Solutions Assignment 2

2.19 (a) The EOM for a particle in a vacuum are

$$\ddot{x} = 0 \text{ and } \ddot{y} = -g.$$

Integrating these equations for a particle starting from the origin with initial velocities v_{xo} and v_{yo} yields

$$x(t) = v_{xo}t \text{ and } y(t) = -\frac{1}{2}gt^2 + v_{yo}t.$$

The trajectory, $y(x)$, is then

$$y(t = x/v_{xo}) = \frac{v_{yo}}{v_{xo}}x - \frac{1}{2}\frac{g}{v_{xo}^2}x^2.$$

(b) Equation 2.37 in the text is

$$y = \frac{v_{yo} + v_{ter}}{v_{xo}}x + v_{ter}\tau \ln\left(1 - \frac{x}{v_{xo}\tau}\right).$$

In the limit of small resistance both τ and $v_{ter} = g\tau$ become very large. This enables us to expand the log function as

$$\ln\left(1 - \frac{x}{v_{xo}\tau}\right) = -\left(\frac{x}{v_{xo}\tau} + \frac{1}{2}\frac{x^2}{v_{xo}^2\tau^2} + \frac{1}{3}\frac{x^3}{v_{xo}^3\tau^3} + \cdots\right).$$

Substituting this expansion into the equation for the trajectory yields

$$\begin{aligned} y &\simeq \frac{v_{yo} + v_{ter}}{v_{xo}}x - v_{ter}\tau\left(\frac{x}{v_{xo}\tau} + \frac{1}{2}\frac{x^2}{v_{xo}^2\tau^2} + \frac{1}{3}\frac{x^3}{v_{xo}^3\tau^3}\right), \\ y &\simeq \frac{v_{yo}}{v_{xo}}x - g\left(\frac{1}{2}\frac{x^2}{v_{xo}^2} + \frac{1}{3}\frac{x^3}{v_{xo}^3\tau}\right) = y_{vac} - \frac{g}{3}\frac{x^3}{v_{xo}^3\tau}. \end{aligned}$$

The leading order term agrees with the vacuum expression, and the first order correction shows that the height is reduced for even small drag effects.

2.39 (a) The EOM and its integral for a cyclist are

$$\begin{aligned} m\frac{dv}{dt} &= -cv^2 - f, \\ \int_{v_0}^{v}\frac{dv}{f + cv^2} &= \frac{1}{c}\int_{v_0}^{v}\frac{dv}{f/c + v^2} = -\frac{1}{m}\int_0^t dt = -\frac{t}{m}. \end{aligned}$$

As a change in variables let $v=\sqrt{f/c}\tan\theta \rightarrow f/c + v^2 = (f/c)/\cos^2\theta$ and $dv = \sqrt{f/c}/\cos^2\theta$. The integral of the EOM then becomes

$$\begin{aligned} -\frac{t}{m} &= \frac{1}{c}\sqrt{\frac{c}{f}}\left(\tan^{-1}\sqrt{c/f}v - \tan^{-1}\sqrt{c/f}v_0\right) \\ t &= \frac{m}{\sqrt{fc}}\left(\tan^{-1}\sqrt{c/f}v_0 - \tan^{-1}\sqrt{c/f}v\right). \end{aligned}$$

(b) For $c = .2N/(m/\sec)^2$, $m = 80kg$, and $f = 3N$, the times to slow are given by the expression

$$t = 103.3\left(\tan^{-1} 5.164 - \tan^{-1} .2582v\right).$$

For an initial speed $v_0 = 20m/s$ the time required to slow to $15m/s$, $10m/s$, $5m/s$, and $0m/s$ are

$v\,(m/s)$	15	10	5	0
$t\,(s)$	6.3	18.4	48.3	142.5

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2.41 For a baseball thrown vertically upward with a velocity of v_0, the EOM is

$$\begin{aligned} m\frac{dv}{dt} &= -mg - cv^2 = -mg\left(1 + v^2/v_{ter}^2\right) \\ \frac{dv}{dt} &= -g\left(1 + v^2/v_{ter}^2\right), \end{aligned}$$

where $v_{ter}^2 = mg/c$. Since $dv/dt = v\,(dv/dy)$ we can write

$$\int_{v_0}^{v} \frac{v}{1 + v^2/v_{ter}^2} dv = -g\int_0^y dy$$

For the maximum height $v = 0$. This leads to

$$\frac{v_{ter}^2}{2}\int_{v_0}^{0} \frac{1}{1 + v^2/v_{ter}^2} d\left(v^2/v_{ter}^2\right) = -\frac{v_{ter}^2}{2}\ln\left(1 + v_0^2/v_{ter}^2\right) = -gy_{\max},$$

or

$$y_{\max} = \frac{v_{ter}^2}{2g}\ln\left(1 + v_0^2/v_{ter}^2\right).$$

In a vacuum $v_{ter}^2 \to \infty$. Expanding the natural log we find

$$\begin{aligned} y_{\max} &= \frac{v_{ter}^2}{2g}\ln\left(1 + v_0^2/v_{ter}^2\right) = \frac{v_{ter}^2}{2g}\left(v_0^2/v_{ter}^2 - v_0^4/2v_{ter}^4 + \cdots\right) \\ y_{\max} &= \frac{v_0^2}{2g}\left(1 - \frac{v_0^2}{2v_{ter}^2} + \cdots\right) \end{aligned}$$

As expected the maximum height is reduced in the presence of drag. For a baseball thrown upward with a velocity of $v_0 = 20m/\sec$ and a terminal velocity of $v_{ter} = mg/c = \sqrt{.15 \times 9.8/(.25 \times 49)} = 35m/s$,the maximum height is $y_{\max} = 17.1m$ as compared to $v_0^2/2g = 20.4m$ in a vacuum.

2.42 For a baseball dropped from the elevation $y_{\max}$ of problem 2.41 (y is now positive going down) the EOM and its integral are

$$\begin{aligned}
m\frac{dv}{dt} &= mg - cv^2 = mg\left(1 - v^2/v_{ter}^2\right) \\
\frac{dv}{dt} &= v\frac{dv}{dy} = g\left(1 - v^2/v_{ter}^2\right), \\
\int_0^V \frac{vdv}{1 - v^2/v_{ter}^2} &= \frac{v_{ter}}{2}\int_0^V \frac{1}{1 - (v^2/v_{ter}^2)} d\left(v^2/v_{ter}^2\right) = \int_0^{y_{\max}} gdy, \\
gy_{\max} &= -\frac{v_{ter}}{2}\ln\left(1 - V^2/v_{ter}^2\right)
\end{aligned}$$

where V is the velocity when the ball returns to the ground. Using the expression for $y_{\max}$ obtained in problem 2.41 yields

$$\begin{aligned}
&\frac{v_{ter}}{2}\ln\left(1 - V^2/v_{ter}^2\right) = -\frac{v_{ter}}{2}\ln\left(1 + v_0^2/v_{ter}^2\right) \\
&1 - V^2/v_{ter}^2 = 1/\left(1 + v_0^2/v_{ter}^2\right), \\
&V^2/v_{ter}^2 = 1 - 1/\left(1 + v_0^2/v_{ter}^2\right) = v_0^2/v_{ter}^2/\left(1 + v_0^2/v_{ter}^2\right), \\
&V = v_0/\sqrt{1 + v_0^2/v_{ter}^2} = v_0 v_{ter}/\sqrt{v_{ter}^2 + v_0^2}
\end{aligned}$$

It is of interest to examine this result for small drag, $v_0/v_{ter} << 1$, and large drag, $v_{ter}/v_0 << 1$. These results are

$$\begin{aligned}
v_0/v_{ter} &< < 1 : V = v_0/\sqrt{1 + v_0^2/v_{ter}^2} \simeq v_0\left(1 - \frac{1}{2}\frac{v_0^2}{v_{ter}^2}\right), \\
v_0/v_{ter} &> > 1 : V = v_{ter}/\sqrt{1 + v_{ter}^2/v_0^2} \simeq v_{ter}\left(1 - \frac{1}{2}\frac{v_{ter}^2}{v_0^2}\right).
\end{aligned}$$

In the case of small drag the return velocity is slightly less than the initial velocity while for large drag the return velocity is slightly less than the terminal velocity. For the case of the baseball in problem 2.41;

$$V = 20 \times 35/\sqrt{20^2 + 35^2} = 17.4m/s,$$

as compared to a return velocity of $20m/s$ in a vacuum.

2.49 (a) From Euler's theorem

$$z = e^{i\theta} = \cos\theta + i\sin\theta.$$

Thus we can write

$$\begin{aligned}
e^{2i\theta} &= \cos 2\theta + i\sin 2\theta = \left(e^{i\theta}\right)^2 = (\cos\theta + i\sin\theta)^2 \\
e^{2i\theta} &= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.
\end{aligned}$$

Equating real and imaginary parts we find

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \text{ and } \sin 2\theta = 2\sin\theta\cos\theta.$$

2.55 (a) Now consider crossed E and B fields, $\overrightarrow{E} = E\widehat{y}$ and $\overrightarrow{B} = B\widehat{z}$. The EOM is

$$m\frac{d\overrightarrow{v}}{dt} = q\left(E\widehat{y} + \overrightarrow{v} \times B\widehat{z}\right).$$

Separating this vector equation into its components yields

$$\frac{dv_x}{dt} = \omega v_y,\ \frac{dv_y}{dt} = \frac{qE}{m} - \omega v_x, \text{and}\ \frac{dv_z}{dt} = 0,$$

where $\omega = qB/m$. The initial conditions are $v_x = v_{x0}$ with all other initial component velocities vanishing. This allows us to integrate the equation for the z component and find

$$v_z = 0 \rightarrow z = z_0.$$

Defining $z_0 = 0$, the particle remains in the plane defined by $z = 0$.

(b) Note from the equation for dv_y/dt, that if $v_{x0} = qE/m\omega = E/B$ our remaining EOM at $t = 0$ reduce to

$$\frac{dv_y}{dt} = 0,\ \frac{dv_x}{dt} = \omega v_y.$$

After a time δt we see these equations imply

$$v_y = v_{y0} + \frac{dv_y}{dt}\delta t = v_{y0}.$$

From the initial condition $v_{y0} = 0$, the expression for v_x after a time δt becomes

$$v_x = v_{x0} + \frac{dv_x}{dt}\delta t = v_{x0} + \omega v_{y0}\delta t = v_{x0}.$$

Continuing this integration process yields $v_y = 0$ and $v_x = v_{x0}$ for all time.

(c) To solve these equations for a general initial v_{x0} it is convenient to define $v_x = u_x + qE/m\omega$ so that u_x is the velocity difference between v_x and the *drift velocity* $v_{x0} = qE/m\omega = E/B$. The EOM of interest then reduce to

$$\frac{du_x}{dt} = \omega v_y,\ \frac{dv_y}{dt} = -\omega u_x.$$

We have already solved these EOM. For this initial condition these solutions are

$$\begin{aligned} u_x &= u_{x0}\cos\omega t \rightarrow v_x = E/B + (v_{x0} - E/B)\cos\omega t \\ v_y &= -u_{x0}\sin\omega t \rightarrow v_y = -\left(v_{x0} - E/B\right)\sin\omega t. \end{aligned}$$

3.4 (a) Assuming that the velocity of the hobos relative to the flatcar is u, then from the conservation of momentum after both hobos jump we find

$$m_{fc}v_f - 2m_h\left(u - v_f\right) = 0 \rightarrow v_f = \frac{2m_h}{m_{fc} + 2m_h}u.$$

(b) If the hobos jump one after the other (with velocity relative to the flatcar remains u for both hobos) then after the first hobo jumps the velocity of the flat car v_f' is found from

$$(m_{fc} + m_h)\, v_f' - m_h \left(u - v_f'\right) = 0 \to v_f' = \frac{m_h}{m_{fc} + 2m_h} u.$$

After the second hobo jumps the conservation of momentum yields

$$\begin{aligned}
m_{fc} v_f - m_h (u - v_f) &= (m_{fc} + m_h)\, v_f' \\
(m_{fc} + m_h)\, v_f &= m_h u + (m_{fc} + m_h) \frac{m_h}{m_{fc} + 2m_h} u \\
v_f &= \left(\frac{m_h}{m_{fc} + m_h} + \frac{m_h}{m_{fc} + 2m_h} \right) u
\end{aligned}$$

Since

$$\frac{m_h}{m_{fc} + m_h} > \frac{m_h}{m_{fc} + 2m_h},$$

we have

$$\frac{m_h}{m_{fc} + m_h} + \frac{m_h}{m_{fc} + 2m_h} > \frac{2m_h}{m_{fc} + 2m_h},$$

and v_f for process (b) is greater than v_f for part (a). Algebraically you could also take the ratio of the final velocity for process (b), v_b, to that for (a), v_a. This results in

$$\begin{aligned}
v_b/v_a &= \left(\frac{m_h}{m_{fc} + m_h} + \frac{m_h}{m_{fc} + 2m_h} \right) \frac{m_{fc} + 2m_h}{2m_h} \\
v_b/v_a &= \left(\frac{m_{fc} + 2m_h}{m_{fc} + m_h} + 1 \right) \frac{1}{2} = \frac{2m_{fc} + 3m_h}{2m_{fc} + 2m_h} > 1.
\end{aligned}$$

3.11 (a) In a time dt the change in momentum from Newton's 2nd law is

$$F^{ext} dt = dp = (m + dm)(v + dv) - mv + dm\,(u - v) = m dv + u dm,$$

where F is any external force. Hence

$$m \frac{dv}{dt} = F^{ext} - u \frac{dm}{dt}.$$

(b) In a gravitational field $F^{ext} = -mg$ and the EOM takes the form

$$m \frac{dv}{dt} = -mg - u \frac{dm}{dt}.$$

Assuming that the rocket ejects mass at a constant rate, $m = m_0 - kt$ we find

$$(m_0 - kt) \frac{dv}{dt} = -(m_0 - kt)\, g + ku.$$

Separating and then integrating to solve for v,

$$\begin{aligned} dv &= \left(-g + \frac{ku}{m_0 - kt}\right) dt \\ v &= u \ln \frac{m_0}{m_0 - kt} - gt = u \ln \frac{m_0}{m(t)} - gt \end{aligned}$$

This is exactly what one would expect given the solution for the velocity of a rocket in the absence of a gravitational field.

(c) For the data $m_0 = 2 \times 10^6 kg$, $m(2\,\text{min}) = 10^6 kg$, and $u = 3000 m/\sec$ the approximate velocity of the shuttle after 2 min is

$$v = -9.8 \times 120 + 3000 \ln 2 = 903 m/s$$

In free space ($g = 0$) the velocity would be

$$v = 3000 \ln 2 = 2080 m/s.$$

(d) The rocket would remain on the launch pad reducing its mass until $udm/dt \geq mg$.

3.13 From problem 3.11 the rocket's height is given by

$$\begin{aligned} y(t) &= \int_0^t \left(u \ln \frac{m_0}{m_0 - kt} - gt\right) dt = -\int_0^t u \ln \frac{m_0/k - t}{m_0/k} dt - \frac{1}{2} g t^2, \\ y(t) &= u (m_0/k - t) \ln \frac{(m_0 - kt)}{m_0} - \frac{1}{2} g t^2 + ut, \\ y(t) &= -\frac{u}{k} m \ln \frac{m_0}{m} - \frac{1}{2} g t^2 + ut. \end{aligned}$$

From the data in problem 3.7, $m_0 = 2 \times 10^6 kg$, $m(t = 2\,\text{min}) = 10^6 kg$, and $v_{ex} = 3000 m/\sec.$, after 2 min

$$\begin{aligned} y(t = 2\,\text{min}) &= -\frac{3000}{10^6/120} 10^6 \ln 2 - \frac{1}{2} 9.8 (120)^2 + 3000 (120) \\ y(t = 2\,\text{min}) &\simeq -2.50 \times 10^5 m - .7 \times 10^5 m + 3.6 \times 10^5 m = 40,000 m \end{aligned}$$

3.22 The CM is found from the expression

$$\overrightarrow{R} = \frac{1}{M} \int \overrightarrow{r} dm.$$

For an object of uniform density, it is convenient to express the mass element as $dm = \rho dV$ where ρ is the mass per unit volume. For this example $\rho = 3M/2\pi R^3$. The volume element in spherical coordinates is $dV = drr \sin\theta d\theta r d\phi = r^2 dr \sin\theta d\theta d\phi$. For a hemisphere the coordinates ϕ, θ, r range from $0 \to 2\pi$, $0 \to \pi/2$, and $0 \to R$ respectively. Due to symmetry the x and y

coordinates for the CM are both zero. The integral for the z coordinate of the CM of a hemisphere is

$$z_{CM} = \frac{3}{2\pi R^3} \int_0^R \int_0^{\pi/2} \int_0^{2\pi} z r^2 dr \sin\theta d\theta d\phi = \frac{3}{R^3} \int_0^R \int_0^{\pi/2} z r^2 dr \sin\theta d\theta$$

In spherical coordinates $z = r\cos\theta$, hence

$$\begin{aligned} z_{CM} &= \frac{3}{R^3} \int_0^R r^3 dr \int_0^{\pi/2} \cos\theta \sin\theta d\theta = \frac{3}{R^3} \int_0^R r^3 dr \int_0^{\pi/2} \frac{1}{2} d\left(\sin^2\theta\right), \\ z_{CM} &= \frac{3}{R^3} \frac{R^4}{4} \frac{1}{2} = \frac{3}{8} R. \end{aligned}$$

Uniform Rod After the impulse the linear momentum of the rod is

$$F\delta t = m v_{cm} \rightarrow v_{cm} = F\delta t / m.$$

The angular momentum about its center of mass is

$$F\delta t l/2 = I\omega = \frac{1}{12} m l^2 \omega.$$

Eliminating $F\delta t$ in terms of v_{cm} we find

$$m v_{cm} l/2 = m l^2 \omega / 12 \rightarrow \omega = 6 v_{cm} / l.$$

Immediately after the impulse the velocity of the end of the rod struck by the impulsive force is

$$v_1 = v_{cm} + \omega l/2 = 4 v_{cm}.$$

The velocity of the opposite end of the rod at this point is

$$v_2 = v_{cm} - \omega l/2 = -2 v_{cm}.$$