Recap

(1) Discussed modern concept of force: virtual bosons mediate interaction between two Fermions

- Bosons with odd integer spin (=1), interaction repulsive between like particles: example is virtual photon mediating repulsive Coulomb force between 2 protons

- Bosons with even integer spin (0, 2) interaction is attractive between like particles: we will talk about virtual pions mediating attractive strong force between 2 nucleons (n, p).

(2) Range of force: virtual Boson created and destroyed, emitted & absorbed, on timescale $\Delta t \leq \hbar / E$.

$\therefore$ range $\Delta d = c\Delta t \leq c\hbar / E$

Example 1 • Virtual photons: $E = h\nu \Rightarrow$

$$\Delta d \leq c/\nu < \lambda$$

Force is long range since E can be arbitrarily small or λ arbitrarily large.

Example 2 Strong Force:

~~Suppose virtual Boson had mass~~

(1) Quarks: interact via massless spin=1 gluons. ~~Difficult~~ Complicated and I won't go into detail. But

- (a) Quarks interact by exchange of gluons, but that exchange does not go on beyond the bound particle (i.e., the n or p).

(2) neutrons and protons: the strong force between n and p mediated by Boson of finite mass, the pion. Before I describe pion, a few words about consequences of Boson with finite mass.

ⓐ Energy: Suppose the exchange Boson has a mass m_0. In that case energy extracted from emitting particle cannot be arbitrarily small since: $\boxed{E \geq m_0 c^2}$: rest-mass energy is lower limit on E.

ⓑ Range: Since maximum speed is c, then maximum distance traveled:

$$\Delta d \leq c \Delta t$$

$$d \leq c\left(\frac{\hbar}{E}\right) = \frac{c\hbar}{m_0 c^2}$$

$$\boxed{\Delta d \leq \frac{\hbar}{m_0 c}}$$ ← Compton wavelength.

ⓒ Pions: the exchange Bosons mediating strong interaction between nucleons (n, p) are π mesons or pions. They are comprised of u, d quark ~~pairs and their~~ anti-~~particles~~ quark pairs.

Recall

quark	charge	spin
u	+2/3	1/2
d	−1/3	1/2

Pair	Charge	Spin	Symbo
$u\bar{u}$ or $d\bar{d}$	$2/3 - 1/3 = 0$ $-1/3 + 1/3 = 0$	$\uparrow\downarrow = 0$	${}_0\pi^0$ (
$u\bar{d}$	$2/3 + 1/3 = 1$	$\uparrow\downarrow = 0$	${}_1\pi^0$ (π^+)
$\bar{u}d$	$-2/3 - 1/3 = -1$	$\uparrow\downarrow = 0$	${}_{-1}\pi^0$ (π^-

~~[illegible]~~ Quark/anti-quark pairs have opposite spins, so net spin is zero. As a result 3 pions

(a) are zero-spin Bosons

(b) mediate attractive force since they posses even integer spin

Result:

$$\begin{pmatrix} n+p \\ n+n \\ p+p \end{pmatrix}$$

← experience attractive force

← crucial to offset Coulomb repulsion

(c) <u>short range</u>

$$\Delta d \leq \frac{h}{m_\pi c}. \quad \text{But } m_\pi = 0.15\, m_p$$

$$\text{Therefore } \Delta d \leq \frac{6.626\times10^{-27}}{.15\times1.67\times10^{-24}\times3\times10^{10}} = 8\times10^{-13} \simeq 10^{-12}\,\text{cm}$$

Example 3 <u>Weak Nuclear Force</u>

n, p also interact through exchange of W and Z Boson. The masses of these particles are heavy $m_W \simeq 100 \cdot m_p$ or $\simeq 10^3 m_\pi$.

Therefore range $d_W \simeq 10^{-3} d_\pi \simeq 10^{-15}$ cm.

Significance of the Weak Interaction:

These interactions are crucial for nucleosynthesis, i.e., fusion reactions that build up the elements. Why? Nuclei of heavy elements contain approximately equal numbers of neutrons, n, and protons, p. But the nucleus of H, the most basic and abundant of all elements is a single proton.

Therefore, to fuse H into heavier elements, the weak force must be invoked to convert half of the interacting protons and the strong force to bind the proton and neutrons together.

Particle Physics:

Proton	neutron
u·u·d	u d d

Weak interaction converts $u \rightleftarrows d$

But $\left.\begin{array}{l} q = +2/3 \text{ for } u \\ q = -1/3 \text{ for } d \end{array}\right\} \Delta q = \pm 1$

So, to conserve charge, e^- or e^+ must be involved. Also to conserve Lepton no., $\nu, \tilde{\nu}$ appear.

Example: A neutron outside the nucleus, i.e., a free neutron, will decay via the weak interaction in about 10 min. into a $p, e^-, \tilde{\nu}$.

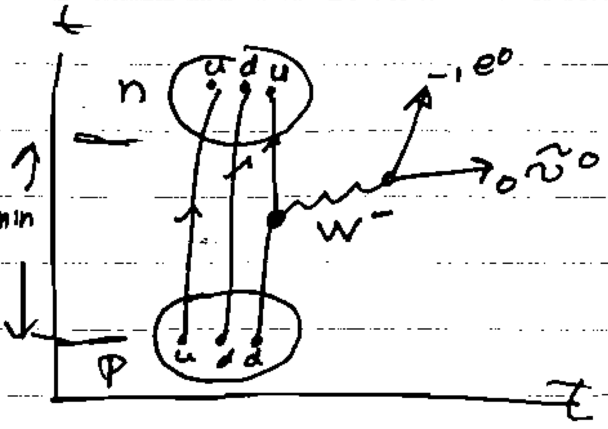
$$ {}_0n^1 \longrightarrow {}_1P^1 + {}_{-1}e^0 + {}_0\tilde{\nu}^0 $$

	${}_0n^1$	${}_1P^1$	${}_{-1}e^0$	${}_0\tilde{\nu}^0$
	u d d	u u d		
charge	0	1	−1	0
Lepton no	L = 0	L = 0	L = 1	L = −1

This reaction is called β decay, which results in emission/or absorption of $_{-1}e^0$, $_{+1}\bar{e}^0$.

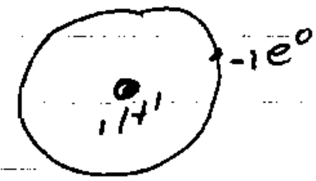
What's going on? a d quark turns into an u quark

$$Q \quad \left.\begin{array}{l} \text{(i)}\ d \rightarrow u + W^- \\ \quad\ -1/3 \quad 2/3 \quad -1 \\ \\ \text{(ii)}\ W^- \rightarrow {}_{-1}e^0 + \tilde{\nu}_e \\ \quad\ -1 \quad\ -1 \quad\ 0 \end{array}\right\}$$



Nuclei Let's first consider Hydrogen & its isotopes

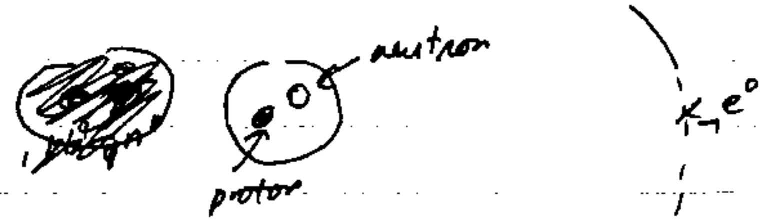
Hydrogen $_1H^1$



nucleus consists of single proton. Surrounded by electron interacting only through electrostatic force.

Deuterium. $_1H^2$

Nucleus consists of bound n, p, surrounded by electron that interacts with proton only through electrostatic interaction



Tritium. $_1H^3$

Nucleus consists of one proton and 2 neutron

Chemical properties of atoms determined by QM properties of bound electrons. therefore, chemical properties of isotopes are essentially the same. But nuclear properties differ -

Binding Energy. Recall n, p in $_1H^2$ as in all nuclei are bound by pion exchange; i.e., system contains negative potential energy:

Resultant rest-mass energy:

$$M_{_1H^2}c^2 = m_n c^2 + m_p c^2 - B(_1H^2)$$

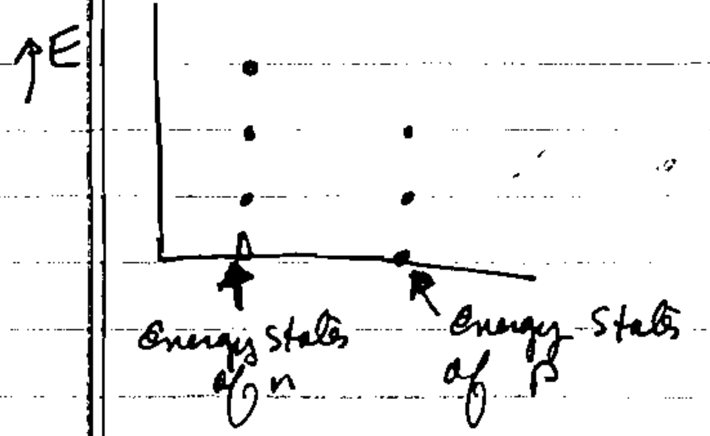
e binding energy of $_1H^2$ nucleus

As a result: $M_{_1H^2}c^2 < m_n c^2 + m_p c^2$

Stability of $_1H^2$.

Free neutron decays in ~10 min. Why doesn't bound n in $_1H^2$ decay?

Nuclear Energy Level Diagram

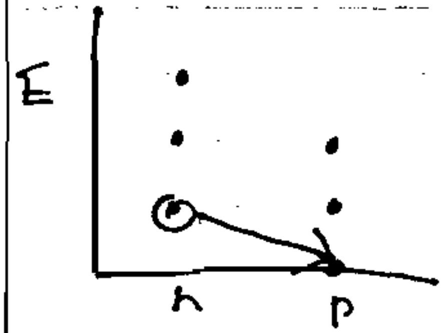


(a) p and n have separate energy states because they are distinguishable particles

(b) Pauli exclusion principle says no 2 n's or p's can occupy same state

(c) n states have higher energy because $m_n c^2 > m_p c^2$

Free neutron Energy Level Diagram

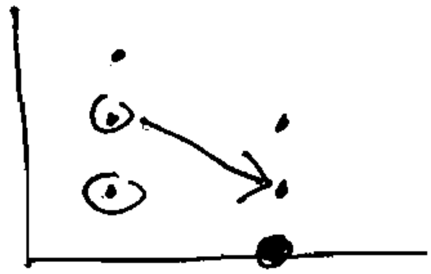


Free n° unstable, becau its energy lies above unoccupied p state

Deuterium Energy Level Diagram:

Lower energy p state is occupied so n decay is forbidden: $_1H^2$ stable

Tritium Energy Level Diagram:



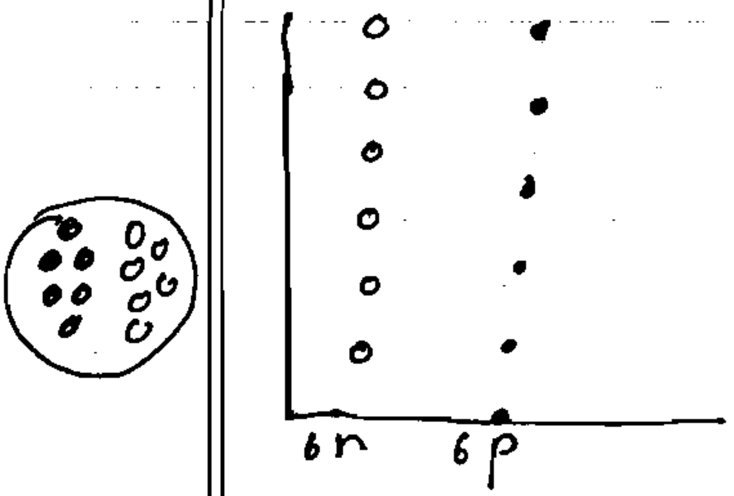
$_1H^3$ unstable since 2^{nd} n can decay into unoccupied excited p state

$$_1H^3 \longrightarrow {}_2He^3 + {}_{-1}e^0 + {}_0\tilde{\nu}^0$$

	$_1H^3$	$_2He^3$	$_{-1}e^0$	$_0\tilde{\nu}^0$
L	0	0	+1	−1
Charge	+1	+2	+1	0

Stability of Nuclei: stability against β decay explains why no. of neutrons ≃ no. of protons in stable nuclei.

Consider $_6C^{12}$ nucleus



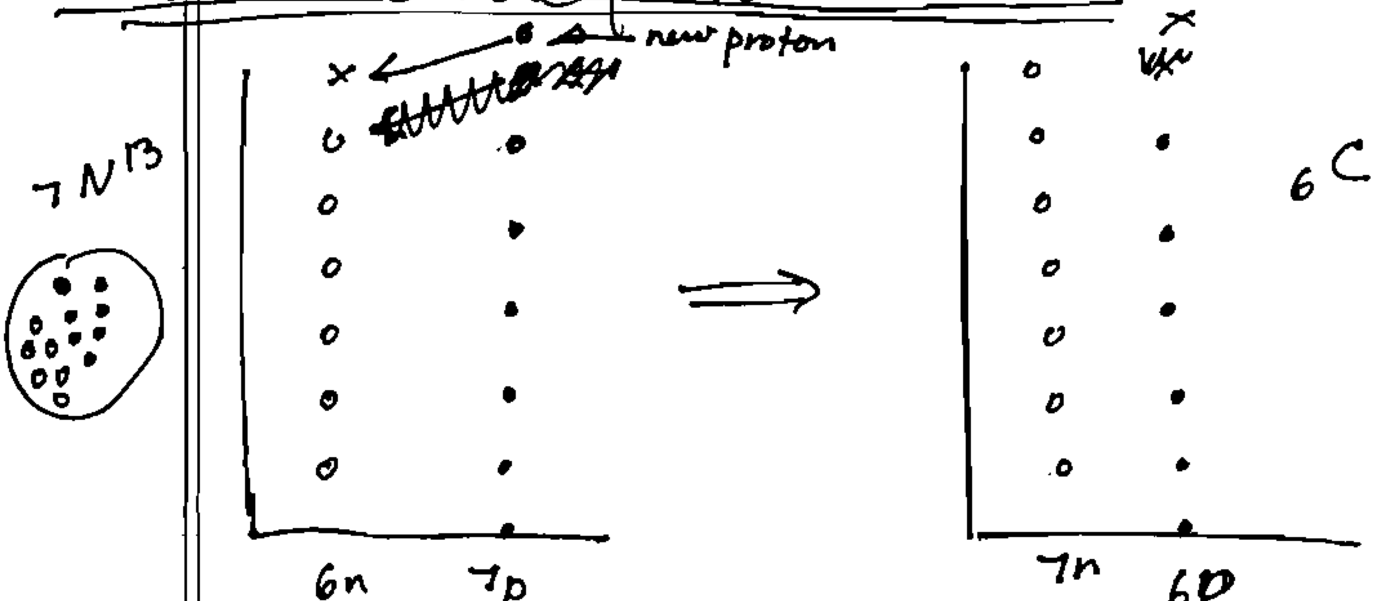
energy separation of protons increases due to increased electrostatic repulsion between positively charged particles

$_6C^{12}$ is stable since all lower p states are filled

no. of protons $\simeq$ no. of neutrons

$$Z \simeq A - Z \Rightarrow A \simeq 2Z$$

Now add one more proton to stack



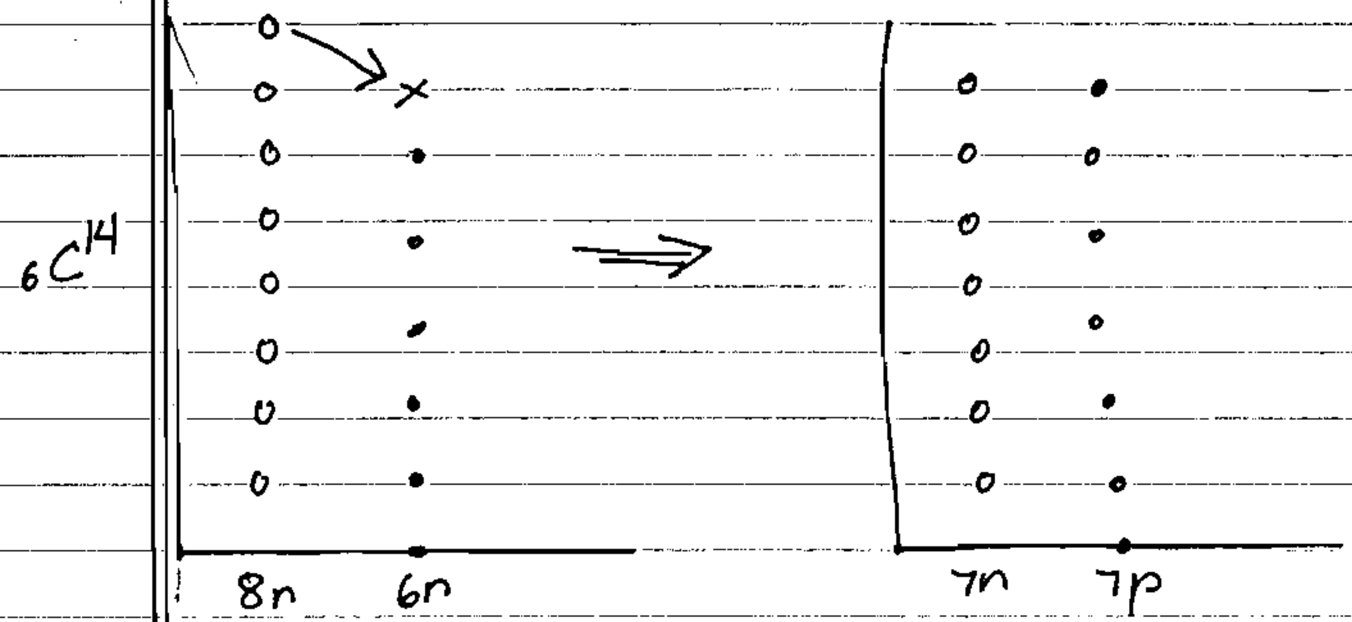
Adding another proton increases ΔE separating p levels. New p state above unfilled n state. Therefore $_7N^{13}$ is unstable

$$_7N^{13} \longrightarrow {}_6C^{13} + {}_{+1}e^0 + {}_0\nu^0$$

L	0	0	−1	+1
Q	7	6	1	0

$_6C^{13}$ is stable since highest energy n state is above filled p state.

Add another n to $_6C^{13}$



$$_6C^{14} \rightarrow {}_7N^{14} + {}_{-1}e^0 + {}_0\bar{\nu}^0$$

L	0	0	+1	−1
Q	6	7	−1	0

Thermonuclear Reactions:

Two obstacles facing fusion reactions:

(1) p → n: Idea is to convert $_1H^1 \rightarrow {}_1H^2$
This implies conversion of p → n.
But $m_nc^2 - m_pc^2 = 1.29$ MeV
How do we create more massive particle?

(2) Coulomb Barrier
Need to take 2 protons within $\sim 10^{-13}$ cm. separation for strong force to turn on.

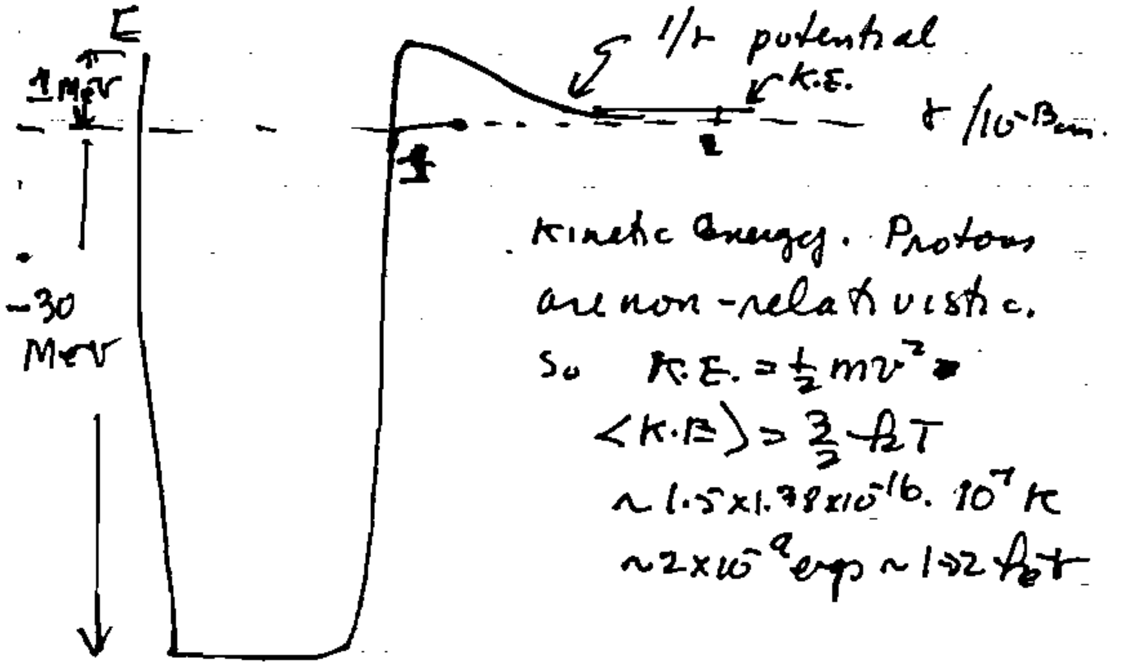
Particles see positive (repulsive) potential energy barrier given by

$$V_{coul} = \frac{Z_1 Z_2 e^2}{r}$$

$$V_{coul} = \frac{Z_1 Z_2 (4.8\times10^{-10})^2}{10^{-13}\,cm \cdot r_f} = \frac{2\times10^{-6} Z_1 Z_2}{r_f}\ \text{erg}$$

$$\approx \left(\frac{Z_1 Z_2}{r_f}\right) \text{MeV} \quad \left(r_f = \frac{r}{10^{-13}cm}\ \text{Fermis}\right)$$

Schematic Energy Level Diagram



Kinetic energy. Protons are non-relativistic.
So $K.E. = \frac{1}{2}mv^2$
$\langle K.E.\rangle = \frac{3}{2}kT$
$\sim 1.5\times1.38\times10^{-16}\cdot10^{7}\,K$
$\sim 2\times10^{-9}$ erg $\sim 1-2$ keV

As a result. $k.e. \ll V_{coul}$. Classically protons would just bounce off Coulomb barrier and be repelled:

$$E = K.E. + V \Rightarrow K.E. = E - V$$

So for K.E. to be positive, $E > V$. But $E = (K.E.)_{r\to\infty}$. Clearly $E \ll V$ where E is total energy

Let's skip answer for now and look at reaction networks!

Fusion Reactions

Nuclei a and b fuse to make c: $a + b \rightarrow c$

Energy Conservation: $E_a + E_b = E_c + Q$

where E's are rest-mass energies of particles

Q is energy released.

Q can take the form of non-nuclei ($\gamma, \nu, \tilde{\nu}$, K.E., e)

If $Q > 0$: energy released: Exothermic Reaction

$Q < 0$: " absorbed: endothermic "

In most cases we require $Q > 0$, since energy input required to drive endothermic reactions is not available. But there are notable exceptions, which we will discuss later on.

Since nuclei are non-relativistic: $m_a c^2 + m_b c^2 = m_c c^2 + Q$

$$\boxed{Q = m_a c^2 + m_b c^2 - m_c c^2}$$

Example: pp cycle: Crucial 1st step in $_2He^4$ production

$$_1H^1 + {}_1H^1 \longrightarrow {}_1H^2 + {}_1\bar{e}^0 + {}_0\nu^0$$

L	0	0	0	−1	+1
chg.	1	1	1	1	0

Reaction is exothermic if: $Q > 0$ or

$$2m_p c^2 > m_D c^2 + m_e c^2$$

$$2m_p c^2 > [(m_n + m_p)c^2 - B_D] + m_e c^2$$

or $B_D > (m_n - m_p)c^2 + m_e c^2$

$> 1.29\ \text{MeV} + 0.51\ \text{MeV} = 1.8\ \text{MeV}$

In fact measured binding energy of $_1H^2$ (D) nucleus $[B_D]_{meas} = 2.22$ MeV

Thus reaction is exothermic and goes spontaneously

Energy Released: $Q = \underbrace{\sum_i m_i c^2}_{\text{Before}} - \underbrace{\sum_f m_f c^2}_{\text{after}}$

$$\therefore\ Q = 2m_p c^2 - (m_D c^2 + m_e c^2)$$

$$= 2m_p c^2 - \left[(m_n + m_p)c^2 - B_D + m_e c^2\right]$$

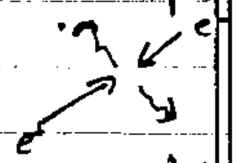
$$Q = B_D - (m_n - m_p)c^2 - m_e c^2$$

$$Q = 2.22 - (1.29 + .51) = 2.22 - 1.80$$

$$Q = 0.42\ \text{MeV}$$

Energy dumped into star:

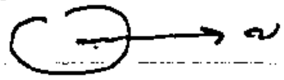
But positron e^+ annihilates ambient e^- producing 2 photons with energy $E_\gamma = 2 \cdot m_e c^2 = 2(.51) = 1.02$ MeV



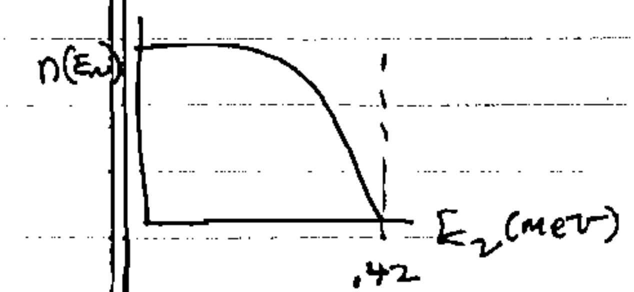
Therefore total energy released

$$Q_{TOT} = 0.42 + E_\gamma = 1.44\ \text{MeV}$$

Neutrinos: Neutrinos escape without dumping any energy into surrounding matter, i.e., without heating the star.



Neutrino Energy spectrum:



Energy released, Q, in form of e^+, ν. Let's look at energetics again. We neglected k.e. of e^-, ${}_1H^2$

Rewrite energy conservation:

$$2m_p c^2 = \gamma m_D c^2 + \gamma m_e c^2 + E_\nu$$

$$= (\gamma - 1)m_D c^2 + (\gamma - 1)m_e c^2 + m_D c^2 + m_e c^2 + E_\nu$$

$$\therefore\ Q \equiv 2m_p c^2 - (m_D c^2 + m_e c^2) = (\gamma - 1)m_D c^2 + (\gamma - 1)m_e c^2 + E_\nu$$

$$\simeq \tfrac{1}{2}m_D v^2 + \tfrac{1}{2}m_e v^2 + E_\nu$$

Recap.

(1) First Step in $_2He^4$ Production

$$_1H^1 + {}_1H^1 \longrightarrow {}_1H^2 + {}_1\bar{e}^0 + {}_0\nu^0$$

(2) Energy Released

$$Q = \underbrace{\sum_i m_i c^2}_{\text{before}} - \underbrace{\sum_f m_f c^2}_{\text{after}}$$

In this case $Q = 2m_p c^2 - (m_D c^2 + m_e c^2)$

Recall $m_D c^2 = m_n c^2 + m_p c^2 - B_d$

$\therefore\ Q = 2m_p c^2 - m_n c^2 - m_p c^2 + B_D - m_e c^2$

$Q = B_D - (m_n - m_p)c^2 - m_e c^2$

$Q = 2.22 - 1.29 - 0.51 = 0.42$ MeV

(3) Total Energy Output:

~~Q~~ Must include photons produced by $e^+ + e^-$ annihation. $Q_{TOT} = Q + 2m_e c^2 = 1.44$ MeV

(4) Energy Available for heating

But since ν's escape from sun, we must subtract E_ν from Q_{TOT} : $Q_{heat} = Q_{TOT} - E_\nu$

(5) Energy Conservation. (Include K.E.)

$$2m_p c^2 + \underset{\text{total k.e of protons}}{E_p} = m_D c^2 + m_e c^2 + \underset{\text{k.e D+e}}{E_{D,e}} + \underset{\nu\text{ energy}}{E_\nu}$$

$$\therefore\ Q = 2m_p c^2 - (m_D + m_e)c^2 = \underbrace{E_{D,e} - E_p}_{\text{increase in K.E}} + E_\nu$$

Therefore increase in KE $\boxed{E_{D,e} - E_{2p} = 0.42 - E_\nu}$

~~Extreme~~ possibilities

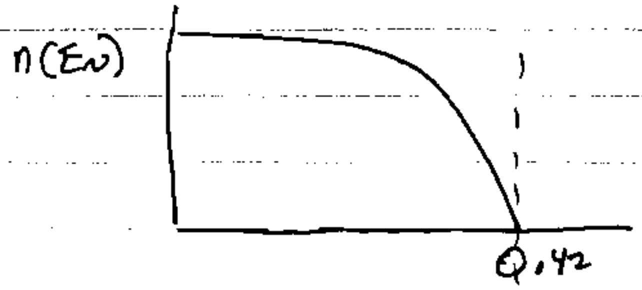
Ⓐ $E_{D,e} = E_p$: no increase in kinetic energy. In that case all the energy goes into the neutrino : $E_\nu = 0.42\,MeV$

Since $Q_{heat} = Q_{TOT} - E_\nu$

$= Q + 2m_ec^2 - E_\nu = 2m_ec^2 = 1.02\,MeV$

Only annihilation energy available

Ⓑ But ~~not~~ neutrinos produced with a range of energies:



$n(E_\nu)dE_\nu$ density of ν's in $(E_\nu, E_\nu + dE_\nu)$

Average energy produced:

$$\langle E_\nu \rangle = \frac{\int E_\nu\, n(E_\nu)\, dE_\nu}{\int n(E_\nu)\, dE_\nu}$$

One finds $\langle E_\nu \rangle = 0.26\,MeV$

Realistic estimate: $Q_{heat} = Q_{TOT} - \langle E_\nu \rangle$

$Q_{heat} = 1.44 - 0.26 = 1.18\,MeV$

Ⓒ Net Energy Input

(i) Annihilation photons

(ii) Increased in kinetic energies of ${}_1H^2$ and ${}_1\bar{e}^0$ with respect to ${}_1H^1 + {}_1H^1$

$\Delta K.E = E_{D,e} - E_p = 0.42 - \langle E_\nu \rangle = 0.42 - .26$

$\Delta K.E = 0.16\,MeV = 160\,keV$

For above average $K.E = \frac{3}{2}kT \sim 2\,keV$

Let's now look a p-p chain of reactions leading to $_2He^4$ production.

Step	Reaction	Energetics Q_{TOT} (MeV)	$\langle E_\nu \rangle$*
①	$_1H^1 + {}_1H^1 \rightarrow {}_1H^2 + {}_1\bar{e}^0 + {}_0\nu^0$	1.44	0.26
②	$_1H^1 + {}_1H^2 \rightarrow {}_2He^3 + {}_0\gamma^0$	5.49	
③	$_2He^3 + {}_2He^3 \rightarrow {}_2He^4 + {}_1H^1 + {}_1H^1$	12.85	

Comments

(A) Reaction ① is slow. It involves weak interaction in which p converted into n.

(B) Photon liberated. No e, ν involved. So this is a strong reaction thus rate of reaction is fast. Presence of γ implies electromagnetic force involved.

(B) Here we have reaction between 2 particles in which $Z_1 = Z_2 = 2$. Coulomb barrier 4 times higher ($\propto Z_1 Z_2$) than in $_1H^2$ production. As we shall see, this slows reaction down.

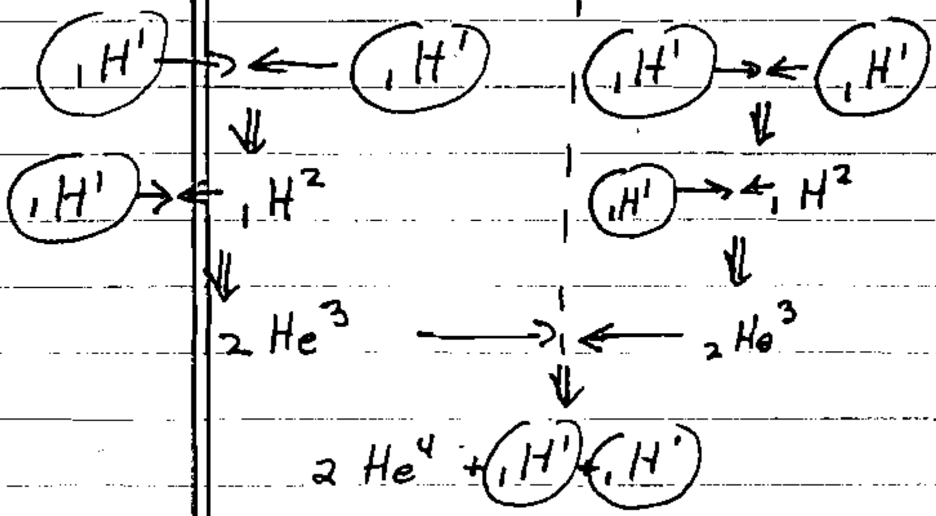
Another possibility for step ③

$${}_1H^1 + {}_2He^3 \rightarrow {}_2He^4 + {}_1e^0 + {}_0\nu^0$$

But since this involves $p \rightarrow n$ conversion, weak interaction is present and rate is slower than interaction between ${}_2He^3 + {}_2He^3$ ∴ We neglect it.

Energy Budget

Reaction ③ requires presence of second ${}_2He^3$ nucleus. Thus real situation is as follows:



Q_{TOT} (MeV)	$\langle E_\nu \rangle$ M
2×1.44	$2 \times .26$
$2(5.49)$	
12.85	

Accounting: 6 protons go in ... come
2 protons + one ${}_2He^4$ nucleus / aa

Result: Net conversion of 4 protons into 1 ${}_2He^4$ nucleus.

Energy Available to heat the Sun

$$Q_{heat} = \sum_{i=1}^{3} (Q_{TOT})_i - \sum_{i=1}^{3} \langle E_\nu \rangle_i \quad \{\text{sum over reactions}\}$$

$$Q_{heat} = 2 \times 1.44 + 2 \times 5.49 + 12.85 - 2 \times 0.26$$

$$Q_{heat} = 26.2 \text{ MeV}$$

This reaction chain is called ppI: proton-proton chain. But there are two other possibilities: ppII and ppIII. Situation looks like this—

$$_1H^1 + {}_1H^1 \rightarrow {}_1H^2 + {}_1\bar{e}^0 + {}_0\nu^0$$
$$_1H^1 + {}_1H^2 \rightarrow {}_2He^3 + \gamma$$

69%

$$_2He^3 + {}_2He^3 \rightarrow {}_2He^4 + {}_1H^1 + {}_1H^1$$

ppI

31%

(from ppI)

$$_2He^3 + {}_2He^4 \rightarrow {}_4Be^7 + \gamma$$

99.7%

$$_4Be^7 + {}_{-1}\bar{e}^0 \rightarrow {}_3Li^7 + {}_0\nu^0$$
$$_3Li^7 + {}_1H^1 \rightarrow {}_2He^4 + {}_2He^4$$

ppII

0.3%

$$_4Be^7 + {}_1H^1 \rightarrow {}_5B^8 + \gamma$$
$$_5B^8 \rightarrow {}_4Be^8 + {}_1\bar{e}^0 + {}_0\nu^0$$
$$_4Be^8 \rightarrow {}_2He^4 + {}_2He^4$$

ppIII

Even though ppIII occurs only 0.03x.31 = .093% of the time, the $_5B^8$ neutrino is very ~~of time~~ significant, since it was first detected ~~and~~ at rate significantly lower than predicted. This gave rise to the solar neutrino problem, and its solution in the form of neutrino color oscillations: more later on this.

<u>CNO Cycle</u>

Hans Bethe who conceived the pp chain also realized that $_2He^4$ can be synthesized with the aid of heavy elements acting as catalysts.

CNO Cycle

Step	Reaction	Q_{TOT} (MeV)	$\langle E_\nu \rangle$ (MeV)
①	${}_1H^1 + {}_6C^{12} \rightarrow {}_7N^{13} + {}_0\gamma^0$	1.95	
②	${}_7N^{13} \rightarrow {}_6C^{13} + {}_1\bar{e}^0 + {}_0\nu^0$	2.22	0.72
③	${}_1H^1 + {}_6C^{13} \rightarrow {}_7N^{14} + \gamma$	7.54	
④	${}_1H^1 + {}_7N^{14} \rightarrow {}_8O^{15} + \gamma$	7.35	
⑤	${}_8O^{15} \rightarrow {}_7N^{15} + {}_1\bar{e}^0 + {}_0\nu^0$	2.71	0.98
⑥	${}_1H^1 + {}_7N^{15} \rightarrow \boxed{{}_6C^{12}} + {}_2He^4$	4.96	
		26.73	1.70

Comments

Ⓐ again Conversion ~~of~~ of 4 ${}_1H^1 \rightarrow 1\ {}_2He^4$

Ⓑ $Q_{heat} = \sum_{i=1}^{6} (Q_{TOT})_i - \sum_{i=1}^{6} \langle E_\nu \rangle = 26.73 - 1.70 = 25.03$

Ⓒ In both pp and CNO cycles,
$Q_{heat} \simeq (4m_p - m_{{}_2He^4})c^2 \simeq 26$ MeV
Qheat slightly lower in CNO cycle, because of larger neutrino losses

Ⓓ CNO cycle: ${}_6C^{12}$ acts like a catalyst.
${}_6C^{12}$ injected in step ① and ejected in step ⑥. So, no net consumption of ${}_6C^{12}$.
Rather; 4 (${}_1H^1$) nuclei consumed and

one (${}_2He^4$) nucleus produced.

Bi Cycle: But this is not completely true. There are competing reactions that result in the net loss of ${}_6C^{12}$ nuclei

reinjection of ${}_6C^{12}$

$${}_1H^1 + {}_6C^{12} \to {}_7N^{13} + {}_0\gamma^0$$
$${}_7N^{13} \to {}_6C^{13} + {}_1\bar{e}^0 + {}_0\nu^0$$
$${}_1H^1 + {}_6C^{13} \to {}_7N^{14} + \gamma$$
$${}_1H^1 + {}_7N^{14} \to {}_8O^{15} + \gamma$$
$${}_8O^{15} \to {}_7N^{15} + {}_1\bar{e}^0 + {}_0\nu^0$$

reinjection of ${}_7N^{14}$ but not ${}_6C^{12}$

99.96% 12 .04%

$${}_1H^1 + {}_7N^{15} \to {}_6C^{12} + {}_2He^4$$

$${}_1H^1 + {}_7N^{15} \to {}_8O^{16} + \gamma$$
$${}_1H^1 + {}_8O^{16} \to {}_9F^{17} + \gamma$$
$${}_9F^{17} \to {}_8O^{17} + {}_1\bar{e}^0 + {}_0\nu^0$$
$${}_1H^1 + {}_8O^{17} \to {}_7N^{14} + {}_2He^4$$

Result is slow conversion of C → N nuclei

$\boxed{\text{Back to reaction Rates}}$

Two-body Fusion Problem

(1) Mechanics

In c.m. frame:

$$m_b \xleftarrow{\ \mathbf{r}_b\ } \bullet \xrightarrow{\ \mathbf{r}_a\ } m_a$$
$$\xrightarrow{\quad \mathbf{r} \quad}$$

Total energy in ~~central~~ field with potential energy $v(r)$: $r = |\mathbf{r}|$

- Center of mass definition. $m_a \mathbf{r}_a + m_b \mathbf{r}_b = 0$

$$\mathbf{r} = \mathbf{r}_a - \mathbf{r}_b$$

Combine 2 equations:

$$m_a \mathbf{r}_a + m_b(\mathbf{r}_a - \mathbf{r}) = 0$$
$$(m_a + m_b)\mathbf{r}_a = m_b \mathbf{r}$$
$$\Rightarrow \boxed{\mathbf{r}_a = \frac{m_b}{m_a + m_b}\mathbf{r}} \quad (1)$$

Since $\mathbf{r}_b = -\frac{m_a}{m_b}\mathbf{r}_a$; $\boxed{\mathbf{r}_b = -\frac{m_a}{m_a + m_b}\mathbf{r}}$ (2)

- Total Energy

$$\boxed{E = \frac{1}{2} m_a \dot{\mathbf{r}}_a^2 + \frac{1}{2} m_b \dot{\mathbf{r}}_b^2 + v(r)} \quad (3)$$

Let $\mathbf{v}_a = \dot{\mathbf{r}}_a$; $\vec{\mathbf{v}}_b = \dot{\mathbf{r}}_b$; $\mathbf{v} = \dot{\mathbf{r}} = \mathbf{v}_a - \mathbf{v}_b$

From eq. (1), $\dot{\mathbf{r}}_a = \frac{m_b}{m_a + m}\dot{\mathbf{r}} \Rightarrow \dot{\mathbf{r}}_a^2 = \frac{m_b^2}{(m_a+m_b)^2}\dot{\mathbf{r}}^2$

$$\dot{\mathbf{r}}_b^2 = \frac{m_a^2}{(m_a+m_b)}\dot{\mathbf{r}}^2$$

From (3) $\therefore E = \frac{1}{2}\frac{m_a m_b^2}{(m_a+m_b)^2}\dot{\mathbf{r}}^2 + \frac{1}{2}\frac{m_b m_a^2}{(m_a+m_b)}\dot{\mathbf{r}}^2 + v(r)$

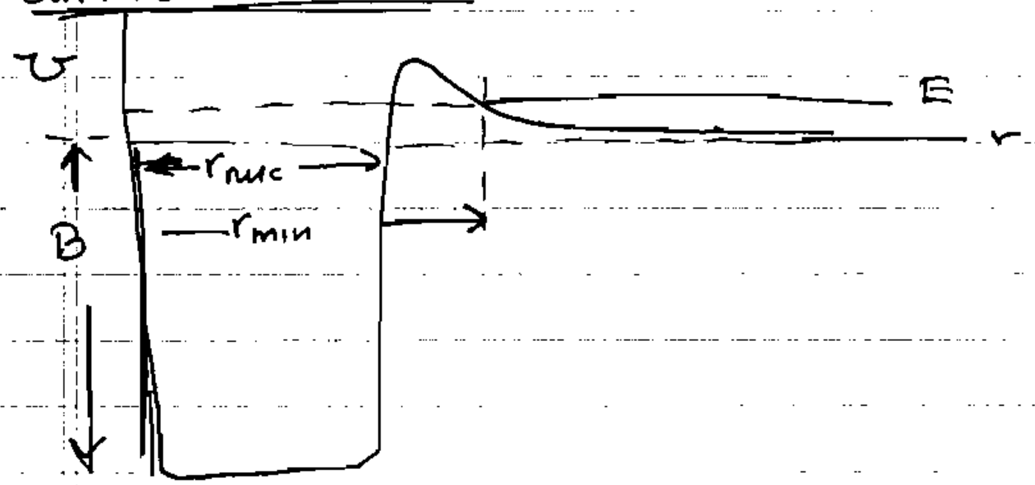
$$E = \frac{1}{2}\frac{m_a m_b}{(m_a+m_b)^2}[m_b + m_a] v^2 + v(r)$$

or $E = \frac{1}{2}\mu_m v^2 + V(r)$

where reduced mass $\mu_m = \frac{M_a M_b}{M_a + M_b}$

relative velocity $v = |\dot{r}|$.

② Barrier Problem



Classical problem: $\frac{1}{2}\mu V^2 = E - V(r)$

only permissible orbits are where

$E > V(r) \Rightarrow \frac{1}{2}\mu_m V^2$

Turning point: as particle with reduced mass μ moves in negative r direction, we see that For a given total energy $E = \frac{1}{2}\mu V_\infty^2$, $V(r)$ increases. As a result $\frac{1}{2}\mu V^2$ decreases until $\frac{1}{2}\mu V^2 =$ at r_{min}. at that point $E = V(r_{min})$.

$$\text{or} \quad \frac{1}{2}\mu_m V_\infty^2 = V(r_{min}) = \frac{Z_1 Z_2 e^2}{r_{min}}$$

Last Lecture I showed that the Coulomb repulsive barrier reaches maximum height at $r \simeq r_{nuc} \simeq 10^{-13}$ cm or $r_F = 1$ Fermi

$$V_{max} \approx \frac{Z_1 Z_2}{r_F} \text{ MeV}$$