PHYSICS 200A : CLASSICAL MECHANICS
PROBLEM SET #5

[1] The figure below shows a string with an attached mass m. The mass point is connected to springs above and below with spring constant κ. In equilibrium, both springs are unstretched. An right-moving disturbance $f(ct-x)$ is incident from the far left. For $x<0$ a left-moving reflected component, $g(ct+x)$ also propagates, while for $x>0$ there is only a right-moving transmitted component $h(ct-x)$, as depicted in the figure below.

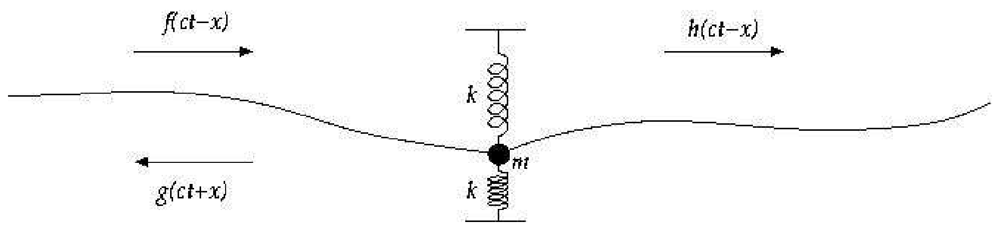

Figure 1: A string with a mass point connected to springs. An incident wave gives rise to reflected and transmitted components.

(a) Define

$$h(\xi)=\int\limits_{-\infty}^{\infty} d\xi'\,\mathcal{T}(\xi-\xi')\,f(\xi')\ .$$

Compute and sketch the kernel $\mathcal{T}(s)$. You may find it convenient to define the quantities $Q\equiv\sigma/m$ and $P\equiv\sqrt{2\kappa\sigma/m\tau}$, both of which have dimension of inverse length. How do the cases $Q<P$ and $Q>P$ differ?

(b) Suppose a square pulse is incident, *i.e.*

$$f(\xi)=b\,\Theta\big(a-|x|\big)\ ,$$

where $\Theta(s)$ is the step function. Derive expressions for the transmitted wave $h(\xi)$ and the reflected wave $g(\xi)$. Plot $f(\xi)$, $g(\xi)$, and $g(\xi)$ *versus* ξ. Choose interesting values of the dimensionless quantities Qa and Pa.

[2] A string of uniform mass density and length ℓ hangs under its own weight in the earth's gravitational field. Consider small transverse displacements $u(x,t)$ in a plane.

(a) Compute the equilibrium tension in the string $\tau(x)$, where x is the distance from the point of suspension.

(b) Show that the normal modes satisfy Bessel's equation.

(c) What are the boundary conditions?

(d) What are the normal mode frequencies?

(e) What are the normal modes?

(f) Construct the general solution to the initial value problem.

[3] A string of length $2a$ is stretched to a constant tension τ with its ends fixed. The mass density of the string is given by

$$\sigma(x) = \sigma_0 \left(1 - \frac{|x|}{a}\right) .$$

(a) Use a zero-parameter trial function to derive a variational estimate of the lowest resonant frequency ω_1. Compare with the numerical value $\omega_1^2 \approx 3.477\,\tau/a^2\sigma_0$.

(b) Devise a one-parameter trial function and show that it leads to a better (*e.g.* lower frequency) estimate.

(c) Repeat part (a) for the next eigenfrequency ω_2, whose numerical value is $\omega_2^2 \approx 18.956\,\tau/a^2\sigma_0$.

[4] A wave travels along an infinite string stretched to a tension τ. The mass density of the string is σ_0 for $|x| > a$ and σ_1 for $|x| < a$.

(a) Solve the wave equation in the regions $|x| > a$ and $|x| < a$, respectively, to find exact expressions for the transmission and reflection amplitudes.

(b) Show that the energy transmission coefficient is given by

$$T = 1 - R = \left\{ 1 + \left(\frac{k_1^2 - k_0^2}{2k_0 k_1} \right)^2 \sin^2(2k_1 a) \right\}^{-1} ,$$

where $k_i = \sqrt{\sigma_i/\tau}\,\omega = \omega/c_i$. Discuss the frequency dependence of T, noting the position and widths of the transmission resonances (where $T = 1$).

[5] Consider a uniform circular membrane of radius a, areal mass density σ, and tension τ.

(a) A point mass m is attached at the center of the membrane. Show that the total density is now

$$\sigma(r, \phi) = \sigma + \frac{m}{\pi r}\,\delta(r) .$$

(b) Use first-order perturbation theory to show that only the circularly symmetric modes are affected by the point mass, in which case

$$k^2a^2 = x_{0,n}^2 \left\{ 1 - \frac{m}{\pi\sigma a^2 J_1^2(x_{0,n})} \right\} ,$$

where $J_\nu(x_{\nu,n}) = 0$, *i.e.* $x_{\nu,n}$ is the n^{th} root of $J_\nu(x)$. Discuss the behavior for large n and compare to the corresponding case of a point mass on a string.