26 Lecture 12-02

26.1 Chapter 11 Coupled Oscillators and Normal Modes (con)

26.1.1 Normal Coordinates

When we studied the system with two equal masses and three identical springs we found that we could replace the two coordinates x_1 and x_2 by two *normal coordinates*

$$\xi_1 = \frac{1}{2}(x_1 + x_2) \quad \text{and} \quad \xi_2 = \frac{1}{2}(x_1 - x_2). \tag{1}$$

These coordinates have the property that they always oscillate at just one of the two normal frequencies, ξ_1 at ω_1 and ξ_2 at ω_2. As it turns out we can do the same thing for any system oscillating about a stable equilibrium (albeit for nonlinear oscillations, they must have small amplitudes). If the system has n degrees of freedom, then it is described by n generalized coordinates $q_1, \cdots, q_n$ (holonomic), and has n normal modes with frequencies $\omega_1, \cdots, \omega_n$. We we shall now show is that each normal coordinate ξ_i oscillates at just one frequency, namely the normal frequency ω_i.

Before we proceed it is useful to review our previous discussion of normal coordinates. The two equations of motion for the case of two equal masses and three springs with the outside springs having spring constant k and the middle spring with spring constant K are

$$\left.\begin{array}{rcl} m\ddot{x}_1 & = & -(k+K)\,x_1 + Kx_2 \\ m\ddot{x}_2 & = & Kx_1 - (k+K)\,x_2 \end{array}\right\} \tag{2}$$

If we add these equations we find

$$m\ddot{\xi}_1 = -k\xi_1, \tag{3}$$

while subtracting yields

$$m\ddot{\xi}_2 = -(k+2K)\,\xi_2. \tag{4}$$

These two equations are uncoupled and show that each normal coordinate oscillates at a single frequency, ξ_1 at ω_1 and ξ_2 at ω_2. In other words the normal coordinates behave just like the coordinates of uncoupled oscillators and by going over to the normal coordinates, we have uncoupled oscillations.

Just as the equations for x_1 and x_2 can be rewritten as a single matrix equation $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$, so too can the equations for ξ_1 and ξ_2 be rewritten as $\mathbf{M}'\ddot{\boldsymbol{\xi}} = -\mathbf{K}'\boldsymbol{\xi}$. The important difference here is that the two matrices $\mathbf{M}'$ and $\mathbf{K}'$ are both diagonal.

$$\mathbf{M}' = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \text{ and } \mathbf{K}' = \begin{bmatrix} k & 0 \\ 0 & k+2K \end{bmatrix}. \tag{5}$$

The transform from the original coordinates to the normal coordinates is said to diagonalize the matrices $\mathbf{M}$ and $\mathbf{K}$. That the new matrices are diagonal is precisely equivalent to the statement that the equations for ξ_1 and ξ_2 are uncoupled and will oscillate independently. It should be noted that with our diagonalization process we have simultaneously diagonalized two separate matrices, $\mathbf{M}$ and $\mathbf{K}$. For those of you with some background in linear algebra that 2 is the maximum number of matrices that can be diagonalized simultaneously.

We can define the two normal coordinates differently, and more generally, in terms of the eigenvectors $\mathbf{a}$ that describe the motion of the normal modes and are determined by the eigenvalue equation $\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{a} = 0$. Now we wish to label each of the column vectors so that

$$\mathbf{a}_{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{6}$$

Two important points need to be made here. The first is that each of these vectors contains an arbitrary multiplier $Ae^{-i\delta}$. However now we will change this and fix $\delta = 0$ and $A = 1$ which we have done in equation (6). Another choice and sometimes a better choice is to normalize the vectors with a factor of $1/\sqrt{2}$. In our case this does not lead to any simplification and for notational simplicity we will stay with $A = 1$. The other point is that each column is made up of two components, two different numbers, which we have labeled as a_1 and a_2. However, now we are discussing two different columns, $\mathbf{a}_{(1)}$ and $\mathbf{a}_{(2)}$, one for each normal mode. For now we will use the parenthesis in the subscripts to emphasize this distinction. Just as we could expand the normal coordinates in terms of the generalized coordinates of the system, we can invert this to expand the generalized coordinates in terms of the normal coordinates. Specifically for the problem we have been discussing this is expressed as

$$\mathbf{x} = \xi_1\mathbf{a}_{(1)} + \xi_2\mathbf{a}_{(2)} = \begin{bmatrix} \xi_1 + \xi_2 \\ \xi_1 - \xi_2 \end{bmatrix}. \tag{7}$$

The first equality defines ξ_1 and ξ_2 in terms as the coefficients in the expansion of $\mathbf{x}$ in terms of the eigenvectors $\mathbf{a}_{(1)}$ and $\mathbf{a}_{(2)}$. The last term in this relation shows that ξ_1 and ξ_2 are precisely the normal coordinates for this problem. That is the normal coordinates can be defined as the coefficients in the expansion of $\mathbf{x}$ in terms of the eigenvectors $\mathbf{a}_{(1)}$ and $\mathbf{a}_{(2)}$. We shall now see that this definition carries over naturally to the general case for coupled oscillators with n degrees of freedom.

The General Case We will now consider the case with n generalized coordinates $q_1, \cdots, q_n$, and n normal modes. In the i^{th} mode the column vector $\mathbf{q}_{(i)}$ oscillates sinusoidally at the normal mode frequency ω_i,

$$\mathbf{q}_{(i)} = \mathbf{a}_{(i)} \cos\left(\omega_i t - \delta_i\right)$$

where the column vector satisfies

$$\mathbf{K}\mathbf{a}_{(i)} = \omega_i^2\mathbf{M}\mathbf{a}_{(i)}. \tag{8}$$

The columns $\mathbf{a}_{(i)}$ are n independent real $n \times 1$ column vectors and any $n \times 1$ column vector can be expanded in terms of them. That is the column vectors $\mathbf{a}_{(i)}$ for a complete set for the space of $n \times 1$ vectors. Thus any solution of the equations of motion $\mathbf{q}(t)$ can be expanded as

$$\mathbf{q}(t) = \sum_i \xi_i(t)\, \mathbf{a}_{(i)}. \tag{9}$$

Now the column vector $\mathbf{q}(t)$ satisfies the equation of motion

$$\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{q}.$$

If we replace $\mathbf{q}(t)$ with the expansion in equation (9) the equation of motion becomes

$$\sum_i \ddot{\xi}_i(t)\, \mathbf{M}\mathbf{a}_{(i)} = -\sum_i \xi_i(t)\, \mathbf{K}\mathbf{a}_{(i)} = -\sum_i \xi_i(t)\, \omega_i^2 \mathbf{M}\mathbf{a}_{(i)}, \tag{10}$$

where the last step follows from equation (8). Now the n column vectors $\mathbf{a}_{(i)}$ are independent and this property is unchanged when the operated on by $\mathbf{M}$, therefore the coefficients on each side of this equation must also be equal. That is

$$\ddot{\xi}_i = \omega_i^2 \xi_i(t). \tag{11}$$

Hence the normal coordinates for a system with n degrees of freedom do in-fact oscillate at their normal mode frequencies independently of each other.

Normal Coordinates for Double Pendulum As another example of normal coordinates we will examine the double pendulum for the case we have been studying with $m_1 = m_2 = m$ and $L_1 = L_2 = L$. For this problem we found the equations of motion to be

$$2mL^2\ddot{\phi}_1 + mL^2\ddot{\phi}_2 = -2mgL\phi_1, \tag{12a}$$

$$mL^2\ddot{\phi}_1 + mL^2\ddot{\phi}_2 = -mgL\phi_2. \tag{12b}$$

For convenience we divide both of these expressions by mL^2 and find

$$2\ddot{\phi}_1 + \ddot{\phi}_2 = -2\omega_o^2\phi_1, \tag{13a}$$

$$\ddot{\phi}_1 + \ddot{\phi}_2 = -\omega_o^2\phi_2, \tag{13b}$$

where $\omega_o^2 = g/L$. The eigenfrequencies for the normal modes were found to be

$$\omega_1^2 = \left(2 - \sqrt{2}\right)\omega_o^2 \;\text{ and }\; \omega_2^2 = \left(2 + \sqrt{2}\right)\omega_o^2 \tag{14}$$

with corresponding eigenvectors

$$\mathbf{a}_{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \quad \text{and} \quad \mathbf{a}_{(2)} = \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}. \tag{15}$$

From our discussion of normal coordinates

$$\boldsymbol{\phi}(t) = \sum_i \xi_i(t)\,\mathbf{a}_{(i)} = \xi_1(t)\begin{bmatrix}1\\ \sqrt{2}\end{bmatrix} + \xi_2(t)\begin{bmatrix}1\\ -\sqrt{2}\end{bmatrix}. \tag{16}$$

Substituting this result into the equations of motion results in

$$\begin{aligned} 2\left(\ddot{\xi}_1+\ddot{\xi}_2\right)+\sqrt{2}\left(\ddot{\xi}_1-\ddot{\xi}_2\right) &= -2\omega_o^2\left(\xi_1+\xi_2\right), & \text{(17a)}\\ \ddot{\xi}_1+\ddot{\xi}_2+\sqrt{2}\left(\ddot{\xi}_1-\ddot{\xi}_2\right) &= -\sqrt{2}\omega_o^2\left(\xi_1-\xi_2\right) & \text{(17b)}\end{aligned}$$

Regrouping terms yields

$$\begin{aligned} \left(2+\sqrt{2}\right)\ddot{\xi}_1+\left(2-\sqrt{2}\right)\ddot{\xi}_2 &= -2\omega_o^2\left(\xi_1+\xi_2\right), & \text{(18a)}\\ \left(1+\sqrt{2}\right)\ddot{\xi}_1+\left(1-\sqrt{2}\right)\ddot{\xi}_2 &= -\sqrt{2}\omega_o^2\left(\xi_1-\xi_2\right). & \text{(18b)}\end{aligned}$$

Solving for $\ddot{\xi}_1$ and $\ddot{\xi}_2$ we find

$$\begin{aligned} \ddot{\xi}_1 &= -\frac{1}{2\sqrt{2}}\det\begin{bmatrix}-2\omega_o^2\left(\xi_1+\xi_2\right) & 2-\sqrt{2}\\ -\sqrt{2}\omega_o^2\left(\xi_1-\xi_2\right) & 1-\sqrt{2}\end{bmatrix} = -\left(2-\sqrt{2}\right)\omega_o^2\xi_1, & \text{(19a)}\\ \ddot{\xi}_2 &= -\frac{1}{2\sqrt{2}}\det\begin{bmatrix}2+\sqrt{2} & -2\omega_o^2\left(\xi_1+\xi_2\right)\\ 1+\sqrt{2} & -\sqrt{2}\omega_o^2\left(\xi_1-\xi_2\right)\end{bmatrix} = -\left(2+\sqrt{2}\right)\omega_o^2\xi_2. & \text{(19b)}\end{aligned}$$

As expected these equations of motion are decoupled with the normal coordinates oscillating at their respective normal mode frequencies. It should be noted that for this problem the normal coordinates are given by

$$\xi_1 = \frac{\phi_1+\phi_2/\sqrt{2}}{2} \quad\text{and}\quad \xi_2 = \frac{\phi_1-\phi_2/\sqrt{2}}{2}, \tag{20}$$

demonstrating that the normal coordinates depend on the expressions for the eigenvectors for each individual scenario.

26.1.2 Three Coupled Pendulums

To further demonstrate some of these concepts we will now consider three identical pendulums coupled by two identical springs as shown in figure 11.10

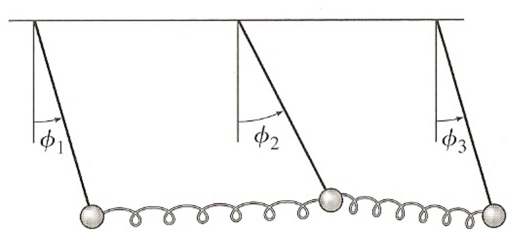

Figure 11.10. Three identical pendulums of length L and mass m are coupled by two identical springs of spring constant k. The natural lengths of the springs are equal to the separation of the supports of the pendulums, so that at equilibrium $\phi_1 = \phi_2 = \phi_3$.

As generalized coordinates it is natural to use the three angles ϕ_1, ϕ_2, and ϕ_3 with equilibrium occurring at $\phi_1 = \phi_2 = \phi_3 = 0$. We now need to find the Lagrangian at least for small displacements. The systematic approach is to write down the exact expressions for T and U and then make the small amplitude approximations. In the present case, finding the potential energy of the springs for any arbitrary angle can be cumbersome. So, as it often happens, we will make the small amplitude approximation for T and U directly and save a lot of tedious algebra.

The kinetic energy of the three pendulums is the same independent of the small angle approximation and is

$$T = \frac{1}{2} m L^2 \left(\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2 \right). \tag{21}$$

The gravitational potential energy of each pendulum has the form $U_i = mgL\left(1 - \cos\phi_i\right)$ and in the small angle approximation becomes $U_i = \frac{1}{2} mgL\phi_i^2$. Hence the total gravitational potential energy is

$$U_{grav} = \frac{1}{2} mgL \left(\phi_1^2 + \phi_2^2 + \phi_3^2 \right). \tag{22}$$

To find the potential energy of the two springs requires us to find how much each is stretched (or compressed). For arbitrary angles this is a messy affair, but for small angles the only appreciable stretching comes from the horizontal displacements of the pendulum bobs each of which moves a horizontal distance given by $L\phi$. With this in mind the total spring potential energy is

$$\begin{aligned} U_{spr} &= \frac{1}{2} k L^2 \left[\left(\phi_2 - \phi_1\right)^2 + \left(\phi_3 - \phi_2\right)^2 \right] \\ U_{spr} &= \frac{1}{2} k L^2 \left(\phi_1^2 + 2\phi_2^2 + \phi_3^2 - 2\phi_1\phi_2 - 2\phi_3\phi_2 \right). \end{aligned} \tag{23}$$

Before we proceed it saves a lot of algebra to choose a set of units such that the uninteresting parameters have the value of 1, a process sometimes described as choosing *natural units*. In this problem we choose the unit of mass to be m and the unit of length to be L. With this choice both m and L disappear from the calculation and simplifies the trivial details of the calculation. Then once the calculation is complete, if it is of interest, we can use the required units of the results to reinsert these quantities. For example we will find that one of the normal modes will be $\omega^2 = g$. However the quantity g/ω^2 has the units of length which we had predetermined to be L. Hence $g/\omega^2 = L$ and the normal mode frequency is $\omega^2 = g/L$. Thus we can put this quantity back into the final results if so required. It should be noted that in special relativity it is often convenient to define the speed of light to be unity, $c = 1$. Then all velocities become some fraction less than one commonly denoted as β. It is a straightforward procedure when the calculation is finished to reinsert c where required.

With this in mind we will choose units such that $m = L = 1$, and our kinetic and potential energies become

$$T = \frac{1}{2}\left(\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2\right) \tag{24}$$

and

$$U = \frac{1}{2}g\left(\phi_1^2 + \phi_2^2 + \phi_3^2\right) + \frac{1}{2}k\left(\phi_1^2 + 2\phi_2^2 + \phi_3^2 - 2\phi_1\phi_2 - 2\phi_3\phi_2\right). \tag{25}$$

We could now write down the Lagrangian followed by the equations of motion, but there is actually no need to do this. We already know that the result will be in the matrix form

$$\mathbf{M}\ddot{\boldsymbol{\phi}} = -\mathbf{K}\boldsymbol{\phi}, \tag{26}$$

where in this case $\boldsymbol{\phi}$ is a 3×1 column vector with components ϕ_1, ϕ_2, and ϕ_3. The components for $\mathbf{M}$ and $\mathbf{K}$ can be read directly from T and U and are

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} g+k & -k & 0 \\ -k & g+2k & -k \\ 0 & -k & g+k \end{bmatrix}. \tag{27}$$

The normal modes of our system have the familiar form $\boldsymbol{\phi}(t) = \operatorname{Re}\mathbf{z}(t) = \operatorname{Re}\mathbf{a}e^{i\omega t}$, where $\mathbf{a}$ and ω are determined by the eigenvalue equation

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{a} = 0. \tag{28}$$

The first step is to find the normal frequencies from the characteristic or secular equation $\det\left(\mathbf{K} - \omega^2\mathbf{M}\right) = 0$. The matrix $\mathbf{K} - \omega^2\mathbf{M}$ is

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} g+k-\omega^2 & -k & 0 \\ -k & g+2k-\omega^2 & -k \\ 0 & -k & g+k-\omega^2 \end{bmatrix}. \tag{29}$$

The determinant is easily determined to be

$$\begin{aligned}
\det\left(\mathbf{K} - \omega^2\mathbf{M}\right) &= \left(g+k-\omega^2\right)^2\left(g+2k-\omega^2\right) - 2k^2\left(g+k-\omega^2\right) \\
\det\left(\mathbf{K} - \omega^2\mathbf{M}\right) &= \left(g+k-\omega^2\right)\left[\left(g+k-\omega^2\right)\left(g+2k-\omega^2\right) - 2k^2\right] \\
\det\left(\mathbf{K} - \omega^2\mathbf{M}\right) &= \left(g+k-\omega^2\right)\left[\left(g-\omega^2\right)^2 + 3k\left(g-\omega^2\right)\right] \\
\det\left(\mathbf{K} - \omega^2\mathbf{M}\right) &= \left(g+k-\omega^2\right)\left(g-\omega^2\right)\left(g-\omega^2+3k\right)
\end{aligned}$$

so that the three normal frequencies are

$$\omega_1^2 = g, \quad \omega_2^2 = g+k, \quad \omega_2^2 = g+3k. \tag{30}$$

For those that are concerned about such issues, in units where m and L are not unity these three frequencies are

$$\omega_1^2 = g/L, \quad \omega_2^2 = g/L + k/m, \quad \omega_2^2 = g/L + 3k/m. \tag{31}$$

Knowing the three normal frequencies we can now find the corresponding three normal modes. Substituting ω_1 into the eigenvalue equation (28) yields

$$\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which has as solutions $a_1 = a_2 = a_3 = A_1 e^{-i\delta_1}$. That is, in the first normal mode

$$\phi_1(t) = \phi_2(t) = \phi_3(t) = A_1 \cos(\omega_1 t - \delta_1), \tag{32}$$

and the three pendulum oscillate in phase and amplitude as shown in figure 11.11(a).

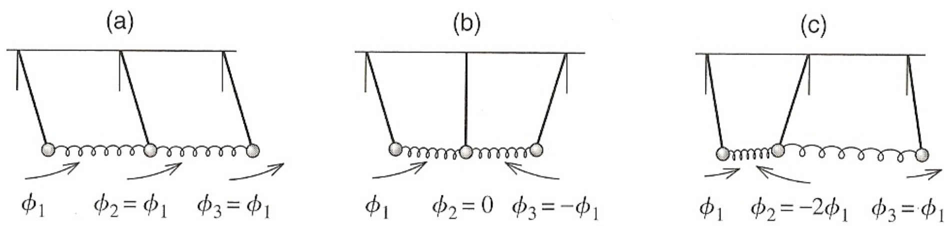

Figure 11.11. The three normal modes for three coupled pendulums. (a) The pendulums swing in unison and the springs remain in equilibrium. (b) The two outer pendulums oscillate exactly out of phase with equal amplitudes while the middle pendulum is stationary. (c) The outer pendulums swing in unison while the middle pendulum swings exactly out of phase with twice the amplitude of the outer pendulums.

In this mode the springs are not stretched (nor compressed) and do not play a role in determining the frequency.

Substituting ω_2 into the eigenvalue equation (28) yields

$$\begin{bmatrix} 0 & -k & 0 \\ -k & k & -k \\ 0 & -k & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which has as solutions $a_2 = 0$ and $a_1 = -a_3 = A_2 e^{-i\delta_2}$. That is, in the second normal mode

$$\phi_2 = 0 \quad \text{and} \quad \phi_1(t) = -\phi_3(t) = A_2 \cos(\omega_2 t - \delta_2), \tag{33}$$

and the two outer pendulums oscillate exactly out of phase with equal amplitudes as shown in figure 11.11(b). Finally substituting ω_3 into the eigenvalue equation (28) yields

$$\begin{bmatrix} -2k & -k & 0 \\ -k & -k & -k \\ 0 & -k & -2k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which has as solutions $a_2 = -2a_1 = -2a_3 = A_3 e^{-i\delta_3}$. That is, in the third normal mode

$$\phi_1(t) = \phi_3(t) = -\phi_2(t)/2 = A_3 \cos(\omega_3 t - \delta_3), \tag{34}$$

and the two outer pendulums oscillate in phase with equal amplitudes while the middle pendulum oscillates exactly out of phase with twice the amplitude as shown in figure 11.11(c). The general solution is an arbitrary linear combination of all three modes.

Normal Modes for Three Coupled Pendulums The equations of motion for the three coupled pendulums were

$$m\ddot{\phi}_1 = -(g+k)\phi_1 + k\phi_2, \tag{35}$$

$$m\ddot{\phi}_2 = k\phi_1 - (g+2k)\phi_2 + k\phi_3, \tag{36}$$

$$m\ddot{\phi}_3 = k\phi_2 - (g+k)\phi_3. \tag{37}$$

Again expanding the three angular coordinates in normal coordinates;

$$\boldsymbol{\phi}(t) = \sum_i \xi_i(t)\,\mathbf{a}_{(i)} = \frac{\xi_1(t)}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\xi_2(t)}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}. + \frac{\xi_3(t)}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \tag{38}$$

The first thing to notice is that these eigenvectors all had different magnitudes, hence it was necessary to normalized them. Substituting these expansions into the equations of motion we find

$$\left(\ddot{\xi}_1/\sqrt{3} + \ddot{\xi}_2/\sqrt{2} + \ddot{\xi}_3/\sqrt{6}\right) = -g\frac{\xi_1}{\sqrt{3}} - (g+k)\frac{\xi_2}{\sqrt{2}} - (g+3k)\frac{\xi_3}{\sqrt{6}}, \tag{39}$$

$$\left(\ddot{\xi}_1/\sqrt{3} - 2\ddot{\xi}_3/\sqrt{6}\right) = -g\frac{\xi_1}{\sqrt{3}} + 2(g+3k)\frac{\xi_3}{\sqrt{6}}, \tag{40}$$

$$\left(\ddot{\xi}_1/\sqrt{3} - \ddot{\xi}_2/\sqrt{2} + \ddot{\xi}_3/\sqrt{6}\right) = -g\frac{\xi_1}{\sqrt{3}} + (g+k)\frac{\xi_2}{\sqrt{2}} - (g+3k)\frac{\xi_3}{\sqrt{6}}. \tag{41}$$

Solving for $\ddot{\xi}_1$ yields

$$\ddot{\xi}_1 = -\det\begin{bmatrix} -g\xi_1/\sqrt{3} - (g+k)\xi_2/\sqrt{2} - (g+3k)\xi_3/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{6} \\ -g\xi_1/\sqrt{3} + 2(g+3k)\xi_3/\sqrt{6} & 0 & -2/\sqrt{6} \\ -g\xi_1/\sqrt{3} + (g+k)\xi_2/\sqrt{2} - (g+3k)\xi_3/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} = -g\xi_1. \tag{42}$$

Solving for $\ddot{\xi}_2$ yields

$$\ddot{\xi}_2 = -\det\begin{bmatrix} 1/\sqrt{3} & -g\xi_1/\sqrt{3} - (g+k)\,\xi_2/\sqrt{2} - (g+3k)\,\xi_3/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & -g\xi_1/\sqrt{3} + 2\,(g+3k)\,\xi_3/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{3} & -g\xi_1/\sqrt{3} + (g+k)\,\xi_2/\sqrt{2} - (g+3k)\,\xi_3/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} = -(g+k)\,\xi_2. \tag{43}$$

Finally solving for $\ddot{\xi}_3$ yields

$$\ddot{\xi}_3 = -\det\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -g\xi_1/\sqrt{3} - (g+k)\,\xi_2/\sqrt{2} - (g+3k)\,\xi_3/\sqrt{6} \\ 1/\sqrt{3} & 0 & -g\xi_1/\sqrt{3} + 2\,(g+3k)\,\xi_3/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -g\xi_1/\sqrt{3} + (g+k)\,\xi_2/\sqrt{2} - (g+3k)\,\xi_3/\sqrt{6} \end{bmatrix} = -(g+3k)\,\xi_3. \tag{44}$$

Again as (as expected) the normal coordinates are decoupled from each other and oscillate at their respective normal mode frequencies. The one difference to note here is that it was necessary to normalize the eigenvectors prior to expanding $\boldsymbol{\phi}(t)$ in terms of the eigenvectors.

This completes our study of normal modes as well as any new material for Physics 110A for the fall quarter. Hope you found at least some of it interesting.