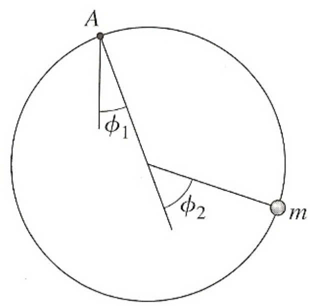
Normal Modes

Problem 11.26 (a) Normal modes

Consider the system in figure in which a bead of mass m is threaded on a frictionless wire hoop of radius R and mass m. The hoop is suspended at point A and is free to swing in its own vertical plane.



A frictionless hoop of mass m and radius R supports a bead of mass m which is free to travel around the hoop. The hoop is suspended from point A and is free to swing in its own vertical plane.

Using the suggested generalized coordinates in the figure, the Cartesian coordinates (y being measured downward) of the bead are

$$x = R\sin\phi_1 + R\sin\phi_2 \quad \text{and} \quad y = R\cos\phi_1 + R\cos\phi_2. \tag{1}$$

The corresponding velocities are

$$\dot{x} = R\cos\phi_1\dot{\phi}_1 + R\cos\phi_2\dot{\phi}_2 \quad \text{and} \quad \dot{y} = -R\sin\phi_1\dot{\phi}_1 - R\sin\phi_2\dot{\phi}_2. \tag{2}$$

Hence the kinetic energy of the bead is

$$T_b = \frac{1}{2}mR^2\left(\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\cos(\phi_1 - \phi_2)\,\dot{\phi}_1\dot{\phi}_2\right) \tag{3}$$

From the parallel axis theorem the momentum of inertia of the hoop about A is $I = 2mR^2$, hence the kinetic energy of the hoop is

$$T_h = mR^2\dot{\phi}_1^2. \tag{4}$$

The gravitational potential energy of the system is

$$U = -mgR\left(2\cos\phi_1 + \cos\phi_2\right). \tag{5}$$

In the limit of small oscillations the Lagrangian (to within some meaningless constants) is

$$\mathcal{L} = \frac{1}{2}mR^2\left(3\dot{\phi}_1^2 + \dot{\phi}_2^2 + 2\dot{\phi}_1\dot{\phi}_2\right) - \frac{1}{2}mgR\left(2\phi_1^2 + \phi_2^2\right). \tag{6}$$

The equations of motion are

$$\phi_1 \quad : \quad \frac{\partial \mathcal{L}}{\partial \phi_1} = -2mgR\phi_1 = mR^2\left(3\ddot{\phi}_1 + \ddot{\phi}_2\right) \tag{7a}$$

$$\phi_2 \quad : \quad \frac{\partial \mathcal{L}}{\partial \phi_2} = -mgR\phi_2 = mR^2\left(\ddot{\phi}_1 + \ddot{\phi}_2\right). \tag{7b}$$

We can now write the the mass and spring matrix as

$$\mathbf{M} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \frac{g}{R}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \omega_o^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \tag{8}$$

where $\omega_o^2 = g/R$. With these matrices the eigenvalue equation is

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right)\mathbf{a} = 0, \tag{9}$$

which has a solution if and only if

$$\det\left(\mathbf{K} - \omega^2\mathbf{M}\right) = \det\begin{bmatrix} 2\omega_o^2 - 3\omega^2 & -\omega^2 \\ -\omega^2 & \omega_o^2 - \omega^2 \end{bmatrix} = 2\omega^4 - 5\omega^2\omega_o^2 + 2\omega_o^4 = 0, \tag{10}$$

or

$$\omega_1^2 = \omega_o^2/2 \quad \text{and} \quad \omega_2^2 = 2\omega_o^2. \tag{11}$$

The eigenvalue equation for the first eigenvector is

$$\begin{aligned} \begin{bmatrix} 2 - 3/2 & -1/2 \\ -1/2 & 1 - 1/2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \\ a_1 - a_2 &= 0 \to a_1 = a_2. \end{aligned} \tag{12}$$

The eigenvalue equation for the first eigenvector is

$$\begin{aligned} \begin{bmatrix} 2 - 6 & -2 \\ -2 & 1 - 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \\ 2a_1 + a_2 &= 0 \to a_1 = -a_2/2. \end{aligned} \tag{13}$$

The normal modes are

$$\boldsymbol{\phi} = A_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos\left(\frac{g}{\sqrt{2}R}t - \delta_1\right) \quad \text{and} \quad \boldsymbol{\phi} = A_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}\cos\left(\frac{\sqrt{2}g}{R}t - \delta_2\right). \tag{14}$$

In the first mode the hoop and the bead are oscillating in phase with equal amplitudes, the bead and hoop oscillate together. In the second mode the hoop and bead oscillate exactly out of phase with the amplitude of ϕ_2 being twice of ϕ_1.

(b) Normal coordinates

For this case expanding $\boldsymbol{\phi}$ in terms of its eigenvectors yields

$$\boldsymbol{\phi} = \frac{\xi_1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\xi_2}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The equations of motion for $\boldsymbol{\phi}$ were

$$\begin{aligned} 3\ddot{\phi}_1 + \ddot{\phi}_2 &= -2\omega_o^2\phi_1, \\ \ddot{\phi}_1 + \ddot{\phi}_2 &= -\omega_o^2\phi_2, \end{aligned}$$

where $\omega_o^2 = g/R$. Substituting for $\boldsymbol{\phi}$ in terms of the normal coordinates yields

$$\begin{aligned} 4\frac{\ddot{\xi}_1}{\sqrt{2}} + \frac{\ddot{\xi}_2}{\sqrt{5}} &= -2\omega_o^2\left(\frac{\xi_1}{\sqrt{2}} + \frac{\xi_2}{\sqrt{5}}\right), \\ 2\frac{\ddot{\xi}_1}{\sqrt{2}} - \frac{\ddot{\xi}_2}{\sqrt{5}} &= -\omega_o^2\left(\frac{\xi_1}{\sqrt{2}} - 2\frac{\xi_2}{\sqrt{5}}\right). \end{aligned}$$

Simply adding these two equations yields

$$6\frac{\ddot{\xi}_1}{\sqrt{2}} = -3\omega_o^2\frac{\xi_1}{\sqrt{2}} \rightarrow \ddot{\xi}_1 = -\frac{\omega_o^2}{2}\xi_1.$$

Multiplying the second EOM by 2 and subtracting them yields

$$3\frac{\ddot{\xi}_2}{\sqrt{5}} = -6\omega_o^2\frac{\xi_2}{\sqrt{5}} \rightarrow \ddot{\xi}_2 = -2\omega_o^2\xi_2.$$

Again we see that the normal coordinates are decoupled and oscillate at their normal mode frequencies.

To solve for the normal coordinates we simply invert the expansion

$$\begin{aligned} \phi_1 &= \frac{\xi_1}{\sqrt{2}} + \frac{\xi_2}{\sqrt{5}}, \\ \phi_2 &= \frac{\xi_1}{\sqrt{2}} - 2\frac{\xi_2}{\sqrt{5}}, \end{aligned}$$

which leads to

$$\begin{aligned} \xi_1 &= \frac{\det\begin{bmatrix}\phi_1 & 1/\sqrt{5} \\ \phi_2 & -2/\sqrt{5}\end{bmatrix}}{\det\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{2} & -2/\sqrt{5}\end{bmatrix}} = \frac{-2\phi_1/\sqrt{5} - \phi_2/\sqrt{5}}{-3/\sqrt{10}} = \frac{\sqrt{2}}{3}\left(2\phi_1 + \phi_2\right), \\ \xi_1 &= \frac{\det\begin{bmatrix}1/\sqrt{2} & \phi_1 \\ 1/\sqrt{2} & \phi_2\end{bmatrix}}{\det\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{2} & -2/\sqrt{5}\end{bmatrix}} = -\frac{\sqrt{5}}{3}\left(\phi_1 - \phi 2\right). \end{aligned}$$