Solutions Assignment 1

1.16 (b) Consider the vector $\overrightarrow{r}+\overrightarrow{s}$ and its invariant scaler product

$$(\overrightarrow{r}+\overrightarrow{s})\cdot(\overrightarrow{r}+\overrightarrow{s}) = r^2+s^2+2\overrightarrow{r}\cdot\overrightarrow{s}.$$

Since we know that r^2 and s^2 are also invariant, the scalar product $\overrightarrow{r}\cdot\overrightarrow{s}$ must also be invariant.

1.19 Consider the component form of the vector expression in problem 1.19:

$$\begin{aligned}
\frac{d}{dt}\left[\overrightarrow{a}\cdot(\overrightarrow{v}\times\overrightarrow{r})\right]_i &= \frac{d}{dt}a_i\epsilon_{ijk}v_jr_k = \dot{a}_i\epsilon_{ijk}v_jr_k + a_i\epsilon_{ijk}\dot{v}_jr_k + a_i\epsilon_{ijk}v_j\dot{r}_k \\
\frac{d}{dt}\left[\overrightarrow{a}\cdot(\overrightarrow{v}\times\overrightarrow{r})\right]_i &= \dot{a}_i\epsilon_{ijk}v_jr_k + \epsilon_{ijk}a_ia_jr_k + a_i\epsilon_{ijk}v_jv_k
\end{aligned}$$

The last two terms vanish due to the properties of the permutation symbol, Hence

$$\begin{aligned}
\frac{d}{dt}\left[\overrightarrow{a}\cdot(\overrightarrow{v}\times\overrightarrow{r})\right]_i &= \dot{a}_i\epsilon_{ijk}v_jr_k \\
\frac{d}{dt}\left[\overrightarrow{a}\cdot(\overrightarrow{v}\times\overrightarrow{r})\right] &= \dot{\overrightarrow{a}}\cdot(\overrightarrow{v}\times\overrightarrow{r}).
\end{aligned}$$

This problem can also be done by using the product rule when taking derivatives,

$$\frac{d}{dt}\left[\overrightarrow{a}\cdot(\overrightarrow{v}\times\overrightarrow{r})\right] = \dot{\overrightarrow{a}}\cdot(\overrightarrow{v}\times\overrightarrow{r}) + \overrightarrow{a}\cdot\left(\dot{\overrightarrow{v}}\times\overrightarrow{r}\right) + \overrightarrow{a}\cdot\left(\overrightarrow{v}\times\dot{\overrightarrow{r}}\right).$$

The third term vanishes since $\overrightarrow{v}\times\dot{\overrightarrow{r}} = \overrightarrow{v}\times\overrightarrow{v} = 0$. The second term is of the form $\overrightarrow{a}\cdot\left(\dot{\overrightarrow{v}}\times\overrightarrow{r}\right) = \overrightarrow{a}\cdot(\overrightarrow{a}\times\overrightarrow{r})$. The vector product inside the parathensis is normal to both $\overrightarrow{a}$ and $\overrightarrow{r}$. Hence the scalar product with $\overrightarrow{a}$ vanishes and we are left with only the first term.

Vector Identity $\nabla\left(\overrightarrow{A}\cdot\left(\overrightarrow{B}\times\overrightarrow{r}\right)\right) = \overrightarrow{A}\times\overrightarrow{B}$ Consider the component form of $\nabla\left(\overrightarrow{A}\cdot\left(\overrightarrow{B}\times\overrightarrow{r}\right)\right)$:

$$\left[\nabla\left(\overrightarrow{A}\cdot\left(\overrightarrow{B}\times\overrightarrow{r}\right)\right)\right]_i = \partial_iA_j\epsilon_{jk\ell}B_kr_\ell = A_j\epsilon_{jk\ell}B_k\delta_{i\ell}.$$

Here we have recognized that $\partial_ir_\ell = \delta_{i\ell}$. Hence

$$\begin{aligned}
\left[\nabla\left(\overrightarrow{A}\cdot\left(\overrightarrow{B}\times\overrightarrow{r}\right)\right)\right]_i &= A_j\epsilon_{jki}B_k = \epsilon_{ijk}A_jB_k = \left(\overrightarrow{A}\times\overrightarrow{B}\right)_i \quad \text{and} \\
\nabla\left(\overrightarrow{A}\cdot\left(\overrightarrow{B}\times\overrightarrow{r}\right)\right) &= \overrightarrow{A}\times\overrightarrow{B}.
\end{aligned}$$

This is the desired result.

1.23 If $\overrightarrow{b} \cdot \overrightarrow{v} = \lambda$, and $\overrightarrow{b} \times \overrightarrow{v} = \overrightarrow{c}$ where λ, $\overrightarrow{b}$, and $\overrightarrow{c}$ are known then the cross product $\overrightarrow{b} \times \overrightarrow{c}$ is

$$\overrightarrow{b} \times \overrightarrow{c} = \overrightarrow{b} \times \left(\overrightarrow{b} \times \overrightarrow{v}\right) = \overrightarrow{b}\left(\overrightarrow{b} \cdot \overrightarrow{v}\right) - \overrightarrow{v}\left(\overrightarrow{b} \cdot \overrightarrow{b}\right).$$

This leads to

$$\overrightarrow{v}\left(\overrightarrow{b} \cdot \overrightarrow{b}\right) = \lambda\overrightarrow{b} - \overrightarrow{b} \times \overrightarrow{c},$$

$$\overrightarrow{v} = \lambda\frac{\overrightarrow{b}}{\overrightarrow{b} \cdot \overrightarrow{b}} - \frac{\overrightarrow{b} \times \overrightarrow{c}}{\overrightarrow{b} \cdot \overrightarrow{b}} = \lambda\frac{\overrightarrow{b}}{b^2} - \frac{\overrightarrow{b} \times \overrightarrow{c}}{b^2}.$$

1.32 Since the magnetic field at $\overrightarrow{r}_1$ is given by

$$\overrightarrow{B}\left(\overrightarrow{r}_1\right) = \frac{\mu_o}{4\pi}\frac{q_2}{s^2}\left(\overrightarrow{v}_2 \times \widehat{s}\right),$$

we know that the magnetic force on the charge located at $\overrightarrow{r}_1$ is

$$\begin{aligned} F_{12}^{mag} &= q_1\left(\overrightarrow{v}_1 \times \overrightarrow{B}\left(\overrightarrow{r}_1\right)\right) = \frac{\mu_o}{4\pi}\frac{q_1 q_2}{s^2}\left(\overrightarrow{v}_1 \times \left(\overrightarrow{v}_2 \times \widehat{s}\right)\right), \\ F_{12}^{mag} &= \frac{\mu_o}{4\pi}\frac{q_1 q_2}{s^2}\left(\left(\overrightarrow{v}_1 \cdot \widehat{s}\right)\overrightarrow{v}_2 - \left(\overrightarrow{v}_1 \cdot \overrightarrow{v}_2\right)\widehat{s}\right). \end{aligned}$$

Now $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ are orthogonal, $\overrightarrow{v}_1 \cdot \overrightarrow{v}_2 = 0$. Additionally $\widehat{s}$ is a unit vector so that the magnitude of F_{12}^{mag} is

$$\left|F_{12}^{mag}\right| = \frac{\mu_o}{4\pi}\frac{q_1 q_2}{s^2}\left|\overrightarrow{v}_1 \cdot \widehat{s}\right| v_2 \leq \frac{\mu_o}{4\pi}\frac{q_1 q_2}{s^2} v_1 v_2.$$

The ratio of the magnitudes F_{12}^{mag}/F_{12}^{el} satisfies

$$\frac{F_{12}^{mag}}{F_{12}^{el}} \leq \frac{\mu_o}{4\pi}\frac{q_1 q_2}{s^2} v_1 v_2 / \left(\frac{1}{4\pi\epsilon_o}\frac{q_1 q_2}{s^2}\right) = \mu_o \epsilon_o v_1 v_2.$$

Since $\mu_o \epsilon_o = 1/c^2$, we have

$$F_{12}^{mag} \leq \frac{v_1 v_2}{c^2} F_{12}^{el},$$

which is the desired result.

1.39 (a) Using the coordinates suggested in the text, the equation of motion in the x and y directions are

$$m\frac{d^2x}{dt^2} = -mg\sin\phi \text{ and } m\frac{d^2y}{dt^2} = -mg\cos\phi,$$

with initial conditions $x(t=0) = y(t=0) = 0$, $\dot{x}(t=0) = v_o\cos\theta$, and $\dot{y}(t=0) = v_o\sin\theta$. First it is necessary to determine the time the ball is above the plane. Integrating the EOM for the y coordinate we find

$$\frac{dy}{dt} = -\left(g\cos\phi\right)t + v_o\sin\theta.$$

Integrating one more time yields

$$y\left(t\right)=-\frac{g\cos\phi}{2}t^{2}+\left(v_{o}\sin\theta\right)t.$$

From this result it is clear that the ball is at $y=0$ both initially, $t=0$, and at $t=T=2v_{o}\sin\theta/g\cos\phi$, where I have defined T as the total time for the flight of the ball. To find the range up the plane, we now need to integrate the EOM for the x coordinate and then evaluate it at $t=T$. The first integration yields

$$\frac{dx}{dt}=-\left(g\sin\phi\right)t+v_{o}\cos\theta.$$

Integrating once more yields

$$x\left(t\right)=-\frac{g\sin\phi}{2}t^{2}+\left(v_{o}\cos\theta\right)t.$$

The range is then

$$\begin{aligned}R &= x\left(T\right)=-\frac{g\sin\phi}{2}\left(\frac{2v_{o}\sin\theta}{g\cos\phi}\right)^{2}+\left(v_{o}\cos\theta\right)\frac{2v_{o}\sin\theta}{g\cos\phi},\\ R &= 2v_{o}^{2}\sin\theta\left[-\sin\phi\sin\theta+\cos\theta\cos\phi\right]/g\cos^{2}\phi\end{aligned}$$

The term in the square braces simplifies to $\cos\left(\theta+\phi\right)$ and we find

$$R=2v_{o}^{2}\sin\theta\cos\left(\theta+\phi\right)/g\cos^{2}\phi,$$

which is the desired result.

(b) To find the value of θ that results in the maximum range, $R_{\max}$, we find the value of θ for which $dR/d\theta=0$.

$$\begin{aligned}\frac{dR}{d\theta} &= \frac{2v_{o}^{2}}{g\cos^{2}\phi}\left(\cos\theta\cos\left(\theta+\phi\right)-\sin\theta\sin\left(\theta+\phi\right)\right)\\ \left.\frac{dR}{d\theta}\right|_{\theta_{o}} &= \frac{2v_{o}^{2}}{g\cos^{2}\phi}\cos\left(\phi+2\theta_{o}\right)=0\rightarrow\phi+2\theta_{o}=\pi/2\\ \theta_{o} &= \left(\pi/2-\phi\right)/2.\end{aligned}$$

The maximum range, $R_{\max}$, is

$$R_{\max}=R\left(\theta_{o}\right)=2v_{o}^{2}\sin\left(\pi/4-\phi/2\right)\cos\left(\pi/4+\phi/2\right)/g\cos^{2}\phi.$$

Using the trigonometric identity

$$\cos\left(\pi/4+\phi/2\right)=\cos\left(\pi/2-\left(\pi/4-\phi/2\right)\right)=\sin\left(\pi/4-\phi/2\right),$$

we find

$$\begin{aligned}\sin\left(\pi/4-\phi/2\right)\cos\left(\pi/4+\phi/2\right) &= \sin^{2}\left(\pi/4-\phi/2\right)=\frac{1}{2}\left(1-\cos\left(\pi/2-\phi\right)\right)\\ \sin\left(\pi/4-\phi/2\right)\cos\left(\pi/4+\phi/2\right) &= \frac{1}{2}\left(1-\sin\phi\right).\end{aligned}$$

The maximum range is then

$$\begin{aligned}
R_{\max} &= v_o^2 (1 - \sin\phi) / g \cos^2\phi = v_o^2 (1 - \sin\phi) / g \left(1 - \sin^2\phi\right), \\
R_{\max} &= v_o^2 / g (1 + \sin\phi),
\end{aligned}$$

which is the desired result.

1.40 (a) From problem 1.39 we can find the solution for the ball's position as a function of time by taking the limit as $\phi \to 0$. This yields the well known result

$$x(t) = (v_o \cos\theta)\, t \text{ and } y(t) = -\frac{1}{2} g t^2 + (v_o \sin\theta)\, t.$$

(b) If $r(t)$ is the ball's distance from the origin then $r^2(t)$ is given by

$$\begin{aligned}
r^2(t) &= x^2(t) + y^2(t), \\
r^2(t) &= \left(v_o^2 \cos^2\theta\right) t^2 + \frac{1}{4} g^2 t^4 - g v_o \sin\theta t^3 + \left(v_o^2 \sin^2\theta\right) t^2 \\
r^2(t) &= v_o^2 t^2 - g v_o \sin\theta t^3 + \frac{1}{4} g^2 t^4.
\end{aligned}$$

Now if $r(t)$ is continually increasing, then $r^2(t)$ is also continuing to increase. The condition for this is

$$\frac{dr^2(t)}{dt} \geq 0,$$

This leads to

$$\frac{dr^2(t)}{dt} = 2 v_o^2 t - 3 g v_o \sin\theta t^2 + g^2 t^3 = t\left(2 v_o^2 - 3 g v_o \sin\theta t + g^2 t^2\right) \geq 0.$$

Since $t \geq 0$ we only have to concern ourselves with the quadratic $f(t) = 2v_o^2 - 3 g v_o \sin\theta t + g^2 t^2 \geq 0$. If $f(t) \geq 0$ for all time, then this quadratic never crosses the $f = 0$ axis. Hence the discriminate for this quadratic must always be less than zero. This leads to

$$\begin{aligned}
9 g^2 v_o^2 \sin^2\theta - 8 g^2 v_o^2 &\leq 0 \to 9 \sin^2\theta \leq 8 \\
\sin\theta &\leq \sqrt{8/9} = 2\sqrt{2}/3 \\
\theta &\leq 1.23 rad \simeq 70.5^\circ.
\end{aligned}$$

1.45 If a vector $\overrightarrow{v}(t)$ that has a constant magnitude then $\overrightarrow{v}(t) \cdot \overrightarrow{v}(t) = \lambda$ a constant. If we take the time derivative of this quantity then

$$\frac{d}{dt} \overrightarrow{v}(t) \cdot \overrightarrow{v}(t) = 2 \overrightarrow{v}(t) \cdot \frac{d\overrightarrow{v}(t)}{dt} = 0,$$

that is the two vectors $\overrightarrow{v}(t)$ and $d\overrightarrow{v}(t)/dt = \overrightarrow{a}(t)$ are necessarily orthogonal.

Conversely if

$$\overrightarrow{v}(t) \cdot \frac{d\overrightarrow{v}(t)}{dt} = 0,$$

then

$$\int \vec{v}(t)\cdot\frac{d\vec{v}(t)}{dt}dt = \int \vec{v}(t)\cdot d\vec{v}(t) = \frac{1}{2}\vec{v}(t)\cdot\vec{v}(t) = const.$$

This is a handy result as it applies not only to the scenarios discussed by Taylor, but also to the 4-velocity in relativity, special or general. But that is for another discussion.

2.11 (a) For an object thrown upward with linear drag the EOM for the y coordinate (positive in upward direction) is

$$m\frac{dv}{dt} = -mg - bv \rightarrow \frac{dv}{dt} = -g - v/\boldsymbol{\tau} = -\frac{v+v_{ter}}{\tau}$$

where $\tau = m/b$ and $v_{ter} = g\tau = mg/b$. With an initial upward velocity of v_0 we find

$$\begin{aligned}
\frac{dv}{dt} &= -\frac{v+v_{ter}}{\tau} \rightarrow \frac{dv}{v+v_{ter}} = -\frac{dt}{\tau}, \\
\ln\frac{v+v_{ter}}{v_0+v_{ter}} &= -\frac{t}{\tau} \rightarrow v+v_{ter} = (v_0+v_{ter})\,e^{-t/\tau} \\
v &= v_0 e^{-t/\tau} - v_{ter}\left(1-e^{-t/\tau}\right).
\end{aligned}$$

Integrating this expression we find that the y coordinate satisfies

$$y(t) = -v_{ter}t + (v_0+v_{ter})\,\tau\left(1-e^{-t/\tau}\right).$$

(b) The upward velocity reaches its peak at a time T which occurs when $v=0$, or

$$\begin{aligned}
e^{-T/\tau} &= v_{ter}/(v_0+v_{ter}) \\
T &= \tau\ln(v_0+v_{ter})/v_{ter} = \tau\ln(1+v_0/v_{ter}).
\end{aligned}$$

The maximum value, $y_{\max} = y(T)$, is given by

$$\begin{aligned}
y_{\max} &= y(T) = -v_{ter}T + (v_0+v_{ter})\,\tau\left(1-e^{-T/\tau}\right) \\
y_{\max} &= -v_{ter}\tau\ln(1+v_0/v_{ter}) + v_{ter}\tau\,(1+v_0/v_{ter})\left(1-(1+v_0/v_{ter})^{-1}\right) \\
y_{\max} &= v_0\tau - v_{ter}\tau\ln(1+v_0/v_{ter}).
\end{aligned}$$

(c) As the drag is reduced ($\tau\rightarrow\infty$) we find that $v_0/v_{ter} = v_0/g\tau \ll 1$. In this limit we find

$$\ln(1+v_0/v_{ter}) = \ln(1+v_0/g\tau) = \frac{v_0}{g\tau} - \frac{1}{2}\left(\frac{v_0}{g\tau}\right)^2 + \cdots.$$

The maximum height in this limit is

$$y_{\max} = v_0\tau - g\tau^2 \ln\left(1+v_0/g\tau\right) \simeq v_0\tau - g\tau^2\left(\frac{v_0}{g\tau} - \frac{1}{2}\left(\frac{v_0}{g\tau}\right)^2 + \frac{1}{3}\left(\frac{v_0}{g\tau}\right)^3\right)$$

$$y_{\max} \simeq -g\tau^2\left(-\frac{1}{2}\left(\frac{v_0}{g\tau}\right)^2 + \frac{1}{3}\left(\frac{v_0}{g\tau}\right)^3\right) = \frac{1}{2}\frac{v_0^2}{g} - \frac{1}{3}\frac{v_0^3}{g^2\tau} = \frac{1}{2}\frac{v_0^2}{g}\left(1 - \frac{2}{3}\frac{v_0}{v_{ter}}\right).$$

Note that the leading term is what you would expect for zero drag. The first order correction shows that the drag reduces the maximum height.

2.12 In one dimension, if a force only depends on the spatial coordinate, x, then the EOM is

$$F(x) = m\frac{d^2x}{dt^2}.$$

Using the chain rule we can write the second derivative of x as

$$\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{dx}{dt}\right)\frac{dx}{dt} = \frac{1}{2}\frac{d}{dx}\left(\frac{dx}{dt}\right)^2.$$

This allows us to integrate the EOM,

$$F(x) = \frac{m}{2}\frac{d}{dx}\left(\frac{dx}{dt}\right)^2$$

$$\int F(x)\,dx = \frac{m}{2}\int d\left(\frac{dx}{dt}\right)^2 = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 - \frac{m}{2}v_o^2$$

$$\left(\frac{dx}{dt}\right)^2 = v^2 = v_o^2 + \frac{2}{m}\int F(x)\,dx,$$

which is the desired result.

Of interest here is the result that we obtain after multiplying by $m/2$. We then find

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_o^2 = \int F(x)\,dx.$$

For those that have some familiarity with the relation between work and kinetic energy, they should recognize that the change in kinetic energy of the particle is equal to the work done on the particle by the force F.

If in fact F is constant then the term involving the integral becomes

$$\frac{2}{m}\int F(x)\,dx = 2\frac{F}{m}\Delta x = 2a\Delta x,$$

where acceleration a is constant. This leads to the well known kinematic relation

$$v^2 = v_o^2 + 2a\Delta x,$$

whenever a is constant.

2.13 For problem 2.12, when $F = -kx$ with initial conditions $x\,(t=0) = x_o$, $v\,(t=0) = v_o = 0$, the result can be written

$$\begin{aligned} \frac{m}{2}\left(\frac{dx}{dt}\right)^2 &= -\int_{x_o}^{x} kx dx = -\frac{1}{2}kx^2 + \frac{1}{2}kx_o^2 \\ \left(\frac{dx}{dt}\right)^2 &= \omega^2\left(x_o^2 - x^2\right), \end{aligned}$$

where $\omega^2 = k/m$. Taking the square root (using the minus sign for as t increases x decreases) we see that

$$\begin{aligned} \frac{dx}{dt} &= -\omega\sqrt{x_o^2 - x^2} \rightarrow \frac{dx}{\sqrt{x_o^2 - x^2}} = -\omega dt \\ -\omega t &= \int_{x_o}^{x} \frac{dx}{\sqrt{x_o^2 - x^2}}. \end{aligned}$$

Letting $x = x_o \cos\theta$ the integral becomes

$$-\omega t = -\int_0^{\theta} \frac{x_o \sin\theta}{x_o \sin\theta} d\theta = -\theta.$$

Thus the solution is $x = x_o \cos\omega t$.