

22 Lecture 11-18

22.1 Chapter 8 Two-Body Central Force Problem (con)

The Orbital Period, Kepler's Third Law We can now find the period of the elliptical orbits of comets and planets. Conceptually all we have to do is integrate the inverse of the angular rate through one complete orbit. Since $\dot{\phi} = L/\mu r^2 = \ell/r^2$, this integral is

$$T = \int_0^{2\pi} \frac{dt}{d\phi} d\phi = \frac{1}{\ell} \int r^2 d\phi. \quad (1)$$

From our solution to the orbit equation for $r(\phi)$, this integral is given by

$$T = \frac{r_o^2}{\ell} \int_0^{2\pi} \frac{1}{(1 + \epsilon \cos \phi)^2} d\phi. \quad (2)$$

This integral may look daunting but it is easily evaluated using pole residue techniques (Mathematica and/or integral tables also provide the answer) with the result

$$\int_0^{2\pi} \frac{1}{(1 + \epsilon \cos \phi)^2} d\phi = \frac{2\pi}{(1 - \epsilon^2)^{3/2}}. \quad (3)$$

Hence the square of the period is

$$T^2 = \frac{4\pi^2 r_o}{\ell^2} \frac{r_o^3}{(1 - \epsilon^2)^3} = \frac{4\pi^2 r_o}{\ell^2} a^3. \quad (4)$$

Substituting for r_o , $r_o = \ell^2/GM$, we find

$$T^2 = \frac{(2\pi)^2}{GM} a^3. \quad (5)$$

This expression states that the square of the period of rotation is proportional to the cube of the semimajor axis and inversely proportional to the total mass of the interacting bodies. Additionally it is independent of the reduced mass. However, Kepler's third law made the point that the square of the period of rotation is proportional to the cube of the semimajor axis while being independent of the mass of the orbiting body. Of course the mass of the Sun is so much greater than any of the planets, e.g. even the mass of Jupiter is only .001 that of the Sun, that for all practical purposes including the mass of the planet in the calculation of $M = M_\odot + m_{planet}$ (M_\odot is the standard symbol for the mass of the Sun) makes very little difference. So Kepler's observation was very accurate within our solar system. We include the correct expression for the period of two orbiting bodies here as it can make a significant difference to astronomers when they are observing the period of binary stars. Before we leave this subject it should be noted that this law applies equally well to all orbiting satellites. For example all the man made satellites of the Earth obey the same law with M

being the total of the Earth's and the orbiting body's mass, which is completely dominated by the mass of the Earth. However, determining the period of the Moon's orbit about the Earth, the mass of the Moon should be included as it is slightly greater than .01 that of the Earth.

Earth's Period Around the Sun and the Period of a Low Earth Orbit One convenient way to measure the mass of the Sun (or any astronomical object) is with Kepler's third law. Since the mass of the Sun is approximately $10^6/3$ times the mass of the Earth we can ignore the Earth's mass in equation (5). All of the other parameters in equation (5) are known, Earth's period is $T = 3.156 \times 10^7$ sec, the gravitational constant $G = 6.67 \times 10^{-11} Nm^2/kg^2$, and since Earth's orbit is nearly circular the semimajor axis is close to the mean distance which is $a = 150 \times 10^9 m$. We merely substitute these values into Kepler's third law and find

$$M_{\odot} = \frac{(2\pi)^2}{GT^2} a^3 = 2 \times 10^{30} kg, \quad (6)$$

which is the accepted value for the Sun's mass. When determining the individual masses of binary stars, equation (5) only results in the sum of the masses for both stars and additional means are required to separate the masses, but interesting though that may be it is a story for another class (astronomy).

Since the mass of the Earth is many orders of magnitude larger than man made satellites we can use equation (5) with the mass of the Earth to determine the period. We will also approximate the semimajor axis for the low lying satellites as the Earth's radius. Then the period should be $T = 2\pi\sqrt{R_E^3/GM_E}$. Recalling that the gravitational acceleration at the surface of the Earth is $g = 9.8m/sec^2$ allows us to write this expression as

$$T = 2\pi\sqrt{\frac{R_E}{g}} = 2\pi\sqrt{\frac{6.38 \times 10^6 m}{9.8m/sec^2}} = 5070 \text{ sec} \simeq 85 \text{ min},$$

which is in agreement with the period for satellites in low Earth orbits.

Relation Between Energy and Eccentricity Finally we can relate the eccentricity to the orbit to the energy E per unit reduced mass, ε , by simply inverting the equation for the eccentricity with the result

$$\varepsilon = \frac{1}{2} \frac{G^2 M^2}{\ell^2} (\epsilon^2 - 1) = \frac{1}{2} \frac{GM}{r_o} (\epsilon^2 - 1). \quad (7)$$

This expression is a useful relation between the physical properties (ε and ℓ) and the geometrical properties (ϵ) of an orbiting body. For example the minimum energy for a body with a given angular momentum occurs when the eccentricity vanishes, i.e. a circular orbit. From the virial theorem we know that for a circular orbit

$$T = -\frac{1}{2}U \rightarrow \varepsilon = -\frac{1}{2} \frac{GM}{r_o},$$

which is consistent with equation (7) This should have been obvious from the form of the effective potential but the above expression states it explicitly.

22.1.1 Unbounded Kepler Orbits

In the previous sections we found the general Kepler orbit,

$$r(\phi) = \frac{\ell^2/GM}{1 + \epsilon \cos \phi}, \quad (8)$$

and examined it in some detail for the bounded orbits, i.e. those for which $\epsilon < 1$ or equivalently $\epsilon < 0$. Now we shall consider the unbounded orbits for which $\epsilon \geq 1$ and $\epsilon \geq 0$. Note that we no longer associate r_o with ℓ^2/GM as a circular orbit no longer makes sense when discussing unbound orbits.

The boundary between the bounded and unbounded orbits occurs when $\epsilon = 1$ or equivalently $\epsilon = 0$. With $\epsilon = 1$ the denominator in the solution for Kepler orbits vanishes at $\phi = \pm\pi$. Thus, $r(\phi) \rightarrow \infty$ as $\phi \rightarrow \pm\pi$. Clearly if $\epsilon = 1$ then the orbit is unbounded. Recognizing that $r \cos \phi = x$ combined with some elementary algebra shows that in Cartesian coordinates the solution for the orbit is

$$y^2 = -2\frac{\ell^2}{GM}x + \frac{\ell^4}{G^2M^2}. \quad (9)$$

This is the expression for a parabola. This orbit is shown with the long dashes in figure 8.9.

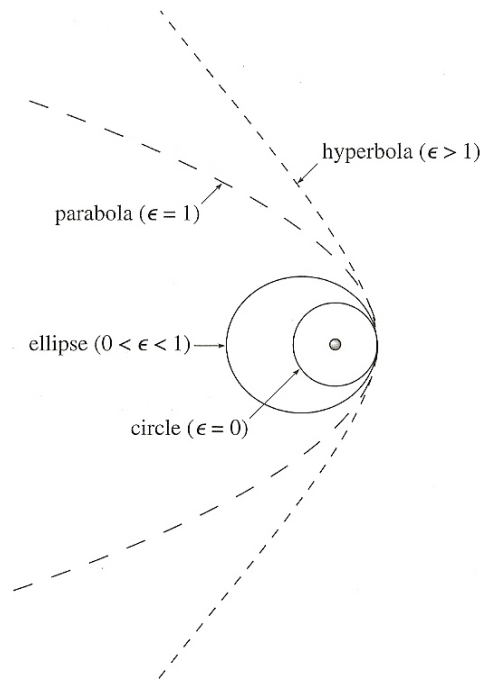


Figure 8.9. Four different Kepler orbits: a circle, an ellipse, a parabola, and a hyperbola.

If $\epsilon > 1$ (or $\varepsilon > 0$) then the denominator in equation (8) vanishes at

$$\epsilon \cos(\phi_{\max}) = -1. \quad (10)$$

Hence, $r(\phi) \rightarrow \infty$ as $\phi \rightarrow \pm\phi_{\max}$ and the orbit is confined to the range of angles $-\phi_{\max} < \phi < \phi_{\max}$. This gives the orbit with a general appearance shown with the dashed lines in figure 3. As an exercise for the student the expression for equation (8) in Cartesian coordinates is of the form

$$\frac{(x - \delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1. \quad (11)$$

This is the equation of a hyperbola and we have proved that the positive energy Kepler orbits are hyperbolas.

Summary of Kepler Orbits At this point it is useful to summarize what we have learned about our results for Kepler orbits. First all of the possible orbits are characterized by one equation,

$$r(\phi) = \frac{\ell^2/GM}{1 + \epsilon \cos \phi}, \quad (12)$$

where the angular momentum ℓ^2 and the eccentricity ϵ are determined by the initial conditions. The energy of a comet depends on these constants through the equation

$$\epsilon = \frac{1}{2} \frac{GM^2}{\ell^2} (\epsilon^2 - 1). \quad (13)$$

It is the eccentricity that determines the shape of the orbit as follows:

eccentricity	energy	orbit
$\epsilon = 0$	$\varepsilon < 0$	circle
$\epsilon < 1$	$\varepsilon < 0$	ellipse
$\epsilon = 1$	$\varepsilon = 0$	parabola
$\epsilon > 1$	$\varepsilon > 0$	hyperbola

As you can see from the expression for $r(\phi)$, the quantity ℓ^2/GM is a scale factor that determines the size of the orbit. It has the dimensions of length and is the distance from the CM of both particles (in our solar system the CM is close to the center of the Sun) to the comet when $\phi = \pi/2$.

22.1.2 Relativistic Corrections To The Orbit Equation

Bending of Light Shortly after Einstein developed and published his famous field equations which allowed for one to solve for the curvature of spacetime in the presence of massive bodies, an astrophysicist named Karl Schwarzschild

found the solution that described the curvature induced by static spherically symmetrical massive bodies. This solution is now referred to as the "Schwarzschild geometry", and it very accurately describes the vacuum geometry outside a nonrotating (or slowly rotating) star. It is the solutions to the orbit equations derived from the Schwarzschild geometry or metric that give rise to the bending of light as it passes the Sun as well as the precession of the perihelion of an elliptical orbit. These results comprised the first confirmation of Einstein's theory of general relativity. It is these orbit equations that we wish to study in this section. We will start with the orbit equation for the path of a photon, i.e. the bending of light as it passes close to the Sun. This orbit equation for a photon is very similar to the path of a particle in the absence of any interaction, equation (??) and is given by

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{1}{\rho_o^2} + \frac{2GM}{c^2}u^3, \quad (14)$$

where c is the speed of light and M is the mass of the spherical body. Again ρ_o is the point of closest approach often called the impact parameter for a particle traveling on a straight-line, or in the language of general relativity, a particle traveling in flat spacetime. The quantity GM/c^2 has the units of length and for the mass of the Sun it is equal to $1.477km$. Since the actual radius of the Sun is approximately $696,000km$ which is also the distance of closest approach, the largest possible value of the quantity GMu/c^2 for a photon as it grazes the Sun is approximately 2×10^{-6} . Clearly this allows us to treat this term perturbatively. Before we proceed it is helpful notationally to nondimensionalize the orbit equation in the form

$$\left(\frac{d\rho_o u}{d\phi}\right)^2 + (\rho_o u)^2 = 1 + 2\frac{GM}{\rho_o c^2}(\rho_o u)^3. \quad (15)$$

The solution that we found for a free particle, i.e. a particle that travels in a straight path, in equation (??) was

$$\rho_o = r \cos(\phi - \delta).$$

For simplicity we will take the arbitrary phase δ to equal $\pi/2$ so that far from the Sun the photon is traveling along a path that corresponds to $y = \rho_o$ a constant, as in figure 8.10.

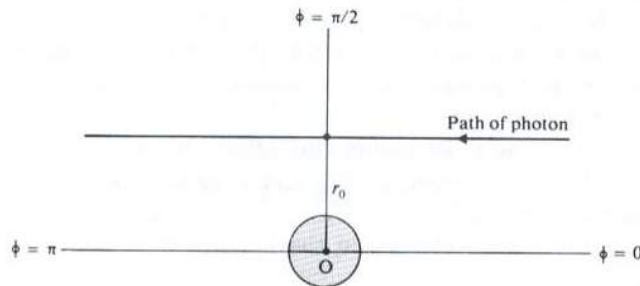


Figure 8.10. Path of a photon in flat spacetime.

Since the perturbative term is small we are justified in considering a trial solution with a linear correction term that deviates from the straight line of the form,

$$\rho_o u = \sin \phi + \frac{GM}{c^2 \rho_o} \rho_o v, \quad (16)$$

where v is yet to be determined. Substituting this into the orbit equation for a photon, equation (15), while keeping only first order terms yields

$$\begin{aligned} \left(\cos \phi + \frac{GM}{c^2 \rho_o} \frac{d\rho_o v}{d\phi} \right)^2 + \left(\sin \phi + \frac{GM}{c^2 \rho_o} \rho_o v \right)^2 &= 1 + 2 \frac{GM}{\rho_o c^2} \sin^3 \phi, \\ \cos \phi \frac{d\rho_o v}{d\phi} + \rho_o v \sin \phi &= \sin^3 \phi. \end{aligned} \quad (17)$$

The quantity on the left hand side of this expression is a perfect differential, $\cos \phi d\rho_o v/d\phi + \rho_o v \sin \phi = \cos^2 \phi d(\rho_o v / \cos \phi) / d\phi$. This allows to simplify this expression,

$$\frac{d(\rho_o v / \cos \phi)}{d\phi} = \frac{\sin^3 \phi}{\cos^2 \phi} = \frac{\sin \phi (1 - \cos^2 \phi)}{\cos^2 \phi}. \quad (18)$$

This expression is easily integrated with the solution

$$\rho_o v = (1 + \cos^2 \phi) + A \cos \phi, \quad (19)$$

where A is a constant of integration. If we insist on $v = 0$ at $\phi = 0$, so that the photon starts out on the horizontal path $y = \rho_o$, then $A = -2$ and $\rho_o v = (1 - \cos \phi)^2$. The complete solution describing the path of the photon is then given by

$$\rho_o u = \sin \phi + \frac{GM}{c^2 \rho_o} (1 - \cos \phi)^2. \quad (20)$$

Now one of the the solutions at $u = 0$ ($r = +\infty$) is given by $\phi = 0$, to find the other solution we let $\phi = \pi + \alpha$, where α is the deviation from a straight-line, as shown in figure 8.11,

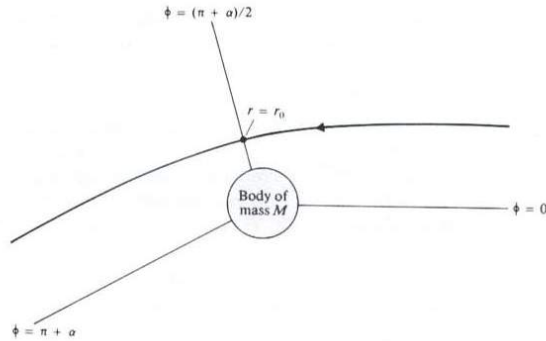


Figure 8.11. Path of a photon as it passes close to the Sun.

and must be of the same order as $GM/c^2\rho_o$. To first order in α we find

$$\sin(\pi + \alpha) + 4\frac{GM}{c^2\rho_o} = 0 \rightarrow \sin\alpha = 4\frac{GM}{c^2\rho_o}. \quad (21)$$

For starlight passing by the Sun, the distance of closest approach is the Sun's radius, thus the maximum deviation of a photon as it grazes the Sun is

$$\alpha = 4 \times 1.477 / (6.96 \times 10^5) = 8.49 \times 10^{-6} \text{ rad} = 1.75 \text{ arcsec}. \quad (22)$$

This is in excellent agreement with the astronomical observations of Sir Arthur Eddington's expedition in which he photographed the relative positions of stars during the solar eclipse of May 29, 1919. The apparent shifting of these positions confirmed Einstein's predictions and were published in newspapers and magazines outside of the scientific arena. It was shortly afterwards that Einstein became a public figure.

Perihelion Advance Although the total energy is defined somewhat differently and is measured with respect to the rest mass energy, in GR the equivalent expression for the conservation of energy is very similar to the Kepler orbit equation. Analogous to the orbit equation for the photon it includes an additional u^3 term,

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2u}{r_o} - 2\frac{GM}{c^2}u^3 = \frac{2\varepsilon}{\ell^2}. \quad (23)$$

When we were considering the orbit of a photon GMu/c^2 was approximately 2×10^{-6} when the largest value of u corresponded to $1/R$ where R was the radius of the Sun. Here the orbit that we wish to describe is that of Mercury for which $u \sim 1/R_{Mer}$ where R_{Mer} is the orbital radius of Mercury and GMu/c^2 is approximately 3×10^{-8} . Hence the effects should be even smaller, but the advance of the perihelion is cumulative and eventually it is observable. This orbit equation does not have an analytical solution, however we can make the assumption that the orbit is nearly circular and define

$$y = u - 1/r_o. \quad (24)$$

This leads to a cubic equation in y in which we can neglect terms of order y^3 . Then assuming a solution of the form

$$y = a + b \cos k\phi, \quad (25)$$

we can solve for k and the resulting rate of precession. For the cases where k is not unity the orbit is not closed and the ellipse precesses about the Sun. For the case of Mercury we find that the perihelion advances at the rate

$$\delta\phi = 5 \times 10^{-7} \text{ radians/orbit} \quad (26)$$

The period for Mercury's orbit is $T = .24\text{yr}$ so

$$\delta\phi = 43''/\text{century}. \tag{27}$$

This may seem to be a very small rate, but due to its accumulative nature astronomers were well aware of it. Prior to Einstein's theory of general relativity, many explanations had been offered including intersolar dust, an unobserved planet inside the orbit of Mercury, etc. All of these hypothesis proved to be unsatisfactory. This was the first observational test of Einstein's model, and after he predicted the correct rate to within the errors at the time he was said to be so excited that, "He could not work for several days."