**Solutions Assignment 8**

**7.38 (a)** Using *spherical polar coordinates* for an inverted cone with a half angle $\alpha$ the relation between $z$ and $r$ is $z = r\cos\alpha$.

The Cartesian coordinates are

$$x = r\sin\alpha\cos\phi,\ y = r\sin\alpha\sin\phi,\ z = r\cos\alpha.$$

The Cartesian components of the velocity are

$$\dot{x} = \dot{r}\sin\alpha\cos\phi - r\sin\alpha\sin\phi\dot{\phi},\ \dot{y} = \dot{r}\sin\alpha\sin\phi + r\sin\alpha\cos\phi\dot{\phi},\ \dot{z} = \dot{r}\cos\alpha.$$

The kinetic energy is then

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\sin^2\alpha\dot{\phi}^2\right)$$

In a uniform gravitational field the potential energy is $U = mgz = mgr\cos\alpha$. Hence the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\sin^2\alpha\dot{\phi}^2\right) - mgr\cos\alpha.$$

**(b)** The Lagrange equations of motion are

$$\begin{aligned} r &: \quad \frac{\partial\mathcal{L}}{\partial r} = mr\sin^2\alpha\dot{\phi}^2 - mg\cos\alpha = \frac{d}{dt}m\dot{r} = m\ddot{r} \\ \phi &: \quad \frac{\partial\mathcal{L}}{\partial\phi} = 0 \to \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\sin^2\alpha\dot{\phi} = \ell_z \quad \text{(const.)} \end{aligned}$$

Rewriting the radial equation we find

$$\frac{\ell_z^2}{m^2r^3\sin^2\alpha} - g\cos\alpha = \ddot{r}.$$

If $\ell_z = 0$ the the acceleration parallel to the surface of the cone is $\ddot{z} = -g\cos\alpha$ which is exactly what you would obtain sliding down a frictionless surface with this incline. At equilibrium $\ddot{r} = 0$ and

$$\frac{\ell_z^2}{m^2r_o^3\sin^2\alpha} = g\cos\alpha \to r_o^3 = \frac{\ell_z^2}{m^2g\sin^2\alpha\cos\alpha}.$$

**(c)** If the particle is in equilibrium and given a slight kick so that $r = r_o + \epsilon$, the radial equation becomes

$$-3\frac{\ell_z^2}{m^2r_o^4\sin^2\alpha}\epsilon = \ddot{\epsilon}.$$

The solution to this equation is a stable simple harmonic oscillator with frequency

$$\omega^2 = 3\frac{\ell_z^2}{m^2r_o^4\sin^2\alpha} = \frac{3g\cos\alpha}{r_o}\frac{\ell_z^2}{m^2gr_o^3\sin^2\alpha\cos\alpha} = \frac{3g\cos\alpha}{r_o}.$$

**7.41** In cylindrical polar coordinates with $z = k\rho^2$ the Cartesian coordinates are

$$x = \rho \cos \omega t, \;\; y = \rho \sin \omega t, \;\; z = k\rho^2.$$

The Cartesian components for the velocity are

$$\dot{x} = \dot{\rho} \cos \omega t - \rho\omega \sin \omega t, \;\; y = \dot{x} = \dot{\rho} \sin \omega t + \rho\omega \cos \omega t, \;\; \dot{z} = 2k\rho\dot{\rho}$$

Hence the kinetic energy is

$$T = \frac{1}{2} m \left( \dot{\rho}^2 + \rho^2\omega^2 + 4k\rho^2\dot{\rho}^2 \right)$$

Since the potential energy is $U = mgz = mgk\rho^2$ the Lagrangian is

$$\mathcal{L} = \frac{1}{2} m \left( \dot{\rho}^2 + \rho^2\omega^2 + 4k\rho^2\dot{\rho}^2 \right) - mgk\rho^2.$$

The equation of motion is

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \rho} &= m\rho\omega^2 + 4mk\rho\dot{\rho}^2 - 2mgk\rho = m\frac{d}{dt}\left( \dot{\rho} + 4k\rho^2\dot{\rho} \right) \\
\rho\omega^2 + 4k\rho\dot{\rho}^2 - 2gk\rho &= \left(1 + 4k\rho^2\right)\ddot{\rho} + 8k\rho\dot{\rho}^2 \\
\rho\omega^2 - 2gk\rho &= \left(1 + 4k\rho^2\right)\ddot{\rho} + 4k\rho\dot{\rho}^2.
\end{aligned}$$

At equilibrium $\ddot{\rho} = \dot{\rho} = 0$. This means

$$\left(\omega^2 - 2gk\right)\rho_o = 0.$$

One of the solutions occurs at $\rho_o = 0$. For small fluctuations, $\rho = \epsilon$, about this position we find

$$\left(\omega^2 - 2gk\right)\epsilon = \ddot{\epsilon}$$

This is stable only if $2gk > \omega^2$. In that case the system oscillates at a frequency $\Omega^2 = 2gk - \omega^2$.

**7.47 (a)** The transformation between the Cartesian coordinates for $N$ particles and the generalized coordinates only depends on a single generalized coordinate. Hence $\overrightarrow{r}_\alpha = \overrightarrow{r}_\alpha(q)$. This allows us to write the kinetic energy as

$$T = \frac{1}{2}\sum_\alpha m_\alpha \frac{\partial \overrightarrow{r}_\alpha}{\partial q} \cdot \frac{\partial \overrightarrow{r}_\alpha}{\partial q} \dot{q}^2 = \frac{1}{2} A \dot{q}^2,$$

where $A$ is defined as

$$A \equiv \sum_\alpha m_\alpha \frac{\partial \overrightarrow{r}_\alpha}{\partial q} \cdot \frac{\partial \overrightarrow{r}_\alpha}{\partial q}.$$

With this definition is it clear that $A$ is a positive definite quantity and may depend on $q$ but not $\dot{q}$. The Lagrangian is then

$$\mathcal{L}=\frac{1}{2}A\dot{q}^2-U(q).$$

The Lagrange equation of motion is

$$\frac{\partial\mathcal{L}}{\partial q}=\frac{1}{2}\frac{dA}{dq}\dot{q}^2-\frac{dU}{dq}=\frac{d}{dt}\left(A(q)\dot{q}\right)=\frac{dA}{dq}\dot{q}^2+A\ddot{q}.$$

The equation of motion then becomes

$$A\ddot{q}=-\frac{1}{2}\frac{dA}{dq}\dot{q}^2-\frac{dU}{dq}.$$

(b)

At equilibrium $\ddot{q}=\dot{q}=0$. Hence

$$\frac{dU(q_o)}{dq}=0,$$

where $q_o$ is a position of equilibrium (there may be no solutions or numerous possible solutions).

**(c)** Given a solution $q_o$ then for small fluctuations about $q_o$, $q=q_o+\epsilon$ we find

$$A\ddot{\epsilon}=-\frac{\partial U(q_o+\epsilon)}{\partial q}=-\frac{d^2U(q_o)}{dq^2}\epsilon.$$

This is a stable equilibrium only if

$$\frac{d^2U(q_o)}{dq^2}>0,$$

which implies that $U(q_o)$ is at least a local minimum for $U$.

**7.50** The modified Lagrangian (with Lagrange multiplier) for this problem is

$$\mathcal{L}=\frac{1}{2}m_1\dot{x}^2+\frac{1}{2}m_2\dot{y}^2+m_2gy+\lambda(x+y).$$

The modified Lagrange equations for $x$ and $y$ are

$$\begin{aligned} x &: \lambda=m_1\ddot{x} \\ y &: m_2g+\lambda=m_2\ddot{y} \end{aligned}$$

From the constraint we must have $\ddot{x}=-\ddot{y}$. Multiplying the $x$ equation by $-1$ and adding the two expressions while taking into account that $\ddot{x}=-\ddot{y}$, we find

$$\begin{aligned} m_2g &= (m_2+m_1)\ddot{y} \\ \ddot{y} &= \frac{m_2}{m_2+m_1}g. \end{aligned}$$

The Lagrange multiplier is given by

$$\begin{aligned}\lambda &= m_2\left(\ddot{y}-g\right)=m_2\left(\frac{m_2}{m_2+m_1}-1\right)g\\ \lambda &= \frac{m_1m_2}{m_2+m_1}g.\end{aligned}$$

Since

$$F_x^{cstr}=\lambda\left(t\right)\frac{\partial f}{\partial x},\ \text{and}\ \frac{\partial f}{\partial x}=1,$$

we find that the constraint force is

$$F^{cstr}=\lambda=\frac{m_1m_2}{m_2+m_1}g.$$

This is the tension in the rope. All of these results match those obtained from free body diagrams using Newton's second law.

**7.51 (a)** The Lagrangian for the simple pendulum in terms of $x$ and $y$ ($y$ positive in the downward direction) subject to the constraint $\sqrt{x^2+y^2}=\ell$ is

$$\mathcal{L}=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)+mgy+\lambda\sqrt{x^2+y^2}.$$

The modified Lagrange equations are

$$\begin{aligned}\lambda\frac{x}{\sqrt{x^2+y^2}} &= m\ddot{x}\to\lambda\frac{x}{\ell}=m\ddot{x},\\ mg+\lambda\frac{y}{\sqrt{x^2+y^2}} &= m\ddot{y}\to mg+\lambda\frac{y}{\ell}=m\ddot{y}.\end{aligned}$$

Now

$$x/\ell=\sin\phi\ \text{and}\ y/\ell=\cos\phi,$$

Hence

$$\begin{aligned}\ddot{x} &= \ell\frac{d}{dt}\left(\cos\phi\dot{\phi}\right)=\ell\left(-\sin\phi\dot{\phi}^2+\cos\phi\ddot{\phi}\right),\\ \ddot{y} &= \ell\frac{d}{dt}\left(-\sin\phi\dot{\phi}\right)=\ell\left(-\cos\phi\dot{\phi}^2-\sin\phi\ddot{\phi}\right).\end{aligned}$$

The two equations of motion are now

$$\begin{aligned}\lambda\sin\phi &= m\ell\left(-\sin\phi\dot{\phi}^2+\cos\phi\ddot{\phi}\right),\\ mg+\lambda\cos\phi &= m\ell\left(-\cos\phi\dot{\phi}^2-\sin\phi\ddot{\phi}\right).\end{aligned}$$

If we multiply the $x$ equation by $\cos\phi$, the $y$ equation by $-\sin\phi$, and add we find

$$-mg\sin\phi = m\ell\ddot{\phi},$$

which is the usual pendulum equation.

To solve for the Lagrange multiplier multiply the $x$ equation by $\sin\phi$, the $y$ equation by $\cos\phi$, and add with the result

$$mg\cos\phi + \lambda = -m\ell\dot{\phi}^2 \rightarrow \lambda = -mg\cos\phi - m\ell\dot{\phi}^2.$$

Since

$$F_x^{cstr} = \lambda(t)\frac{\partial f}{\partial x} \text{ and } F_y^{cstr} = \lambda(t)\frac{\partial f}{\partial y}$$

we find

$$F_x^{cstr} = \lambda\frac{x}{\ell} = \lambda\sin\phi \text{ and } F_y^{cstr} = \lambda\frac{y}{\ell} = \lambda\cos\phi.$$

Hence the magnitude of the constraint force is $F^{cstr} = |\lambda| = mg\cos\phi + m\ell\dot{\phi}^2$. This is the expression for the tension in the rod.

**Area under string** The maximum area under a string of length $\ell$ can be found via the method of Lagrange multipliers. Expressing the area as

$$A = \int ydx + \lambda\int\sqrt{1+y'^2}dx.$$

The first integral of the Euler Lagrange equation results in

$$y + \lambda\sqrt{1+y'^2} - \lambda y'\frac{y'}{\sqrt{1+y'^2}} = y_o \rightarrow (y-y_o)\sqrt{1+y'^2} = -\lambda$$

$$y'^2 = \frac{\lambda^2 - (y-y_o)^2}{(y-y_o)^2} \rightarrow \frac{(y-y_o)\,dy}{\sqrt{\lambda^2-(y-y_o)^2}} = \pm dx$$

$$(x-x_o)^2 + (y-y_o)^2 = \lambda^2 \rightarrow y = y_o + \sqrt{\lambda^2-(x-x_o)^2}$$

The last expression is the equation of a semicircle of radius $\lambda = R$ whose center is located at $(x_o, y_o)$. If the circle is to pass through the origin $(0,0)$ then

$$0 = y_o + \sqrt{R^2 - x_o^2},$$

and both $y_o$ and the term inside the square root must vanish. This means that $y_o = 0$ and $x_o = R$. The expression is then

$$y = \sqrt{R^2 - (x-R)^2}.$$

The area under this semicircle is $A = \pi R^2/2$ and the length of the string is $\ell = \pi R$.

**8.3** The Lagrangian for this problem is ($y$ is measured upward from the table top) is

$$\mathcal{L} = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 - \frac{1}{2}k\left(y_1 - y_2 - L\right)^2 - m_1 g y_1 - m_2 g y_2.$$

Defining the center of mass coordinate, $Y = (m_1 y_1 + m_2 y_2)/(m_1 + m_2)$, and the relative coordinate, $y = y_1 - y_2$, the Lagrangian takes the form

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}M\dot{Y}^2 + \frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}k\left(y - L\right)^2 - m_1 g\left(Y + \frac{m_2}{M}y\right) - m_2 g\left(Y - \frac{m_1}{M}\right) \\
\mathcal{L} &= \frac{1}{2}M\dot{Y}^2 + \frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}k\left(y - L\right)^2 - MgY.
\end{aligned}$$

The two Lagrange equations of motion with their solutions are

$$\begin{aligned}
Y &: \frac{\partial \mathcal{L}}{\partial Y} = -Mg = M\ddot{Y} \rightarrow Y(t) = -\frac{1}{2}gt^2 + V_0 t + Y_0 \\
y &: \frac{\partial \mathcal{L}}{\partial y} = -k\left(y - L\right) = \mu\ddot{y} \rightarrow y = L + A\sin\left(\omega t - \delta\right),
\end{aligned}$$

where $\omega^2 = k/\mu$. The intial conditions for the relative coordinate lead to

$$\begin{aligned}
y(t=0) &= L \rightarrow \delta = 0, \\
\dot{y}(t=0) &= -v_0 \rightarrow A = -v_0/\omega, \\
y(t) &= L - v_0/\omega \sin\omega t.
\end{aligned}$$

The initial conditions for the center of mass coordinate lead to

$$\begin{aligned}
Y(t=0) &= \frac{m_2 L}{M} = Y_0 \\
\dot{Y}(t=0) &= \frac{m_1 v_0}{M} = V_0 \\
Y(t) &= \frac{m_2 L}{M} + \frac{m_1 v_0}{M}t - \frac{1}{2}gt^2.
\end{aligned}$$

**8.9 (a)** The Lagrangian in terms of $\vec{r}_1$ and $\vec{r}_2$ for this problem ($m_1 = m_2 = m$) is

$$\mathcal{L} = \frac{1}{2}m\left(\dot{\vec{r}}_1^2 + \dot{\vec{r}}_2^2\right) - \frac{1}{2}k\left(|\vec{r}_1 - \vec{r}_2| - L\right)^2.$$

In terms of $\vec{R}$ and $\vec{r}$ the Lagrangian is

$$\mathcal{L} = m\dot{\vec{R}}^2 + \frac{1}{4}m\dot{\vec{r}}^2 - \frac{1}{2}k\left(r - L\right)^2 = \mathcal{L}_{CM} + \mathcal{L}_{rel}.$$

**(c)** In terms of $r$ and $\phi$ the Lagrangian for the relative coordinates is

$$\mathcal{L}_{rel} = \frac{1}{4}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{1}{2}k\left(r-L\right)^2.$$

The two equations of motion are

$$\begin{aligned}
\phi &: \quad \frac{\partial \mathcal{L}_{rel}}{\partial \dot{\phi}} = \frac{1}{2}mr^2\dot{\phi} = \ell = \text{const} \\
r &: \quad \frac{\partial \mathcal{L}_{rel}}{\partial r} = \frac{1}{2}mr\dot{\phi}^2 - k\left(r-L\right) = \frac{1}{2}m\ddot{r} \\
r &: \quad \frac{1}{2}\frac{\ell^2}{mr^3} - k\left(r-L\right) = \frac{1}{2}m\ddot{r}.
\end{aligned}$$

If $r = r_o$ and $\dot{r} = 0$ then the motion is a circle with radius $r_o$ where

$$\frac{1}{2}\frac{\ell^2}{mr_o^3} = k\left(r_o - L\right).$$

If $\dot{\phi} = 0$ then the angular momentum vanishes, $\ell = 0$, and

$$-k\left(r-L\right) = \frac{1}{2}m\ddot{r}.$$

This is the equation for oscillating motion about $r = L$ with a frequency $\omega^2 = 2k/m$.

**8.12 (a)** The effective potential is

$$U_{eff} = -\frac{G\mu M}{r} + \frac{1}{2}\frac{L^2}{\mu r^2} = \mu\left(-\frac{GM}{r} + \frac{1}{2}\frac{\ell^2}{r^2}\right).$$

To find the radius for a circular orbit we must satisfy

$$\frac{dU_{eff}\left(r_o\right)}{dr} = \frac{GM}{r_o^2} - \frac{\ell^2}{r_o^3} = 0 \rightarrow r_o = \frac{\ell^2}{GM}.$$

**(b)** The second derivative evaluated at this radius is

$$\frac{d^2U_{eff}\left(r_o\right)}{dr^2} = \mu\left(-2\frac{GM}{r_o^3} + 3\frac{\ell^2}{r_o^4}\right) = \mu\left(\frac{GM}{\ell^2}\right)^3\left(-2GM + 3GM\right) = \mu\frac{\left(GM\right)^4}{\ell^6}.$$

The curvature is positive, hence circular orbits are stable. The frequency small radial oscillations about $r_o$ is

$$\omega = \frac{2\pi}{\tau_{osc}} = \sqrt{\left(GM\right)^4/\ell^6} = \left(GM\right)^2/\ell^3.$$

The orbital rate, $\dot{\phi}$, is found from

$$\begin{aligned}\ell &= r_o^2\dot{\phi} \rightarrow \dot{\phi} = \ell/r_o^2 \\ \dot{\phi} &= \frac{2\pi}{\tau_{orb}} = \ell\left(GM\right)^2/\ell^4 = \left(GM\right)^2/\ell^3.\end{aligned}$$

The angular velocities are identical, which means the orbital period is equal to the period of radial oscillations. Hence the orbit must be closed.

**8.14 (a)** For a potential $U = kr^n$, the force is given by

$$F = -\frac{dU}{dr} = -nkr^{n-1}.$$

As long as $nk > 0$ this force is attractive.

**(b)** The effective potential is

$$U_{eff} = kr^n + \frac{L^2}{2\mu r^2}.$$

At equilibrium (circular orbit) $dU_{eff}/dr$ vanishes, hence

$$\frac{dU_{eff}\left(r_o\right)}{dr} = nkr_o^{n-1} - \frac{L^2}{\mu r_o^3} = 0 \rightarrow r_o^{n+2} = \frac{L^2}{\mu nk}.$$

To determine if this orbit is stable we need to find $d^2U_{eff}/dr^2$,

$$\begin{aligned}\frac{d^2U_{eff}\left(r_o\right)}{dr^2} &= n\left(n-1\right)kr_o^{n-2} + 3\frac{L^2}{\mu r_o^4} = \frac{1}{r_o^4}\left(n\left(n-1\right)kr_o^{n+2} + 3L^2/\mu\right) \\ \frac{d^2U_{eff}\left(r_o\right)}{dr^2} &= \frac{1}{r_o^4}\left(n\left(n-1\right)k\frac{L^2}{\mu nk} + 3L^2/\mu\right) = \frac{1}{r_o^4}\left(\left(n+2\right)\frac{L^2}{\mu}\right).\end{aligned}$$

Hence this is greater than zero only if $n > -2$.

**(c)** The oscillation frequency is

$$\omega = \frac{L}{\mu r_o^2}\sqrt{n+2}.$$

The orbital frequency is

$$\dot{\phi} = \frac{L}{\mu r_o^2}.$$

Hence the ratio of the periods is

$$\tau_{osc} = \tau_{orb}/\sqrt{n+2}.$$

This is consistent with the result in problem 8.12 for which $n = -1$. Clearly if $\sqrt{n+2}$ is a rational number, $n/m$, then

$$n\tau_{osc} = m\tau_{orb} = T.$$

This time $T$ is the time required for the radial oscillations to undergo $n$ complete oscillations. If this is equal to an integer number of orbital periods then the position of the orbit is idential to its position at time $T$ previously and the orbit is closed. Note if $n = -1, 2, 7$ then $\sqrt{n+2} = 1, 2, 3$. For these values the period for the radial oscillations is equal to, 1/2, or 1/3 of the orbital period. These orbits are closed.