Solutions Assignment 3

3.27 (a) The angular momentum of an object is $\overrightarrow{\ell} = \overrightarrow{r} \times \overrightarrow{p}$. Choosing the orbit of the planet to lie in the $x - y$ plane then $\overrightarrow{r}$ and $\overrightarrow{p}$ both lie in this plane as well. In polar coordinates we find that

$$\begin{aligned} \dot{\overrightarrow{r}} &= \frac{d}{dt}(r\widehat{r}) = \frac{dr}{dt}\widehat{r} + r\frac{d\phi}{dt}\frac{d}{d\phi}\widehat{r}, \\ \dot{\overrightarrow{r}} &= \dot{r}\widehat{r} + r\dot{\phi}\widehat{\phi}. \end{aligned}$$

From the expressions for $\widehat{r}$ and $\widehat{\phi}$ we find $\widehat{r} \times \widehat{\phi} = \widehat{z}$. Thus the angular momentum is

$$\overrightarrow{\ell} = \overrightarrow{r} \times \overrightarrow{p} = mr^2\dot{\phi}\widehat{z} = mr^2\omega\widehat{z},$$

where $\omega = d\phi/dt$.

(b) For a differential angular displacement, $d\phi$, the area inside the triangle is

$$dA = \frac{1}{2}r^2 d\phi.$$

Dividing by dt we find

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\phi}{dt} = \frac{1}{2}r^2\omega = \frac{\ell}{2m}.$$

Since angular momentum is conserved for central forces, we see that dA/dt is constant. Thus an orbiting planet sweeps out equal areas in equal times.

3.32 The moment of inertia of a uniform solid sphere of mass M and radius R rotating about an axis (taken to be the z axis) that passes through its center is found from

$$I = \int r_z^2 dM = \rho \int \rho_z^2 dV = \frac{3M}{4\pi R^3}\int_0^R \int_0^\pi \int_0^{2\pi} r_z^2 r^2 \sin\theta d\phi d\theta dr.$$

where r_z is the perpendicular distance from the z axis. In spherical coordinates $r_z = r\sin\theta$ and the momentum of inertia is

$$\begin{aligned} I &= \frac{3M}{4\pi R^3}\int_0^R \int_0^\pi \int_0^{2\pi} r^4 \sin^3\theta d\phi d\theta dr = \frac{3M}{2R^3}\int_0^R \int_0^\pi r^4\left(1 - \cos^2\theta\right)\sin\theta d\theta dr, \\ I &= \frac{3M}{2R^3}\frac{R^5}{5}\left(2 - \frac{2}{3}\right) = \frac{2}{5}MR^2. \end{aligned}$$

3.35 (b) The torque about the point P, the point of contact between the disc and the inclined plane, for a disk of mass M and radius R is

$$\overrightarrow{\Gamma}^{ext} = \overrightarrow{r} \times \overrightarrow{F} = R\widehat{y} \times Mg\left(\sin\gamma\widehat{x} - \cos\gamma\widehat{y}\right),$$

where γ is the angle of the plane relative to horizontal. Note that both the normal force and the frictional force pass through the point P and contribute no additional torques. We have defined the x axis to point down the inclined plane and the y axis normal to the plane. Performing the cross product we find

$$\dot{\overrightarrow{L}} = \overrightarrow{\Gamma}^{ext} = -MgR\sin\gamma\widehat{z}.$$

The minus sign comes from defining the z axis to point out of the plane and from the right hand rule, the disc rotates in the negative z direction. The rate of change for the magnitude of the angular momentum is

$$\dot{L} = I\dot{\omega} = MgR\sin\gamma.$$

As stated in the problem, the momentum of inertia for a disk rotating about its circumference is $I = 3MR^2/2$, thus

$$\frac{3}{2}MR^2\dot{\omega} = MgR\sin\gamma \to \dot{\omega} = \frac{2\sin\gamma}{3R}.$$

From the no slip condition

$$\dot{v} = R\dot{\omega} = \frac{2}{3}\sin\gamma.$$

(c) The torque about the CM for the disc is

$$\overrightarrow{\Gamma}^{ext} = \overrightarrow{r} \times \overrightarrow{F} = -R\widehat{y} \times F_f\left(-\widehat{x}\right) = -RF_f\widehat{z},$$

where F_f is the magnitude of the frictional force. From the x component of Newton's second law

$$M\dot{v} = Mg\sin\gamma - F_f \to F_f = M\dot{v} - Mg\sin\gamma.$$

Since the momentum of interia for the disc about its CM is $I = MR^2/2$ we have (including the no slip condition, $R\dot{\omega} = \dot{v}$)

$$\begin{aligned}
\Gamma^{ext} &= \dot{L} = I\dot{\omega} = \frac{1}{2}MR^2\dot{\omega} = -R\left(M\dot{v} - Mg\sin\gamma\right) \\
\frac{1}{2}M\dot{v} &= -M\dot{v} + Mg\sin\gamma \\
\dot{v} &= \frac{2}{3}\sin\gamma.
\end{aligned}$$

Note that in keeping track of signs, $\dot{\omega}$ is defined to be positive when the disk is rotating so that it proceeds up the plane (think of the right hand rule).

3.37 The derivation of

$$\frac{d}{dt}\overrightarrow{L}\,(\text{about CM}) = \overrightarrow{\Gamma}^{\text{ext}}\,(\text{about CM})\,,$$

was done in the discussion session, but is repeated here for reference. Defining $\overrightarrow{r}'_\alpha$ as the position of mass particle α relative to the center of mass, $\overrightarrow{r}'_\alpha = \overrightarrow{r}_\alpha - \overrightarrow{R}$, the rate of change of the angular momentum about the center of mass (CM) is expressed as

$$\begin{aligned}
\frac{d}{dt}\overrightarrow{L}_{\text{CM}} &= \sum_\alpha \frac{d}{dt}\left(\overrightarrow{r}'_\alpha \times m_\alpha \dot{\overrightarrow{r}}'_\alpha\right) = \sum_\alpha \overrightarrow{r}'_\alpha \times m_\alpha \ddot{\overrightarrow{r}}'_\alpha, \\
\frac{d}{dt}\overrightarrow{L}_{\text{CM}} &= \sum_\alpha m_\alpha \overrightarrow{r}'_\alpha \times \left(\ddot{\overrightarrow{r}}_\alpha - \ddot{\overrightarrow{R}}\right), \\
\frac{d}{dt}\overrightarrow{L}_{\text{CM}} &= \sum_\alpha \overrightarrow{r}'_\alpha \times \overrightarrow{F}_\alpha - \left(\sum_\alpha m_\alpha \overrightarrow{r}'_\alpha\right) \times \ddot{\overrightarrow{R}}.
\end{aligned}$$

It is very straightforward to show that from the definition of the CM that $\sum_\alpha m_\alpha \overrightarrow{r}'_\alpha = 0$. Thus the expression reduces to

$$\frac{d}{dt}\overrightarrow{L}_{\text{CM}} = \sum_\alpha \overrightarrow{r}'_\alpha \times \overrightarrow{F}_\alpha = \overrightarrow{\Gamma}^{\text{ext}}_{\text{CM}}$$

Note that this is true even if the CM is accelerating, $\ddot{\overrightarrow{R}} \neq 0$!

4.3 (a) First consider the line intergal for $\overrightarrow{F} = -y\widehat{x} + x\widehat{y}$ along the path POQ. This path integral is expressed as

$$W = \int_P^Q \overrightarrow{F} \cdot d\overrightarrow{r} = -\int_1^0 (y=0)\,dx + \int_0^1 (x=0)\,dy = 0.$$

(b) Now consider the line intergal for $\overrightarrow{F} = -y\widehat{x} + x\widehat{y}$ along the path PQ. This path integral is expressed as

$$\begin{aligned}
W &= \int_P^Q \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{(1,0)}^{(0,1)} \left(F_x\,(x,y)\,dx + F_y\,(x,y)\,dy\right), \\
W &= \int_{(0,1)}^{(0,1)} -ydx + xdy.
\end{aligned}$$

Along this path, $y = 1 - x$, so that

$$W = \int_1^0 -(1-x)\,dx - \int_1^0 xdx = -\int_1^0 dx = 1.$$

(c) Now consider the line integral for $\overrightarrow{F} = -y\widehat{x} + x\widehat{y}$ along the circular path PQ. First we will rewrite the force in polar coordinates along the unit circle. In these coordinates $x = \cos\phi$ and $y = \sin\phi$; and $\widehat{x} = \cos\phi\widehat{r} - \sin\phi\widehat{\phi}$ and $\widehat{y} = \sin\phi\widehat{r} + \cos\phi\widehat{\phi}$ so that

$$\begin{aligned}\overrightarrow{F} &= -y\widehat{x} + x\widehat{y} = -\sin\phi\left(\cos\phi\widehat{r} - \sin\phi\widehat{\phi}\right) + \cos\phi\left(\sin\phi\widehat{r} + \cos\phi\widehat{\phi}\right), \\ \overrightarrow{F} &= \widehat{\phi}.\end{aligned}$$

The path integral then becomes

$$W = \int_P^Q \overrightarrow{F} \cdot d\overrightarrow{r} = \int_0^{\pi/2} d\phi = \pi/2.$$

4.4 (a) Since it is a radial force, the angular momentum is conserved so that

$$\ell = mr_o^2\omega_o = mr^2\omega \rightarrow \omega = \frac{\ell}{mr^2} = \frac{r_o^2}{r^2}\omega.$$

(b) From lecture 2, we know that (assuming $\ddot{r}$ is small) the force on the string is $F = ma_\phi = mr\omega^2$. The work pulling on the string is then

$$W = \int_{r_o}^{r} F dr = -\int_{r_o}^{r} mr\omega^2 dr = -\int_{r_o}^{r} \frac{\ell^2}{mr^3} dr = \frac{1}{2}\frac{\ell^2}{m}\left(\frac{1}{r^2} - \frac{1}{r_o^2}\right) = \frac{1}{2}mr_o^4\omega_o^2\left(\frac{1}{r^2} - \frac{1}{r_o^2}\right),$$

where the minus sign comes from F being in the opposite direction of dr.

(c) The change in KE is

$$\Delta KE = \frac{1}{2}\frac{\ell^2}{m}\left(\frac{1}{r^2} - \frac{1}{r_o^2}\right) = \frac{1}{2}mr_o^4\omega_o^2\left(\frac{1}{r^2} - \frac{1}{r_o^2}\right),$$

which is the same as the work done as it had to be.

4.7 (a) The force of gravity on Planet X is $\overrightarrow{F} = -m\gamma y^2\widehat{y}$. The work done by gravity moving the mass m from $\overrightarrow{r}_1$ to $\overrightarrow{r}_2$ is

$$\begin{aligned}W\left(\overrightarrow{r}_1 \rightarrow \overrightarrow{r}_2\right) &= \int_{\overrightarrow{r}_1}^{\overrightarrow{r}_2} \overrightarrow{F} \cdot d\overrightarrow{r} = -\int_{\overrightarrow{r}_1}^{\overrightarrow{r}_2} m\gamma y^2 dy = -\int_{y_1}^{y_2} m\gamma y^2 dy, \\ W\left(\overrightarrow{r}_1 \rightarrow \overrightarrow{r}_2\right) &= -\frac{1}{3}m\gamma\left(y_2^3 - y_1^3\right).\end{aligned}$$

Since the work done only depends on the end points, it is a conservative force. The potential energy for this gravitational field is

$$U(y) = \frac{1}{3}m\gamma y^3$$

(c) The energy for a stationary mass at a height h is

$$E = \frac{1}{3}m\gamma h^3 = \frac{1}{2}m\dot{y}^2 + \frac{1}{3}m\gamma y^3,$$

where y is measured from the ground. When $y=0$, the velocity is

$$\dot{y} = \sqrt{2\gamma h^3/3}.$$

4.18 (a) From equation (4.35) in the text $df = \nabla f \cdot d\overrightarrow{r}$. If the differential displacement vector $d\overrightarrow{r}$ lies in a surface defined by $f = const.$ then $df = 0$. From that we see

$$\nabla f \cdot d\overrightarrow{r} = 0,$$

when $d\overrightarrow{r}$ lies in a surface defined by $f = const.$, thus ∇f is $\perp$ to a surface of constant f.

(b) Now let $d\overrightarrow{r} = \epsilon\widehat{u}$ where ϵ is small and $\widehat{u}$ is a unit vector that points in an arbitrary direction. Thus

$$df = \nabla f \cdot d\overrightarrow{r} = \epsilon \nabla f \cdot \widehat{u} = \epsilon \left|\nabla f\right| \cos\theta,$$

where df is the change in f in the direction of $\widehat{u}$. This is a maximum when $\cos\theta = 1$ which occurs when $\widehat{u}$ points in the direction of ∇f.

4.19 (a) A surface defined by $f = x^2 + 4y^2 = const.$ is a elliptic surface. The intersection of any plane orthogonal to the z axis with this surface will form a ellipse with the semimajor axis, which is parallel to the x axis, being twice that of the semiminor axis, which is parallel to the y axis.

(b) The unit normal to this surface is in the direction of ∇f which yields

$$\begin{aligned} \nabla f = \quad & 2x\widehat{x} + 8y\widehat{y} \rightarrow \widehat{n} = \frac{x\widehat{x} + 4y\widehat{y}}{\sqrt{x^2 + 16y^2}} \\ & \widehat{n}\left(1,1,1\right) = \frac{\widehat{x} + 4\widehat{y}}{\sqrt{17}}. \end{aligned}$$

Moving in the direction of $\widehat{n}$ will maximize the rate of change in f.

4.23 First all three forces only depend on position. It only remains to check $\nabla \times \overrightarrow{F}$.

(a) If $\overrightarrow{F} = kx\widehat{x} + 2ky\widehat{y} + 3kz\widehat{z}$, then

$$\nabla \times \overrightarrow{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\widehat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\widehat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\widehat{z} = 0.$$

This force is conservative. The corresponding potential is

$$U = -k\left(\frac{1}{2}x^2 + y^2 + \frac{3}{2}z^2\right).$$

(b) If $\overrightarrow{F} = ky\widehat{x} + kx\widehat{y}$, then

$$\begin{aligned} \nabla \times \overrightarrow{F} &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\widehat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\widehat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\widehat{z}, \\ \nabla \times \overrightarrow{F} &= k\left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}\right)\widehat{z} = 0. \end{aligned}$$

This force is conservative. The potential is

$$U(x,y) = -kxy.$$

(c) If $\overrightarrow{F} = -ky\widehat{x} + kx\widehat{y}$, then

$$\begin{aligned}
\nabla \times \overrightarrow{F} &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\widehat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\widehat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\widehat{z}, \\
\nabla \times \overrightarrow{F} &= \left(k\frac{\partial x}{\partial x} + k\frac{\partial y}{\partial y}\right)\widehat{z} = 2k\widehat{z} \neq 0.
\end{aligned}$$

This force is not conservative.

4.34 (a) The vertical distance that the mass hangs down on a pendulum of length ℓ is $\ell\cos\phi$. Since at equilibrium it is a length ℓ below the pivot the potential energy measured above equilibrium is

$$U(\phi) = mg\ell(1-\cos\phi).$$

The total energy is

$$E = \frac{1}{2}m\left(\ell\dot{\phi}\right)^2 + mg\ell(1-\cos\phi).$$

(b) Differentiating this expression wrt t yields

$$m\ell^2\dot{\phi}\ddot{\phi} + mg\ell\sin\phi\dot{\phi} = 0,$$

or

$$m\ell^2\ddot{\phi} + mg\ell\sin\phi = 0 \rightarrow I\alpha = -F\ell\sin\phi = -\left|-\overrightarrow{r} \times mg\widehat{y}\right| = \Gamma.$$

(c) If $\phi << 1$ then $\sin\phi \simeq \phi$ and our EOM becomes

$$\ddot{\phi} + \frac{g}{\ell}\phi = 0.$$

This differential equation has as solutions

$$\phi = A\sin\omega t + B\cos\omega t,$$

where $\omega^2 = g/\ell$ or a period of $\tau = 2\pi\sqrt{\ell/g}$.