + 2\gamma\cos\chi}}\ , \tag{11.45}$$

we derive

$$\frac{d\cos\vartheta}{d\cos\chi} = \frac{1 + \gamma\cos\chi}{\left(1 + \gamma^2 + 2\gamma\cos\chi\right)^{3/2}} \tag{11.46}$$

and, accordingly,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{L}} = \frac{\left(1 + \gamma^2 + 2\gamma\cos\chi\right)^{3/2}}{1 + \gamma\cos\chi} \cdot \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}}\ . \tag{11.47}$$