1 Lecture 9-25

1.1 Chapter 1 Newton's Laws of Motion

1.1.1 Vectors

We will start with defining the three basis vectors that align themselves along the three Cartesian coordinate axes such that the position vector $\overrightarrow{r}$ is given by,

$$\overrightarrow{r} = x\widehat{x} + y\widehat{y} + z\widehat{z} = x\widehat{e}_x + y\widehat{e}_y + z\widehat{e}_z. \tag{1}$$

It is common to use the carrot indicating a unit vector. It should be clear from this defining equation that $\widehat{x} = \widehat{e}_x$ etc. The reason for the choice of $\widehat{e}_i$ to represent the basis comes from German for one, ein. On occasion (when it is convenient) we will use the definition

$$\overrightarrow{r} = r_1\widehat{e}_1 + r_2\widehat{e}_2 + r_3\widehat{e}_3 = \sum_{i=1}^{3} r_i\widehat{e}_i \equiv r_i\widehat{e}_i. \tag{2}$$

Here the index $i = 1, 2, 3$ references x, y, z. Also we will use the Einstein notation (actually a close approximation to his notation) in which repeated indicies representing components are summed over. The advantage of this notation is clear from the equation above, the use of subscripts allows us to represent the position vector (or any other vector) as a simple sum over indicies. If convenient, we may use the short hand notation to indicate the set of all the components as

$$\{r_i\} = (r_1, r_2, r_3). \tag{3}$$

We will define our vector to satisfy the linear addition property

$$\overrightarrow{r} + \overrightarrow{s} = (r_1 + s_1)\widehat{x} + (r_2 + s_2)\widehat{y} + (r_3 + s_3)\widehat{z} = (r_i + s_i)\widehat{e}_i \tag{4}$$

So to add vectors we simply add their components, the parallelogram rule. Vectors also satisfy the linear property

$$c\overrightarrow{r} = c(r_1\widehat{x} + r_2\widehat{y} + r_3\widehat{z}) = cr_1\widehat{x} + cr_2\widehat{y} + cr_3\widehat{z} = c(r_i\widehat{e}_i). \tag{5}$$

For example, if an object of mass m has an acceleration $\overrightarrow{a}$, then the resultant force $\overrightarrow{F}$ will equal $m\overrightarrow{a}$.

In defining a scalar (or dot) product between a pair of vectors, we first define the scalar product of our Cartesian basis to satisfy

$$\widehat{e}_i \cdot \widehat{e}_j = \delta_{ij}. \tag{6}$$

Here δ_{ij} is the *Kronecker delta symbol* which has the properties

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{7}$$

With this definition we see immediately that

$$\overrightarrow{r} \cdot \overrightarrow{s} = r_1 s_1 + r_2 s_2 + r_3 s_3 = r_i s_i. \tag{8}$$

The magnitude of a vector is found by taking the scalar product of the vector with itself. The magnitude of the vector squared is given by

$$r^2 = \overrightarrow{r} \cdot \overrightarrow{r} = r_1^2 + r_2^2 + r_3^2 = \sum_i r_i^2, \tag{9}$$

where in this equation we have also defined $r = \left|\overrightarrow{r}\right|$. Note that we have used the summation sign to avoid confusion with r_i^2 which we will define to be the square of a single component. With these definitions, it can easily be shown that the scalar product satisfies

$$\overrightarrow{r} \cdot \overrightarrow{s} = rs\cos\theta, \tag{10}$$

where θ is the angle between the direction of the two vectors. With this result it is clear that

$$\overrightarrow{r} \cdot \widehat{e}_i = r_i = r\cos\theta_i, \tag{11}$$

where θ_i is the angle between the vector and the corresponding coordinate axes. The cosine of this angle is usually called the direction cosine, and they have the property

$$\sum_i \cos^2\theta_i = 1. \tag{12}$$

With these properties the expansion of a vector in its direction cosines is

$$\overrightarrow{r} = r\cos\theta_i \widehat{e}_i. \tag{13}$$

One point that I wish to make here is that the components of a vector depend on the coordinates used to describe the vector. However a vector is a geometrical object and, as such, is independent of the coordinates. This is because a vector depends not only on its components but also its set of basis vectors which define the coordinate system. So sometimes you see the notation

$$\overrightarrow{r} \rightarrow_O (r_1, r_2, r_3). \tag{14}$$

Here the symbol O indicates that (r_1, r_2, r_3) are the components of $\overrightarrow{r}$ in the coordinate system O. This notation serves to emphasize that the components of the vector are given in a specific coordinate system.

For example, under a rotation of the coordinate axes, the components change even though the geometrical object, the vector itself, remains unchanged. This is because the basis vectors also change. To illustrate this we will consider a vector, $\overrightarrow{V}$, lying in the $x - y$ plane, with an angle α between the directions of the x axis, $\widehat{e}_x$, and $\overrightarrow{V}$, so that

$$\begin{aligned}
\overrightarrow{V} &= V_x \widehat{e}_x + V_y \widehat{e}_y = V\left(\cos\alpha \widehat{e}_x + \cos\left(\pi/2 - \alpha\right)\widehat{e}_y\right) \\
\overrightarrow{V} &= V\left(\cos\alpha \widehat{e}_x + \sin\alpha \widehat{e}_y\right).
\end{aligned} \tag{15}$$

If we were to rotate the coordinate axis through an angle θ, defining a coordinate system $\overline{O}$ through its basis set $\widehat{e_{\overline{x}}}$ and $\widehat{e_{\overline{y}}}$, then the vector would be expanded as

$$\overrightarrow{V} = V_{\overline{x}}\widehat{e_{\overline{x}}} + V_{\overline{y}}\widehat{e_{\overline{y}}} = V\left(\cos(\alpha-\theta)\,\widehat{e_{\overline{x}}} + \sin(\alpha-\theta)\,\widehat{e_{\overline{y}}}\right). \tag{16}$$

This transformation is easily visualized if you think of the case when $\theta < \alpha$, but it is also true for any value of θ. Using the concept of direction cosines we can expand the $\overline{O}$ basis in terms of the O basis and find

$$\widehat{e_{\overline{x}}} = \cos\theta\widehat{e_x} + \sin\theta\widehat{e_y} \text{ and } \widehat{e_{\overline{y}}} = \cos\theta\widehat{e_y} - \sin\theta\widehat{e_x} \tag{17}$$

Substituting this basis expansion into the expansion of $\overrightarrow{V}$ in $\overline{O}$ we find

$$\begin{aligned}
\overrightarrow{V} &= V\left(\cos(\alpha-\theta)\left(\cos\theta\widehat{e_x} + \sin\theta\widehat{e_y}\right) + \sin(\alpha-\theta)\left(\cos\theta\widehat{e_y} - \sin\theta\widehat{e_x}\right)\right), \\
\overrightarrow{V} &= V\left(\cos(\alpha-\theta)\cos\theta - \sin(\alpha-\theta)\sin\theta\right)\widehat{e_x} \\
&\quad + V\left(\sin(\alpha-\theta)\cos\theta + \cos(\alpha-\theta)\sin\theta\right)\widehat{e_y}, \\
\overrightarrow{V} &= V\left(\cos\alpha\widehat{e_x} + \sin\alpha\widehat{e_y}\right),
\end{aligned} \tag{18}$$

which was our original expansion confirming that the original vector was unchanged, invariant. As an aside it is useful to note that the components of the vector in $\overline{O}$ can also be expressed as

$$\begin{aligned}
V_{\overline{x}} &= V\cos(\alpha-\theta) = V\cos\alpha\cos\theta + V\sin\alpha\sin\theta \\
&= V_x\cos\theta + V_y\sin\theta, && \text{(19a)} \\
V_{\overline{y}} &= V\sin(\alpha-\theta) = V\sin\alpha\cos\theta - V\cos\alpha\sin\theta \\
&= -V_x\sin\theta + V_y\cos\theta. && \text{(19b)}
\end{aligned}$$

The second kind of product between a pair of vectors is the vector (or cross) product. If the vector $\overrightarrow{p}$ is given by $\overrightarrow{p} = \overrightarrow{r} \times \overrightarrow{s}$, then it has components

$$\begin{aligned}
p_x &= r_y s_z - r_z s_y, && \text{(20a)} \\
p_y &= r_z s_x - r_x s_z, && \text{(20b)} \\
p_z &= r_x s_y - r_y s_x. && \text{(20c)}
\end{aligned}$$

The vector product can also be written equivalently as

$$\overrightarrow{r} \times \overrightarrow{s} = \det\begin{bmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{bmatrix}, \tag{21}$$

where det stands for the determinant. Since interchanging a pair of rows (or columns) changes the sign of a determinant, this expression makes clear that interchanging the order of the vectors in the vector product changes the sign of the resultant vector. The components of the resultant vector can also be given by

$$p_i = \sum_{j,k=1}^{3} \epsilon_{ijk} r_j s_k, \tag{22}$$

where ϵ_{ijk} is the Levi-Civita permutation symbol often just called the permutation symbol. It has the properties

$$\epsilon_{ijk} = \begin{matrix} 0 & \text{if any pair of indices are equal} \\ +1 & \text{if } i,j,k \text{ form an even permutation of } 1,2,3 \\ -1 & \text{if } i,j,k \text{ form an odd permutation of } 1,2,3 \end{matrix} . \tag{23}$$

An even permutation has an even number of exchanges in the position of the indices. Cyclic permutations are always even, e.g. $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$. An odd permutation has an odd number of exchanges in the position of the indices. These all amount to a single permutation from one of the cyclic permutations, e.g. $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$. An important property of the permutation symbol is:

$$\boxed{S_{ij\ell n} = \epsilon_{kij}\epsilon_{k\ell n} = \delta_{i\ell}\delta_{jn} - \delta_{in}\delta_{j\ell}.} \tag{24}$$

This property is often extremely useful in solving a variety of vector identities involving vector products. As an example, we will use the permutation symbol to express the components of a vector that results from the vector product of three vectors. Consider the vector product $\overrightarrow{t} = \overrightarrow{p} \times (\overrightarrow{s} \times \overrightarrow{r})$. If we define $\overrightarrow{u} = \overrightarrow{s} \times \overrightarrow{r}$ then the components of $\overrightarrow{u}$ are given by

$$u_k = \epsilon_{k\ell n} s_\ell r_n, \tag{25}$$

where we have used ℓ and n as our dummy summation indices. Then the components of $\overrightarrow{t} = \overrightarrow{p} \times \overrightarrow{u}$ are expressed as

$$t_i = \epsilon_{ijk} p_j \epsilon_{k\ell n} s_\ell r_n. \tag{26}$$

We can simplify this expression by first cyclically permuting the indices in the first permutation index and note that $\epsilon_{ijk} = \epsilon_{kij}$. Since none the components of the vectors in the vector product depend on the k index we can perform that sum independent of those components. Substituting the the result in equation (24) we find the components of $\overrightarrow{t}$ are

$$t_i = p_j s_\ell r_n \left(\delta_{i\ell}\delta_{jn} - \delta_{in}\delta_{j\ell}\right) = s_i \left(p_j r_j\right) - r_i \left(p_j s_j\right). \tag{27}$$

The terms inside the parentheses are the scalar products $\overrightarrow{p} \cdot \overrightarrow{r}$ and $\overrightarrow{p} \cdot \overrightarrow{s}$. This expression for the components of $\overrightarrow{t}$ enables us to express the vector $\overrightarrow{t}$ via the identity

$$\overrightarrow{t} = \overrightarrow{p} \times (\overrightarrow{s} \times \overrightarrow{r}) = (\overrightarrow{p} \cdot \overrightarrow{r})\,\overrightarrow{s} - (\overrightarrow{p} \cdot \overrightarrow{s})\,\overrightarrow{r}. \tag{28}$$

It is useful to note that this expresion makes it clear that the vector product, $\overrightarrow{p} \times (\overrightarrow{s} \times \overrightarrow{r})$, is orthogonal to $\overrightarrow{p}$, as

$$\overrightarrow{p} \cdot \overrightarrow{t} = \overrightarrow{p} \cdot ((\overrightarrow{p} \cdot \overrightarrow{r})\,\overrightarrow{s} - (\overrightarrow{p} \cdot \overrightarrow{s})\,\overrightarrow{r}) = 0. \tag{29}$$

Another useful notation in proving vector identities is to represent the partial derivative $\partial/\partial x_i$ as ∂_i. As an example of this notation consider the vector quantity $\nabla \times \left(\nabla \times \overrightarrow{A}\right)$. The i^{th} component of this vector expression is

$$\begin{aligned}\left[\nabla \times \left(\nabla \times \overrightarrow{A}\right)\right]_i &= \epsilon_{ijk}\partial_j\epsilon_{k\ell n}\partial_\ell A_n = \left(\delta_{i\ell}\delta_{jn} - \delta_{in}\delta_{j\ell}\right)\partial_j\partial_\ell A_n \\ \left[\nabla \times \left(\nabla \times \overrightarrow{A}\right)\right]_i &= \partial_i\left(\partial_j A_j\right) - \partial_j\partial_j A_i = \left[\nabla\left(\nabla \cdot \overrightarrow{A}\right) - \nabla^2\overrightarrow{A}\right]_i, \end{aligned} \tag{30}$$

and we have easily verified this well known vector identity.

Before we leave our discussion of the vector product we note that the amplitude of $\overrightarrow{p} = \overrightarrow{r} \times \overrightarrow{s}$ is given by

$$|\overrightarrow{p}| = p = rs\sin\theta, \tag{31}$$

where θ is the angle between the two vectors $\overrightarrow{r}$ and $\overrightarrow{s}$. We have already shown that the vectors $\overrightarrow{r}$ and $\overrightarrow{s}$ are geometrical objects and independent of the coordinate system used to describe them. As a result their scaler product and angle between them is invariant. This argument allows us to conveniently define the x axis to be parallel to $\overrightarrow{r}$. A rotation about this axis is all that is required to have $\overrightarrow{s}$ lie in the $x-y$ plane. In this coordinate system the vector product $\overrightarrow{r} \times \overrightarrow{s}$ is perpendicular to the $x-y$ plane (parallel to the z axis) and $p_z = r_x s_y = rs\cos\theta_y$ where $\cos\theta_y$ is the direction cosine between $\overrightarrow{s}$ and the y axis. This angle is invariant and is the complement of the angle between $\overrightarrow{r}$ and $\overrightarrow{s}$. Hence equation (31) is true in any coordinate system. The direction of $\overrightarrow{p}$ is given by the usual right-hand rule. The vector product plays an important role in the discussion of rotational motion. For example, the torque on a body about the origin is defined as the vector product $\overrightarrow{T} = \overrightarrow{r} \times \overrightarrow{F}$.

1.1.2 Differentiation of vectors

Since much of physics involves the differentiation of vectors we need to define how we differentiate vectors. Initially we will consider a vector that only depends on time and take its time derivative. For a scaler, the time derivative is given by

$$\frac{dx}{dt} = \lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t}. \tag{32}$$

In exactly the same way we define the time derivative of a vector as.

$$\frac{d\overrightarrow{r}}{dt} = \lim_{\Delta t \to 0}\frac{\Delta\overrightarrow{r}}{\Delta t}, \tag{33}$$

where

$$\Delta\overrightarrow{r} = \overrightarrow{r}\left(t + \Delta t\right) - \overrightarrow{r}\left(t\right). \tag{34}$$

From this definition it can easily be shown that

$$\frac{d}{dt}\left(\overrightarrow{r} + \overrightarrow{s}\right) = \frac{d\overrightarrow{r}}{dt} + \frac{d\overrightarrow{s}}{dt}, \tag{35}$$

and

$$\frac{d}{dt}\left(f\overrightarrow{r}\right) = \frac{df}{dt}\overrightarrow{r} + f\frac{d\overrightarrow{r}}{dt}. \tag{36}$$

Since our Cartesian basis does not change in time, the time derivative of our position vector is simply

$$\frac{d\overrightarrow{r}}{dt} = \frac{dx}{dt}\widehat{x} + \frac{dy}{dt}\widehat{y} + \frac{dz}{dt}\widehat{z} = \frac{dr_i}{dt}\widehat{e}_i. \tag{37}$$

Comparing this with the velocity vector

$$\overrightarrow{v} = v_x\widehat{x} + v_x\widehat{y} + v_x\widehat{z} = v_i\widehat{e}_i, \tag{38}$$

we see that

$$v_x = \frac{dx}{dt} \text{ etc.} \tag{39}$$

Note that this is true only because the basis vectors are constant in time (and space). When we have to include the time dependence of the basis vectors things get substantially more complicated.

What we have not mentioned is the choice of reference frames. A proper choice can make the solution of many problems essentially trivial or at least much simpler. (e.g. when proving $\overrightarrow{r} \cdot \overrightarrow{s} = rs\cos\theta$ or $|\overrightarrow{r} \times \overrightarrow{s}| = rs\sin\theta$). There is also the possibility of choosing frames that are moving relative to each other. If they are moving with a uniform velocity then they are called inertial frames and as we shall see Newton's laws hold (at least when the relative velocities are much smaller than the speed of light). If the frames are accelerating or rotating (a form of acceleration) then Newton's laws do not hold in their standard form.

1.1.3 Mass and Force

We will take the point of view that the amount of mass contained in an object is simply the amount of stuff contained in the object. Given a balance in a uniform gravitational field, the relative masses of two objects can be easily determined. An alternate and equivalent way is shown in Figure 1-1. In an inertial balance two masses are equal if and only if a force applied at the rod's midpoint causes

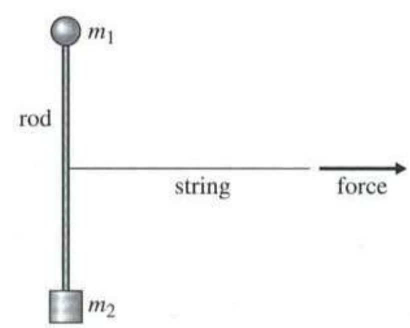

Figure 1-1. An inertial balance compares the masses of m_1 and m_2 that are attached to the opposite ends of a rigid rod. The masses are equal if and only if a force applied to the rod's midpoint causes them to accelerate at the same rate so that the rod does not rotate.

them to accelerate at the same rate. Since in a uniform gravitational field equal masses weigh the same, it is often easier to simply use a spring balance to determine the amount of mass in an object. A spring balance can also be used to measure the relative strengths of separate forces.

1.1.4 Newton's First and Second Laws

First we consider a particle or a point mass. Later on we will expand this concept but NTL there are many situations where this concept is an excellent approximation to the physical problem of interest. Newton's first two laws should be familiar to you. The first is given as

$$\boxed{\text{In the absence of forces, a particle moves with a uniform velocity } \overrightarrow{v}.} \tag{40}$$

This is often referred to the *law of inertia*. The second of Newton's laws states that if a net force $\overrightarrow{F}$ acts on a particle of mass m, this results in an acceleration of the particle given by

$$\boxed{\overrightarrow{F} = m\overrightarrow{a}.} \tag{41}$$

Here $\overrightarrow{a}$ is defined as

$$\overrightarrow{a} = \frac{d\overrightarrow{v}}{dt} \equiv \dot{\overrightarrow{v}}, \text{ or } \overrightarrow{a} = \frac{d^2\overrightarrow{r}}{dt^2} \equiv \ddot{\overrightarrow{r}}. \tag{42}$$

Newton's second law can also be written as

$$\overrightarrow{F} = \dot{\overrightarrow{p}}, \tag{43}$$

where $\overrightarrow{p} = m\overrightarrow{v}$ is defined to be the momentum of our point particle. In fact, with the proper definitions for $\overrightarrow{F}$, $\overrightarrow{p}$, and the time derivative, it is this form that is also correct in special relativity. Clearly both of these laws are intended to be applied in inertial frames and are no longer true in these forms in accelerating frames. Newton's second law is also often referred to as the *equation of motion,* EOM.

Most philosphers of science consider Newton's second law as the definition of a force. The quantities of mass, length, and time are well defined. Hence the quantity force, $\overrightarrow{F}$, is defined in terms of these quantities via Newton's second law.

1.1.5 Newton's Third Law

If you are leaning against a wall, it is clear that the wall is exerting a force back onto you. This is often stated as, *for every action there is an equal and opposite reaction.* To be more precise Newton's third law is stated as, "if object 1 exerts a force $\overrightarrow{F}_{21}$ on object 2, then object 2 always exerts an equal and opposite reaction force on object 1", or

$$\boxed{\overrightarrow{F}_{21} = -\overrightarrow{F}_{12}} \tag{44}$$

Think of the gravitational force between the Earth and the Moon.

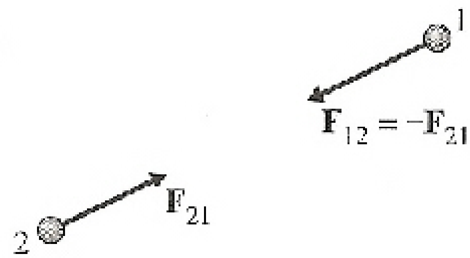

Figure 1-2. Newton's third law states that the reaction force exerted on object 1 by object 2 is equal and opposite to the force exerted by 2 on 1, i.e. $F_{12} = -F_{21}$

As an example we will consider two particles. Assume that an external force is present, and they interact with each other as well. The net force, $\overrightarrow{F}_1$ on particle 1 is

$$\overrightarrow{F}_1 = \overrightarrow{F}_{12} + \overrightarrow{F}_1^{\text{ext}} = \dot{\overrightarrow{p}}_1, \tag{45}$$

where $\dot{\overrightarrow{p}}_1$ is the rate of change in the momentum of particle 1 and similarly

$$\overrightarrow{F}_2 = \overrightarrow{F}_{21} + \overrightarrow{F}_2^{\text{ext}} = \dot{\overrightarrow{p}}_2. \tag{46}$$

Defining the total momentum of the system as $\overrightarrow{P} = \overrightarrow{p}_1 + \overrightarrow{p}_2$, then the rate of change of the total momentum is

$$\dot{\overrightarrow{P}} = \dot{\overrightarrow{p}}_1 + \dot{\overrightarrow{p}}_2 = \overrightarrow{F}_{12} + \overrightarrow{F}_1^{\text{ext}} + \overrightarrow{F}_{21} + \overrightarrow{F}_2^{\text{ext}}. \tag{47}$$

Because of Newton's third law the internal forces cancel and

$$\dot{\overrightarrow{P}} = \overrightarrow{F}_1^{\text{ext}} + \overrightarrow{F}_2^{\text{ext}} = \overrightarrow{F}^{\text{ext}}, \tag{48}$$

where we have defined

$$\overrightarrow{F}^{\text{ext}} = \overrightarrow{F}_1^{\text{ext}} + \overrightarrow{F}_2^{\text{ext}}. \tag{49}$$

This is an important result as it asserts that if there are no external forces, $\overrightarrow{F}^{\text{ext}} = 0$, then $\dot{\overrightarrow{P}} = 0$, and the total momentum for the pair of particles is conserved. Additionally, the rate of change for the total momentum of the system is determined only by the external force acting on the pair of particles.

The analysis for a system of N particles is a straightforward extension of that used for a two particle system. Consider a particle designated by α. The net force on this particle given by

$$\overrightarrow{F}_\alpha = \sum_{\beta \neq \alpha}^{N} \overrightarrow{F}_{\alpha\beta} + \overrightarrow{F}_\alpha^{\text{ext}} = \dot{\overrightarrow{p}}_\alpha. \tag{50}$$

Here the sum over β includes all of the particles other than the α particle as it does not exert a force on itself. This sum is true for any of the N particles in the multiparticle system. The total momentum for this system is given by the sum

$$\overrightarrow{P} = \sum_{\alpha=1}^{N} \overrightarrow{p}_\alpha. \tag{51}$$

The sum here covers all N particles. Differentiating this expression with respect to time we find

$$\dot{\overrightarrow{P}} = \sum_{\alpha=1}^{N} \dot{\overrightarrow{p}}_\alpha = \sum_{\alpha=1}^{N} \overrightarrow{F}_\alpha \tag{52}$$

From equation (50) this sum is given by

$$\sum_{\alpha=1}^{N} \overrightarrow{F}_\alpha = \sum_{\alpha=1}^{N} \sum_{\beta\neq\alpha}^{N} \overrightarrow{F}_{\alpha\beta} + \sum_{\alpha=1}^{N} \overrightarrow{F}_\alpha^{\text{ext}}. \tag{53}$$

The double sum is a sum over α and β such that all terms in which $\alpha = \beta$ are omitted. Imagine a matrix in which you sum over all of the terms except for those on the diagonal. Since α and β are dummy summation indices we can exchange them and write

$$\sum_{\alpha=1}^{N} \sum_{\beta\neq\alpha}^{N} \overrightarrow{F}_{\alpha\beta} = \sum_{\alpha,\beta\ (\alpha\neq\beta)}^{N} \overrightarrow{F}_{\alpha\beta} = \sum_{\beta,\alpha\ (\alpha\neq\beta)}^{N} \overrightarrow{F}_{\beta\alpha}. \tag{54}$$

In this last step, interchanging the dummy summation indices amounts to simply performing the sum in a different order, but it results in the same total sum. From Newton's third law we know that $\overrightarrow{F}_{\alpha\beta} = -\overrightarrow{F}_{\beta\alpha}$. Hence this sum must vanish. It might add some insight to note that the matrix given by the components $\overrightarrow{F}_{\alpha\beta}$ is an antisymmetric matrix in which $\overrightarrow{F}_{\alpha\beta} = -\overrightarrow{F}_{\beta\alpha}$ and as in any antisymmetric matrix the diagonal term, $\overrightarrow{F}_{\alpha\alpha}$, vanishes. Summing all of the terms in this matrix also vanishes. Since this term vanishes equations (52) and (53) become

$$\dot{\overrightarrow{P}} = \sum_{\alpha=1}^{N} \overrightarrow{F}_\alpha = \sum_{\alpha=1}^{N} \overrightarrow{F}_\alpha^{\text{ext}}. \tag{55}$$

This is analogous to the result for the two particle system in that the rate of change for the total momentum of all of the particles is given by the sum of the external forces. Clearly in the absence of any external force the total momentum of the N particle system is conserved.