1 Midterm - Physics 110 A

All problems are worth 10 points.

1. *Making use of the permutation symbol* verify the vector identity

$$b^2 = \overrightarrow{b} \cdot \overrightarrow{b} = \left(\widehat{u} \cdot \overrightarrow{b}\right)^2 + \left(\widehat{u} \times \overrightarrow{b}\right)^2,$$

where the vector $\widehat{u}$ is an arbitrary unit vector.

Solution (i) Using the permutation symbol the expression for $\left(\widehat{u} \times \overrightarrow{b}\right)^2$ is

$$\begin{aligned}\left(\widehat{u} \times \overrightarrow{b}\right)^2 &= \epsilon_{ijk}\epsilon_{i\ell n} u_j b_k u_\ell b_n = \left(\delta_{j\ell}\delta_{kn} - \delta_{jn}\delta_{k\ell}\right) u_j u_\ell b_k b_n \\ \left(\widehat{u} \times \overrightarrow{b}\right)^2 &= u_j u_j b_k b_k - u_j b_j u_\ell b_\ell = \left(\overrightarrow{u} \cdot \overrightarrow{u}\right)\left(\overrightarrow{b} \cdot \overrightarrow{b}\right) - \left(\widehat{u} \cdot \overrightarrow{b}\right)^2.\end{aligned}$$

Since $\overrightarrow{u} \cdot \overrightarrow{u} = 1$, we have the required result,

$$\overrightarrow{b} \cdot \overrightarrow{b} = \left(\widehat{u} \cdot \overrightarrow{b}\right)^2 + \left(\widehat{u} \times \overrightarrow{b}\right)^2.$$

2. A rocket of initial mass m_0 uses its engines to hover stationary above the ground (gravitational acceleration is g). How long does it hover while burning fuel of mass $\left(1 - e^{-2}\right) m_0$? Assume that the exhaust velocity u is a constant.

In the presence of a gravitational field the EOM is

$$m\frac{dv}{dt} = -mg - u\frac{dm}{dt}.$$

Since the rocket is hovering $dv/dt = 0$, and the EOM reduces to

$$u\frac{dm}{dt} = -mg \rightarrow u\frac{dm}{m} = -g$$

Since the fuel is $\Delta m = m_f - m_0$, the mass of the rocket after the fuel is burned is $m_f = m_0 - \Delta m = m_0\left(1 - \left(1 - e^{-2}\right)\right) = e^{-2}m_0$. From the EOM we find the time to be

$$\begin{aligned} u\int_{m_0}^{e^{-2}m_0} \frac{dm}{m} &= u \ln\frac{e^{-2}m_0}{m_0} = -2u = -gt \\ t &= 2u/g.\end{aligned}$$

3. In the presence of the Earth's gravitational field, consider a baseball with a quadratic drag term, $-cv^2$. The baseball is initially thrown directly downward from the top of a tall building at twice the terminal velocity, $v_0 = 2v_{ter}$, where $v_{ter} = \sqrt{mg/c}$. Find the expression that describes the velocity of the object as a function of time (prior to reaching the ground) during the decent and discuss the physical results described by your answer. Hint: you may find the use of parrtial fractions useful for performing the necessary integration.

The EOM is (v and y positive in the downward direction)

$$m\frac{dv}{dt} = mg - cv^2 \rightarrow \frac{dv}{dt} = g\left(1 - \frac{c}{mg}v^2\right) = g\left(1 - v^2/v_{ter}^2\right)$$
$$\frac{dv}{v^2/v_{ter}^2 - 1} = -gdt \rightarrow \frac{1}{2}\left(\frac{1}{v/v_{ter} - 1} - \frac{1}{v/v_{ter} + 1}\right)dv = -gt$$
$$\frac{1}{2}v_{ter}\int_{2v_{ter}}^{v}\left(\frac{1}{v - v_{ter}} - \frac{1}{v + v_{ter}}\right)dv = \frac{v_{ter}}{2}\ln\frac{v - v_{ter}}{v_{ter}}\frac{3v_{ter}}{v + v_{ter}} = -gt$$
$$3\frac{v - v_{ter}}{v + v_{ter}} = e^{-2gt/v_{ter}} \rightarrow 3\left(v - v_{ter}\right) = e^{-2gt/v_{ter}}\left(v + v_{ter}\right)$$
$$\left(3 - e^{-2gt/v_{ter}}\right)v = v_{ter}\left(3 + e^{-2gt/v_{ter}}\right) \rightarrow v = v_{ter}\frac{3 + e^{-2gt/v_{ter}}}{3 - e^{-2gt/v_{ter}}}.$$

The solution shows that the velocity of the object is always greater than v_{ter} while approaching the terminal velocity after starting with an initial velocity of $v_0 = 2v_{ter}$.

4. The figure shows the end view of a child's toy of mass m which amounts to a vertical cylinder mounted on top of a hemisphere. As shown in the figure the radius of the hemisphere is R and the CM at equilibrium is a height h above the floor. (a) What is the gravitational potential energy of the toy as it tips through an angle θ from the vertical. (b) For what values of R and h is the equilibrium at $\theta = 0$ stable?

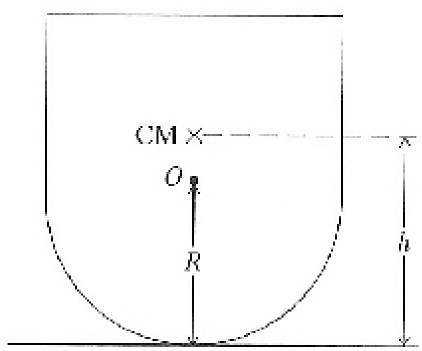

Figure for problem 4.

(a) The height of the center of the hemisphere does not change as the toy tips. Hence the potential energy is determined by the height of the CM above the center of the hemisphere which is $y = (h - R)\cos\theta$. Hence

$$U = mgy = mg\,(h - R)\cos\theta.$$

(b) Equilibrium occurs when

$$\frac{dU}{d\theta} = -mg\,(h - R)\sin\theta = 0 \rightarrow \theta = 0.$$

This equilibrium is stable when $d^2U/d\theta^2 > 0$. This leads to the condition

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0} = -mg\,(h - R)\cos(\theta = 0) = mg\,(R - h) > 0$$

The center of mass must lie below the center of the hemisphere.

5. A uniform wire of mass m and length $\ell = \pi a$ is bent into a perfect semicircle of radius a. Find the vertical height above its diameter for location of the center of mass, y_{cm}, of the wire.

The center of mass in the y direction is

$$y_{cm} = \frac{1}{m}\int y dm.$$

If we define the mass per unit length as $\mu = m/\ell = m/\pi a$ the $dm = \mu ds$, where ds is a differential arc length along the semicircle. Hence

$$dm = \mu a d\theta = \frac{m}{\pi a} a d\theta = \frac{m}{\pi} d\theta.$$

Measuring θ from the positive x axis the height of this differential segment is $y = a\sin\theta$. With these definitions the integral is given by

$$y_{cm} = \frac{1}{\pi}\int_0^{\pi} a\sin\theta d\theta = -\frac{a}{\pi}(\cos\pi - \cos 0) = \frac{2}{\pi}a.$$

6. Consider an overdamped oscillator with $\omega_o = .6\beta$. The oscillator is given a kick so that at $t = 0$ its initial velocity at the origin is v_0. (a) Find the expression $x\,(t)$ for this oscillator including both decay terms. (b) Find the expression for the the maximum distance, $x_{\max}$, from the origin that the oscillator obtains in terms of v_0/β. Comment on the contribution of both terms to $x_{\max}$, specifically the ratio of the term with the decay paramter to the term that decays more rapidly.

(a) The solution for an overdamped oscillator is

$$x\,(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2 - \omega_o^2}t} + C_2 e^{-\sqrt{\beta^2 - \omega_o^2}t}\right),$$

Since $\sqrt{\beta^2 - \omega_o^2} = \sqrt{\beta^2 - .36\beta^2} = \sqrt{.64\beta^2} = .8\beta$, the expression for $x\,(t)$ becomes

$$x\,(t) = C_1 e^{-\beta t/5} + C_2 e^{-9\beta t/5}.$$

From the initial conditions

$$\begin{aligned} x\left(t=0\right) &= C_1 + C_2 = 0 \\ \dot{x}\left(t=0\right) &= -\frac{\beta}{5}\left(C_1 + 9C_2\right) = v_0. \end{aligned}$$

Substituting for C_2 we find

$$-\frac{\beta}{5}\left(C_1 - 9C_1\right) = v_0 \rightarrow C_1 = 5v_0/8\beta.$$

Hence

$$x\left(t\right) = \frac{5v_0}{8\beta}\left(e^{-\beta t/5} - e^{-9\beta t/5}\right).$$

(b) The maximum distance occurs at the time when $\dot{x}\left(t_{\max}\right) = 0$. This leads to the condition

$$\begin{aligned} \dot{x}\left(t_{\max}\right) &= -\frac{v_0}{8}\left(e^{-\beta t_{\max}/5} - 9e^{-9\beta t_{\max}/5}\right) = 0 \\ e^{-\beta t_{\max}/5} &= 9e^{-9\beta t_{\max}/5} \rightarrow -\beta t_{\max}/5 = \ln 9 - 9\beta t_{\max}/5 \\ 8\beta t_{\max}/5 &= \ln 9 \rightarrow t_{\max} = \frac{5}{8\beta}\ln 9 \end{aligned}$$

The distance from the origin at this time is

$$x_{\max} = x\left(t_{\max}\right) = \frac{5v_0}{8\beta}\left(e^{-(\ln 9)/8} - e^{-(9\ln 9)/8}\right) = \frac{5v_0}{8\beta}\left(.760 - .084\right) = .422v_0/\beta.$$

Clearly the term that decays more quickly only contributes about 1/9 as much as the term that decays according to the decay parameter.