16   Lecture 11-2

16.1   Chapter 7 Lagrange's Equations (con)

Two Unconstrained Particles   So far we have shown that Lagrange's equation leads to the correct path for a single unconstrained particle through the appropriate configuration space. That is Lagrange's equations are identical to Newton's second law. The generalized coordinates may be the Cartesian coordinates or some other equivalent set of coordinates, e.g. spherical polar or cylindrical polar. But the path is unchanged by this change in coordinates. So for this simple case of a single unconstrained particle we have verified Hamiltonian's principle as the action integral is stationary along these paths.

Here we will discuss the situation for two particles, mainly to examine the form of Lagrange's equations for $N > 1$. For two particles the Lagrangian is defined as before as $\mathcal{L} = T - U$. This means that the Lagrangian is now given by

$$\mathcal{L} = \frac{1}{2}m\dot{\vec{r}}_1^2 + \frac{1}{2}m\dot{\vec{r}}_2^2 - U\left(\vec{r}_1, \vec{r}_2\right), \tag{1}$$

where as usual the forces on the particles are $\vec{F}_1 = -\nabla_1 U$ and $\vec{F}_2 = -\nabla_2 U$. Hence Newton's second law can be expressed as

$$F_{1x} = \dot{p}_{1x}, \quad F_{1y} = \dot{p}_{1y}, \quad \cdots \quad F_{2z} = \dot{p}_{2z}. \tag{2}$$

Each of these six equations is equivalent to a corresponding Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1}, \quad \frac{\partial \mathcal{L}}{\partial y_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}_1}, \quad \cdots \quad \frac{\partial \mathcal{L}}{\partial z_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}_2}. \tag{3}$$

These six equations imply that the action integral $S = \int \mathcal{L} dt$ is stationary. We can transform the integrand to any other suitable set of six coordinates, $q_1, q_2, \cdots, q_6$ and the action integral will still be stationary. This implies that Lagrange's equations must be true with respect to the new coordinates:

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}, \cdots, \frac{\partial \mathcal{L}}{\partial q_6} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_6}. \tag{4}$$

An example of a set of six such generalized coordinates that we shall use repeatedly when we study the two body central force problem is this: The three coordinates of the CM position $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1 + m_2)$, and the three coordinates of the relative position of the two particles, $\vec{r} = \vec{r}_1 - \vec{r}_2$. We will find that this choice leads to a dramatic simplification. For now however, the main point is that Lagrange's equations are automatically true in their standard form, equation (4) with respect to the new generalized coordinates.

The extension of these ideas to the case of N unconstrained particles leads to the $3N$ Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}, \quad [i = 1, 2, \cdots, 3N]. \tag{5}$$

These equations are valid for any choice of the $3N$ coordinates $q_1, q_2, \cdots, q_{3N}$ needed to describe the N particles. So now we see that Hamilton's principle and the resulting Lagrange equations leads to the correct equations of motion for N unconstrained particles as well for a single unconstrained particle.

16.1.1 Constrained System; an Example

One of the great advantages of the Lagrangian approach is the effortless way in which it handles constraints. There are numerous examples of this (almost too many too count), but to get the flavor of how easily this is handled we will again consider a simple example, the plane pendulum. A bob of mass m is attached to a massless rod of length ℓ which rotates without friction in the $x-y$ plane about a point which we will take to be the origin. The bob moves in both the x and y directions, but it is constrained by the rod so that $\sqrt{x^2 + y^2} = \ell$. However only one of the coordinates is independent for as x changes y is predetermined by the constraint equation, or vice versa. Thus the system has only one degree of freedom. One way to express this is to eliminate one of the coordinates, for example we could write $y = \sqrt{\ell^2 - x^2}$ so that we could express everything in terms of x. A much simpler way to proceed is to eliminate both x and y in terms of the angle ϕ, the angle between the pendulum and its equilibrium position.

Both the kinetic and potential energy can be expressed in terms of ϕ. The kinetic energy is $T = \frac{1}{2} m \ell^2 \dot{\phi}^2$. The potential energy is given by $U = mgh$ where h is the height above equilibrium. A little trigonometry shows that this height is $h = \ell \left(1 - \cos \phi \right)$. We can now write the Lagrangian as

$$\mathcal{L} = \frac{1}{2} m \ell^2 \dot{\phi}^2 - mg\ell \left(1 - \cos \phi \right). \tag{6}$$

Now it is a fact that once a system is expressed in terms of a single generalized coordinate (for a system with only one degree of freedom), the evolution of this constrained system also satisfies Lagrange's equation. With ϕ as our generalized coordinate Lagrange's equation is

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}}. \tag{7}$$

These derivatives are easily evaluated to give

$$-mg\ell \sin \phi = m \ell^2 \ddot{\phi}. \tag{8}$$

The quantity $-mg\ell \sin \phi$ is just the torque Γ exerted by gravity on the pendulum, while $m\ell^2$ is the pendulum's momentum of inertia. This equation is then a specific example of an equation on motion in cylindrical coordinates and demonstrates that the correct equation of motion is obtained in presence of a constraint.

16.1.2 Constrained Systems in General

Generalized Coordinates Consider a system of N particles with positions $\overrightarrow{r}_\alpha$. The set of parameters $q_1, q_2, \cdots, q_n$ are a set of generalized coordinates for the system if each position $\overrightarrow{r}_\alpha$ can be expressed as a function of $q_1, q_2, \cdots, q_n$, and possibly the time, t,

$$\overrightarrow{r}_\alpha = \overrightarrow{r}_\alpha(q_1, q_2, \cdots, q_n, t) \quad [\alpha = 1, \cdots, N] \tag{9}$$

Conversely each q_i can be expressed in terms of the $\overrightarrow{r}_\alpha$ and possibly t,

$$q_i = q_i(\overrightarrow{r}_1, \cdots, \overrightarrow{r}_N, t) \quad [i = 1, \cdots, n]. \tag{10}$$

In addition we require that the number of the generalized coordinates (n) is the smallest number that allows for the system to parameterized. In our three dimensional world, n is certainly no more than $3N$ and for a constrained system it may be much less. For example in a rigid body, the number of particles is on the order of Avogadro's number, whereas the number of generalized coordinates is 6, three coordinates to give the location of the CM and three more to give the orientation of the body.

To illustrate the relation between the location of particles and the generalized coordinates, we will again consider the simple pendulum. There is one particle, the bob, and two Cartesian coordinates. As we saw there is just one generalized coordinate, which we chose to be the angle ϕ. So the analog of equation (9) is

$$\overrightarrow{r} = (x, y) = \overrightarrow{r}(\phi) = (\ell \sin \phi, \ell \cos \phi). \tag{11}$$

Here the two Cartesian coordinates are expressed in terms of just one generalized coordinate ϕ.

The double pendulum shown in figure 7.1 has two bobs, both confined to a plane, so it has four Cartesian coordinates, all of which can be expressed in
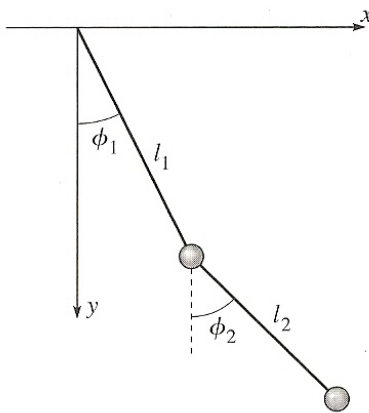

Figure 7.1 The double pendulum is uniquely specified by two generalized coordinates ϕ_1 and ϕ_2 which can be varied independently.

terms of the two generalized coordinates ϕ_1 and ϕ_2. So if we define the origin to be located at the suspension point of the top pendulum we have

$$\overrightarrow{r}_1 = \overrightarrow{r}_1(\phi_1) = (\ell_1 \sin \phi_1, \ell_1 \cos \phi_1), \tag{12}$$

and

$$\overrightarrow{r}_2 = \overrightarrow{r}_2(\phi_1, \phi_2) = (\ell_1 \sin \phi_1 + \ell_2 \sin \phi_2, \ell_1 \cos \phi_1 + \ell_2 \cos \phi_2). \tag{13}$$

Here the components of $\overrightarrow{r}_2$ depend on both ϕ_1 and ϕ_2.

In these two examples the transformation between the Cartesian and generalized coordinates did not depend on time. However it is easy to construct examples in which it does. Consider the railroad car in figure 7.2. which has a pendulum fixed to the ceiling and is being forced to accelerate at a constant acceleration a. It is natural to specify the position of the pendulum by the angle of the bob as usual. However we must recognized that this gives the position of the pendulum relative to the accelerating (non inertial) frame of the car. If we wish to specify the position of the bob relative to an inertial frame, e.g. the ground, then we note that the position
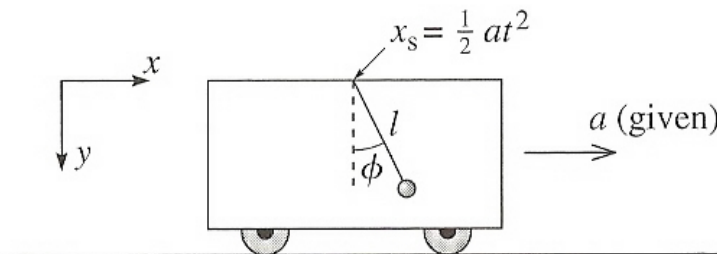

Figure 7.2 A pendulum suspended from the roof of a railroad car that is accelerating with an acceleration a.

of the bob must have the additional term $x = \frac{1}{2}at^2$ added to its position inside the car. Its position is now given by

$$\overrightarrow{r} = (x, y) = \overrightarrow{r}(\phi) = \left(\ell \sin \phi + \frac{1}{2}at^2, \ell \cos \phi \right). \tag{14}$$

Now the relation between $\overrightarrow{r}$ and the generalized coordinate ϕ includes a dependence on t, which was a possibility that we allowed for in equation (9).

We shall sometimes describe a set of coordinates $q_1, q_2, \cdots, q_n$ as *natural* if the relation between the Cartesian coordinates $\overrightarrow{r}_\alpha$ and the generalized coordinates does not involve the time t. We shall find certain convenient properties of natural coordinates that do not apply to coordinates for which the relation in equation (9) does involve the time. Fortunately, as the name implies, there are many problems for which the most convenient choice of coordinates is natural.

Degrees of Freedom The number of degrees of freedom of a system is the number of coordinates that can be independently varied in a small displacement. For example, the simple pendulum has just one degree of freedom, while the double pendulum has two. A particle that is free to move anywhere in three dimensions has three degrees of freedom, while a gas of N particles has $3N$.

When the number of degrees of freedom of an N-particle in three dimensions is less than $3N$, then the system is constrained. Of course this number is $2N$ in two dimensions. The bob in a simple pendulum is constrained. The two masses of the double pendulum is also constrained. Other examples include the N atoms of a rigid body ($3N$ coordinates→degrees of freedom), a bead constrained to move on a fixed wire, and a particle constrained to move on a fixed surface in three dimensions.

In all of the examples we have given so far, the number of degrees of freedom was equal to the number of generalized coordinates needed to describe the system's configuration. A system with this natural-seeming property is said to be *holonomic*. That is a holonomic system has n degrees of freedom and can be described by n generalized coordinates, $q_1, q_2, \cdots, q_n$. Holonomic systems are easier to treat than nonholonomic systems and in this course we shall restrict ourselves to holonomic systems.

At a first glance you might think that all systems would be holonomic, or at least that nonholonomic systems would be rare and extremely complicated. If fact, there are some quite simple examples of nonholonomic systems. Consider a hard rubber ball that is free to roll without slipping on a horizontal plane. Starting at any position it can move in only two independent directions. Therefore it has two degrees of freedom. You might well imagine that its configuration could be uniquely specified by two coordinates, x and y, of its center. But clearly the orientation of the ball as it travels to another position on the plane depends on the path taken to reach that position. Evidently two coordinates is not enough to specify a unique configuration. In fact we need three more coordinates to specify the configuration completely, two to determine an axis of rotation and another to determine the angular rotation about that axis. The ball has two degrees of freedom but five generalized coordinates. Clearly the ball is a nonholonomic system.

From the example of the ball, it is clear that holonomic systems exist, however they are more complicated to analyze than holonomic systems and we shall not discuss them any further. For any holonomic system with generalized coordinates $q_1, q_2, \cdots, q_n$ and potential energy $U(q_1, q_2, \cdots, q_n, t)$, the evolution in time is determined by the n Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}, \quad [i = 1, 2, \cdots, n], \tag{15}$$

where the Lagrangian $\mathcal{L}$, is defined as usual as $\mathcal{L} = T - U$.

16.1.3 Review of Hamilton's Principle and Lagrange's Equations to this Point

Given a particle with a kinetic energy T and a potential energy U, *Hamilton's principle* states that the so-called action integral

$$\boxed{S = \int \mathcal{L}\left(x, y, z, \dot{x}, \dot{y}, \dot{z}, t\right) dt},\tag{16}$$

where $\mathcal{L}$ is the Lagrangian defined by $\mathcal{L} = T - U$, is stationary along the particle's path. It was straightforward to show that the Euler-Lagrange equations, or simply the Lagrange equations when applied to the action integral,

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i},\tag{17}$$

reproduced Newton's law of motion for a particle in Cartesian coordinates. Since the integral is unchanged under a coordinate transformation, we can consider a set of generalized coordinates q_1, q_2, q_3 that may be more appropriate for the system under consideration and the action integral will still be stationary along the particle's path. This means that the solution to the Lagrange equations in these coordinates

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i},\tag{18}$$

will still describe the path of the particle. We also noted that the Lagrange equations in polar cylindrical coordinates reproduced Newton's second law for those coordinates as they had to if they were to correctly describe the path of the particle. However they were arrived at in a natural way by simply expressing the kinetic and potential energy in polar coordinates.

We found that the Lagrange equations when expressed in generalized coordinates take the form

$$(\text{generalized force}) = \frac{d}{dt}(\text{generalized momentum}),\tag{19}$$

where

$$\frac{\partial \mathcal{L}}{\partial q_i} = (i\text{th component of a generalized force}),\tag{20}$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = (i\text{th component of a generalized momentum}).\tag{21}$$

When expressed this way if the Lagrangian is independent of any of the generalized coordinates, then that component of the generalized momentum is constant. To find a conservation law we merely have to note whether or not the Lagrangian depends on that coordinate.

The extension to N unconstrained particles was also straightforward and we saw that Lagrange's equations again reproduced Newton's laws of motion for a

multiparticle system. Again the Cartesian coordinates used in the action integral could be transformed into a generalized set of coordinates without changing the integral and consequently the paths of the particles. In these generalized coordinates the form of Lagrange's equations remains unchanged as for a stationary integral they must still satisfy the necessary Euler-Lagrange equations.

We also demonstrated for the simple case of the pendulum that Lagrange's equations reproduced the correct equations of motion for this constrained system. However, verifying that the Lagrange's equations are the correct equations for a single system with a constraint is far from a proof that the Lagrange equations apply for constrained systems in general. Thus the only problem that remains is to prove that Lagrange's equations are the correct equations systems with constraints.

16.1.4   Proof of Lagrange's Equations with Constraints

With these concepts, we are now ready to prove Lagrange's equations for any holonomic system. To start we will consider the case of just one particle (the extension to an arbitrary number of particles is fairly straightforward). We will propose that the particle is constrained to move on a surface. This means that it has two degrees of freedom and can be described by two generalized coordinates q_1 and q_2.

First we will consider the forces of constraint on the particle. In general the forces of constraint are not necessarily conservative, but that is not relevant since one of the main objectives of the Lagrangian approach is to find equations that don't involve the constraint forces (which we usually don't want to know anyway). Notice however that if the constraining forces are nonconservative, then the Lagrange equations in the simple unconstrained form certainly do not apply. We shall denote the net constraining force by $\overrightarrow{F}_{cstr}$, which for our example is just the of the surface to which the particle is confined.

Second, there are the "nonconstraint forces" on the particle, such as gravity. These are the forces with which we are usually concerned in practice, and we shall denote the sum of these forces by $\overrightarrow{F}$. We will assume that these forces all satisfy at least the second condition for a force to be conservative so that they are derivable from a potential energy, $U(\overrightarrow{r}, t)$, and

$$\overrightarrow{F} = -\nabla U(\overrightarrow{r}, t). \tag{22}$$

(It turns out that if all of the nonconstraint forces are conservative, then U is time independent, but we don't need to assume this.) The total force on the particle is then $\overrightarrow{F}_{tot} = \overrightarrow{F}_{cstr} + \overrightarrow{F}$.

Finally we will assume that the Lagrangian is defined as usual to be $\mathcal{L} = T - U$. Here U excludes the constraint forces. This reflects that Lagrange's equations for a constrained system eliminates the need to know the constraint forces as we shall soon see.

The Action Integral is Stationary at the Right Path Consider two points $\overrightarrow{r}_1$ and $\overrightarrow{r}_2$ through which the particle passes at times t_1 and t_2. Now $\overrightarrow{r}(t)$ will be the correct path, the actual path that the particle follows between the two points, and $\overrightarrow{R}(t) = \overrightarrow{r}(t) + \overrightarrow{\epsilon}(t)$ is any nearby wrong path. Since both $\overrightarrow{r}(t)$ and $\overrightarrow{R}(t)$ are both constrained to lie in the surface to which the particle is confined, the small vector $\overrightarrow{\epsilon}(t)$ which connects the correct path and some nearby incorrect path must also be constrained to lie in the surface. Additionally, since both $\overrightarrow{r}(t)$ and $\overrightarrow{R}(t)$ must pass through the same endpoints, $\overrightarrow{\epsilon}(t_1) = \overrightarrow{\epsilon}(t_2) = 0$.

We again denote the action integral along the correct path as S,

$$S = \int_{t_1}^{t_2} \mathcal{L}\left(\overrightarrow{r}, \dot{\overrightarrow{r}}, t\right) dt. \tag{23}$$

In a manner analogous to the scheme we used to derive the Euler-Lagrange equations we denote as δS as the small change in the action integral from the correct path to a nearby incorrect path as

$$\delta S = \int_{t_1}^{t_2} \mathcal{L}\left(\overrightarrow{R}, \dot{\overrightarrow{R}}, t\right) dt - \int_{t_1}^{t_2} \mathcal{L}\left(\overrightarrow{r}, \dot{\overrightarrow{r}}, t\right) dt \tag{24}$$

We also denote the change in the Lagrangian from the correct path to the incorrect path as

$$\delta\mathcal{L} = \mathcal{L}\left(\overrightarrow{R}, \dot{\overrightarrow{R}}, t\right) - \mathcal{L}\left(\overrightarrow{r}, \dot{\overrightarrow{r}}, t\right). \tag{25}$$

If we substitute $\overrightarrow{R}(t) = \overrightarrow{r}(t) + \overrightarrow{\epsilon}(t)$ and

$$\mathcal{L}\left(\overrightarrow{r}, \dot{\overrightarrow{r}}, t\right) = \frac{1}{2}m\dot{\overrightarrow{r}}^2 - U\left(\overrightarrow{r}, t\right),$$

then via a Taylor's series expansion equation (25) becomes

$$\delta\mathcal{L} = \frac{1}{2}m\left[\left(\dot{\overrightarrow{r}}(t) + \dot{\overrightarrow{\epsilon}}(t)\right)^2 - \dot{\overrightarrow{r}}^2(t)\right] - \left[U\left(\overrightarrow{r} + \overrightarrow{\epsilon}(t), t\right) - U\left(\overrightarrow{r}, t\right)\right],$$

$$\delta\mathcal{L} = m\dot{\overrightarrow{r}} \cdot \dot{\overrightarrow{\epsilon}} - \overrightarrow{\epsilon} \cdot \nabla U + O\left(\epsilon^2\right), \tag{26}$$

where $O\left(\epsilon^2\right)$ denotes terms of second order in ϵ and $\dot{\epsilon}$. Then the change in the action integral, equation (24), becomes

$$\delta S = \int_{t_1}^{t_2} \left(m\dot{\overrightarrow{r}} \cdot \dot{\overrightarrow{\epsilon}} - \overrightarrow{\epsilon} \cdot \nabla U\right) dt. \tag{27}$$

As usual the first term is integrated by parts and we find

$$\delta S = - \int_{t_1}^{t_2} \left(m\ddot{\overrightarrow{r}} + \nabla U\right) \cdot \overrightarrow{\epsilon}\, dt. \tag{28}$$

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Now the right path, $\overrightarrow{r}(t)$, satisfies Newton's second law. Therefore $m\overset{..}{\overrightarrow{r}}$ is equal to the total force on the particle, $\overrightarrow{F}_{tot} = \overrightarrow{F}_{cstr} + \overrightarrow{F}$. Meanwhile $\nabla U = -\overrightarrow{F}$. This means that in equation (28) we are left with

$$\delta S = -\int_{t_1}^{t_2} \overrightarrow{F}_{cstr} \cdot \overrightarrow{\epsilon}\, dt. \qquad (29)$$

But the constraint force is normal to the surface in which the particle moves, while $\overrightarrow{\epsilon}$ lies in the surface. Therefore $\overrightarrow{F}_{cstr} \cdot \overrightarrow{\epsilon} = 0$, and we have proved that $\delta S = 0$. That is the action integral is stationary for the right path. It should be noted that the observation that $\overrightarrow{F}_{cstr} \cdot \overrightarrow{\epsilon} = 0$, is the crucial step in the proof. When you consider the generalization of the proof to an arbitrary constrained system, problem 7.13 for the student, you will find a corresponding step and that the corresponding term is zero. This is because the forces of constraint do no work in a displacement that is consistent with the constraints. In fact, this is one possible definition of a force of constraint.

The Final Proof We have proved that Hamilton's principle, the action integral is stationary for the path that the particle actually takes. However, we have proved it, not for arbitrary variations of the path, but rather variations that are consistent with the constraints - i.e. paths which lie on the surface to which the particle was constrained. This means that we cannot prove Lagrange's equations with respect to the three Cartesian coordinates. On the other hand we can prove them with respect to the appropriate generalized coordinates. Since we assumed that the particle was confined by holonomic constraints to move on a surface This means that the particle has two degrees of freedom and can be described by two generalized coordinates q_1 and q_2. Any variation of q_1 and q_2 is consistent with the constraints. Accordingly we can rewrite the action integral as

$$S = \int_{t_1}^{t_2} \mathcal{L}\left(q_1, q_2, \dot{q}_1, \dot{q}_2, t\right) dt, \qquad (30)$$

and this integral is stationary for any variations of q_1 and q_2 about the correct path. Therefore, the Euler-Lagrange equations apply, and we have the two Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} \qquad (31)$$

The proof presented here applies only to a single particle in three dimensions constrained to move on a two dimensional surface. However the main ideas of the general case are all present. The generalization is straightforward and I hope that enough has been said to convince you of the truth of the general result. That is, for any holonomic system with n degrees of freedom and n generalized coordinates, and with the nonconstraint forces derivable from a potential energy

$U\left(q_1, q_2, \cdots, q_n, t\right)$, the path followed by the system is determined by the n Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad [i = 1, \cdots, n], \tag{32}$$

where $\mathcal{L}$ is the Lagrangian $\mathcal{L} = T - U$ and $U = U\left(q_1, q_2, \cdots, q_n, t\right)$ is the total potential energy corresponding to all the forces excluding the forces of constraint.

It was essential to the proof that the nonconstraint forces be conservative or at least be derivable from a potential via $\overrightarrow{F} = -\nabla U$, i.e. path independent but not necessarily independent of time. If this is not true, then Lagrange's equations may not hold at lease in the form of equation (32). An obvious example of this is sliding friction. Sliding friction cannot be regarded as a force of constraint (it is not normal to the surface) and it cannot be derived from a potential energy. Thus when sliding friction is present they do not hold in the form of equation (32). Lagrange's equations can be modified to include forces like friction, but the result is clumsy and we will confine ourselves to situations where equation (32) does hold.

Before we move on and consider several examples of Lagrange's equations, it is worthwhile to again list the important advantages of Lagrange's equations versus Newton's equations of motion. First, they take the same form in any generalized coordinate system, i.e. the form of the Euler-Lagrange equations in which the integrand of the stationary integral of interest (the action integral) is the Lagrangian, $\mathcal{L} = T - U$. Second, we need only to find the kinetic and potential energy in generalized coordinates. Since these are scalar quantities, this is typically much easier than determining the vector quantities required in Newton's second law. Third and most important, they eliminate the forces of constraint. This greatly simplifies most problems and as it turns out this simplification comes at almost no cost since we usually do not want to know these forces anyway.